FAST DISTRIBUTED SUBSPACE PROJECTION VIA GRAPH FILTERS

{thilina.weerasinghe, daniel.romero, cesar.asensio, baltasar.beferull}@uia.no

Department of ICT, University of Agder, Norway

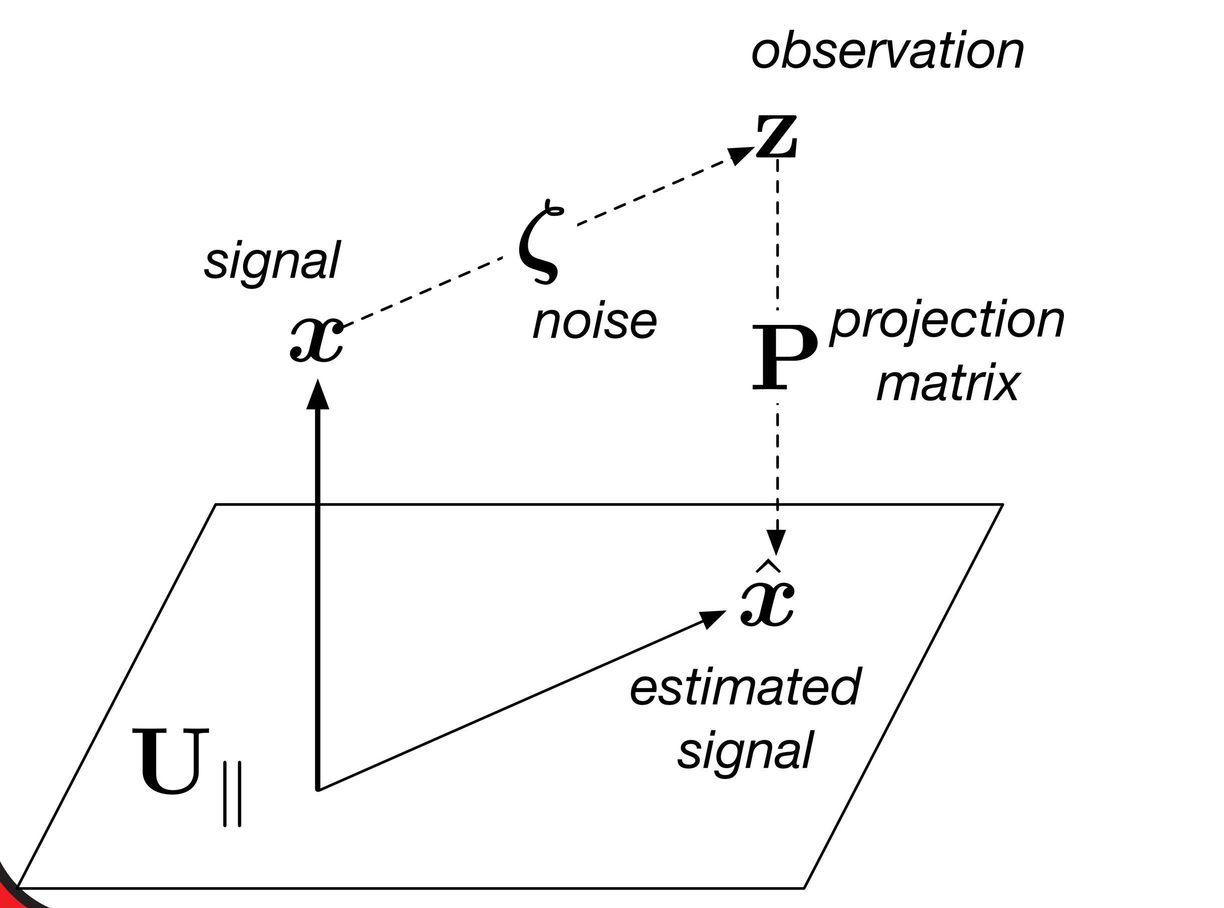
WISENET

Contributions

Result: Subspace projection
- in a decentralized fashion
- in a finite number of iterations.

Novelty: Based on Graph filters
- finds valid shift matrix when it exists and
- is the one that approximately minimizes order
$\rightarrow$ number of communications between nodes.

Subspace projection example



Problem formulation

A graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is considered
- $\mathcal{V} = \{v_1, \dots, v_N\}$ represent N sensors
- edge $(v_n, v_{n'})$ iff sensors communicate.
- self loops $(v_n, v_n) \in \mathcal{E},\ n = 1, \dots, N$ included.

$\mathbf{A}$ is the adjacency matrix:
- $(\mathbf{A})_{n,n'} = 1$ if $(v_n, v_{n'}) \in \mathcal{E}$
- $(\mathbf{A})_{n,n'} = 0$ otherwise.

Goal: estimate signal vector $\boldsymbol{x} \in \mathbb{R}^N$ from observation vector $\mathbf{z} = [z_1, \dots, z_N]^T = \boldsymbol{x} + \boldsymbol{\zeta}$
- $z_n \in \mathbb{R}$ denotes observation of node $v_n \in \mathcal{V}$
- $\boldsymbol{\zeta} \in \mathbb{R}^N$ stands for additive noise.

Knowledge: $\boldsymbol{x}$ is known to lie in the subspace spanned by $\mathbf{U}_\parallel \in \mathbb{R}^{N \times r}$, where $r < N$;
$\rightarrow \boldsymbol{x} = \mathbf{U}_\parallel \boldsymbol{\alpha}$ for some $\boldsymbol{\alpha} \in \mathbb{R}^r$.

Problem: find $\hat{\boldsymbol{x}} \triangleq [\hat{x}_1, \dots, \hat{x}_N]^T = \mathbf{U}_\parallel \mathbf{U}_\parallel^T \mathbf{z} \triangleq \mathbf{P}\mathbf{z}$ given $\mathbf{z}$ and $\mathbf{U}_\parallel$ in a decentralized fashion.

Motivation

Least squares estimation, denoising, weighted consensus, and distributed detection, can be cast as **subspace projection**.

Robustness, scalability, and energy consumption motivate **decentralized algorithms**.

Existing approaches:

(i) Only converges asymptotically to desired result or
(ii) Do not provide shift matrix and do not consider number of steps to converge.

Proposed methodology

Previous work: have shown that a graph filter allows decentralized implementation.
Problem reduced to: find a graph filter $\mathbf{H} := c_0\mathbf{I} + \sum_{l=1}^{L-1} c_l \mathbf{S}^l$ such that $\mathbf{P}\mathbf{z} = \mathbf{H}\mathbf{z},\ \forall \mathbf{z}$.

Our solution: find shift matrix $\mathbf{S}$ and filter coefficients c_l that ensure L nearly minimal.

First step: characterize set of feasible shift matrices

$$\mathcal{S} = \Big\{\mathbf{S} \in \mathbb{R}^{N \times N} : \mathbf{S} = \mathbf{S}^T,\ (\mathbf{S})_{n,n'} = 0 \text{ if } (v_n, v_{n'}) \notin \mathcal{E},$$
$$\exists \mathbf{c} = [c_0, \dots, c_{N-1}]^T \text{ satisfying } \mathbf{U}_\parallel \mathbf{U}_\parallel^{\mathrm{T}} = c_0\mathbf{I} + \sum_{l=1}^{N-1} c_l \mathbf{S}^l\Big\}$$

Key point 1: Matrices in $\mathcal{S}$ can be expressed as $\mathbf{S} = \mathbf{S}_\parallel + \mathbf{S}_\perp$

- $\mathbf{S}_\perp$ is a symmetric matrix satisying $\mathbf{S}_\perp^T \mathbf{U}_\parallel = \mathbf{0}$

- $\mathbf{S}_\parallel = \mathbf{U}_\parallel \mathbf{F} \mathbf{U}_\parallel^T$ for some symmetric $\mathbf{F} \in \mathbb{R}^{r \times r}$.

Second step: Ensure L is nearly minimal

- Requirement: Given a minimal L, filter $\mathbf{H}\mathbf{z} = \mathbf{P}\mathbf{z}\ \forall \mathbf{z}$ must exist.

- Initial result: Minimal L equals the number of different eigenvalues of $\mathbf{F}$ plus $\mathbf{S}_\perp$.

- Difficulty: finding $\mathbf{F}$ and $\mathbf{S}_\perp$ minimizing number of different eigenvalues is non-convex.

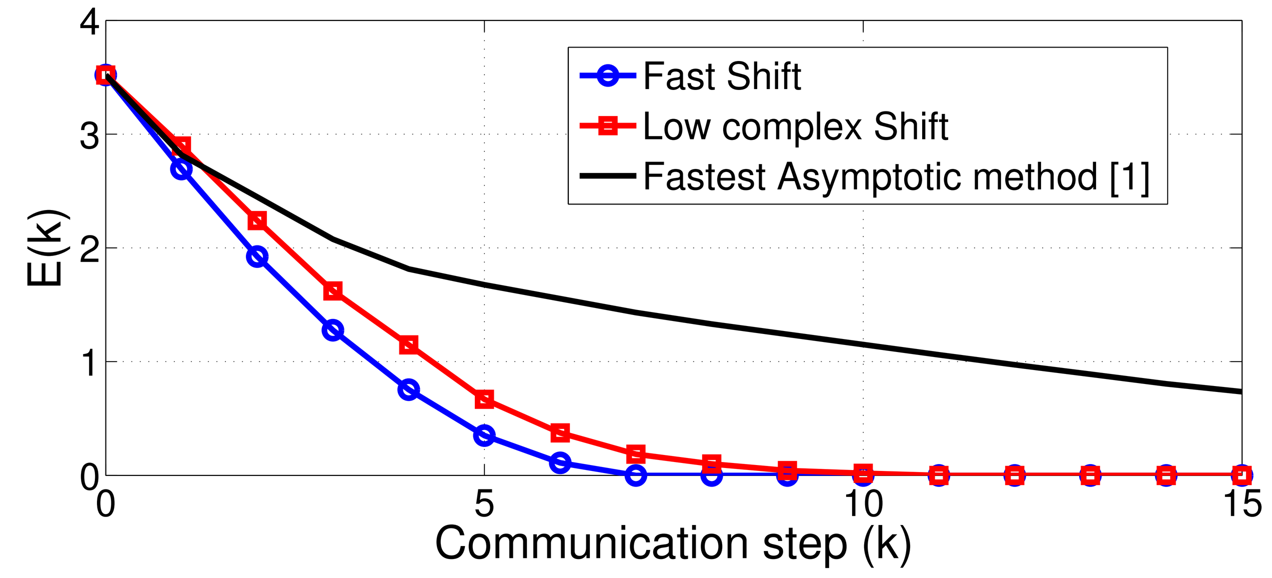
Key point 2: convex surrogate for objective. Similar to ℓ_1-norm replacing zero norm.

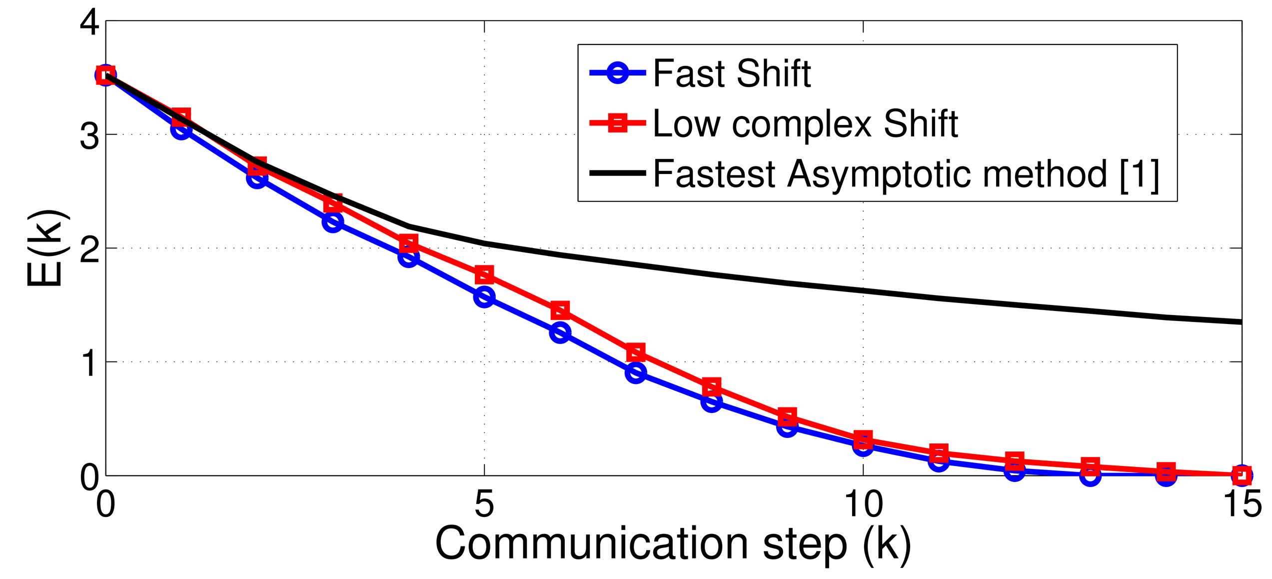
Solution: convex problem:

$$\begin{aligned}
\underset{\mathbf{F}, \mathbf{S}, \mathbf{S}_\parallel, \mathbf{S}_\perp}{\text{minimize}} \quad & ||\mathbf{F} \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{F}||_\star + ||\mathbf{S}_\perp \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{S}_\perp||_\star \\
\text{s. t.} \quad & (\mathbf{S})_{n,n'} = 0 \text{ if } (v_n, v_{n'}) \notin \mathcal{E}, n, n' = 1, \dots, N \\
& \mathbf{S} = \mathbf{S}_\parallel + \mathbf{S}_\perp, \quad \mathbf{S}_\perp = \mathbf{S}_\perp^{\mathrm{T}}, \quad \mathbf{S}_\parallel = \mathbf{S}_\parallel^{\mathrm{T}}, \quad \mathbf{S}_\parallel = \mathbf{U}_\parallel \mathbf{F} \mathbf{U}_\parallel^T, \quad \mathbf{S}_\perp^T \mathbf{U}_\parallel = \mathbf{0}, \\
& \mathrm{tr}(\mathbf{F}) = r, \quad \mathrm{tr}(\mathbf{S}_\perp) \leq N - r - \epsilon
\end{aligned}$$

$\epsilon > 0$ is small positive constant and last two constraints needed to avoid trivial solutions.

Numerical Results





Setting: Monte Carlo simulation.
- Topology $\mathcal{E}$ and matrix $\mathbf{U}_\parallel$ random.
- $N = 25$, $r = [5, 10]$.

What is compared: error $||\mathbf{y} - \mathbf{P}\mathbf{z}||_2$
- exact and approximate solutions vs
- fastest asymptotic method [1].

Error definition:
- For two objectives proposed

$$E(k) = \mathbb{E}_{\mathbf{A},\mathbf{z}} || \sum_{l=0}^{k} c_l^{(k)} \mathbf{S}^l \mathbf{z} - \mathbf{P}\mathbf{z}||_2$$

- For approach in [1]

$$E(k) = \mathbb{E}_{\mathbf{A},\mathbf{z}} ||\mathbf{W}^k \mathbf{z} - \mathbf{P}\mathbf{z}||_2$$

Conclusion: Proposed shifts converge to desired projection in nearly minimal number of steps, outperforming [1].

[1] *S. Barbarossa et al. Distributed signal subspace projection algorithms with maximum convergence rate for sensor networks with topological constraints. ICASSP 2009.*

Funding

The work in this paper was supported by the PETROMAKS Smart-Rig grant 244205/E30, the SFI Offshore Mechatronics grant 237896/O30, and the TOPPFORSK WISECART grant 250910/F20.