Permutation Entropy for Graph Signals

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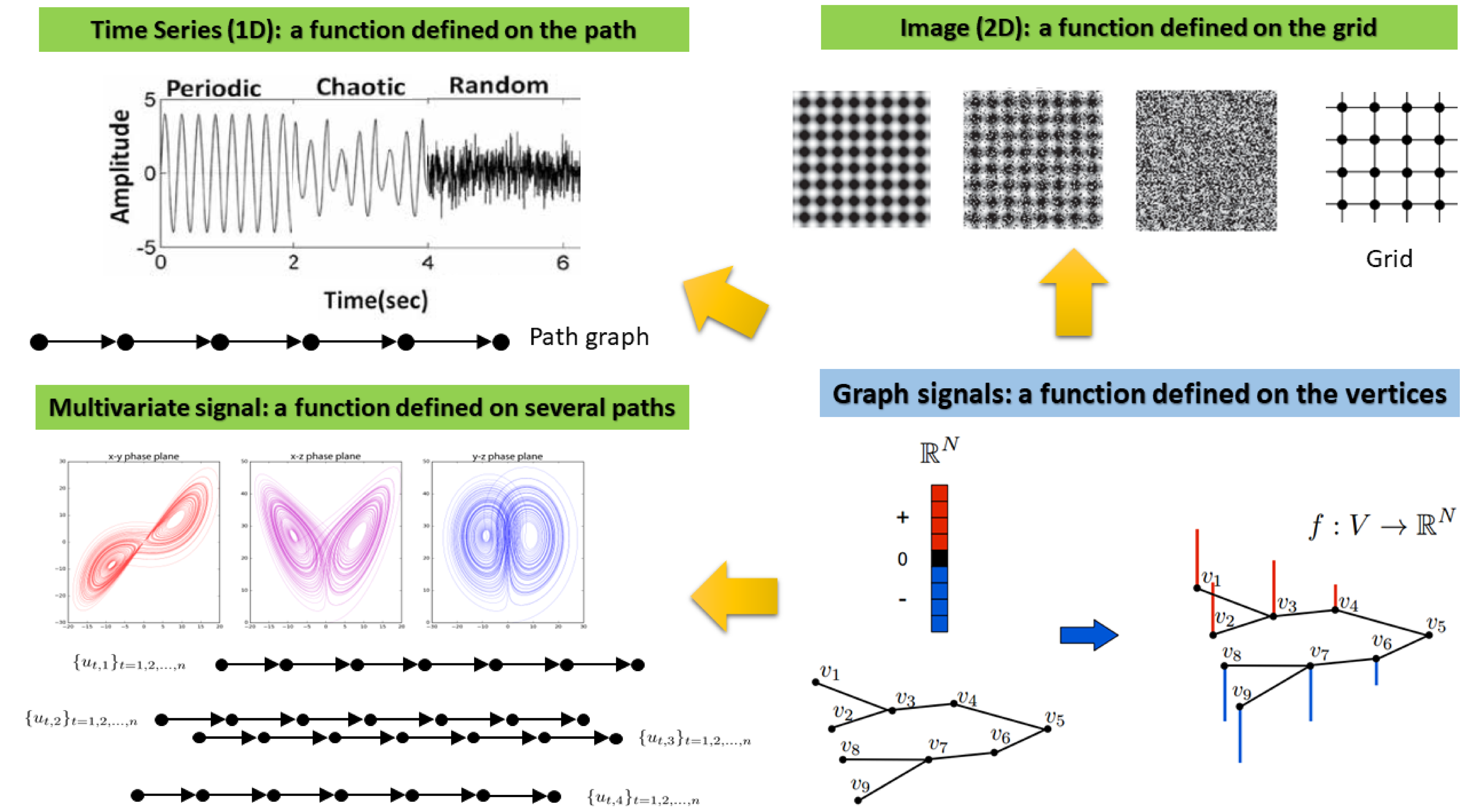
1. Abstract

Entropy metrics (for example, permutation entropy, **PE**) are nonlinear measures of irregularity in time series (one-dimensional data). Some of these entropy metrics can be generalised to data on periodic structures such as a grid or lattice pattern (two-dimensional data) using its symmetry, thus enabling their application to images. However, these metrics have not been developed for signals sampled on irregular domains, defined by a graph. Here, we define for the first time an entropy metric to analyse signals measured over irregular graphs by generalising permutation entropy, a well-established nonlinear metric based on the comparison of neighbouring values within patterns in a time series. Our algorithm, $\mathbf{PE_G}$, is based on comparing signal values on neighbouring nodes, using the adjacency matrix.

We show that this generalisation, **PE**, preserves the properties of classical **PE** for time series and the recent **PE** for images, and it can be applied to any graph structure with synthetic and real signals.

2. Conceptual framework



Entropy metric	Type of signal
PE: Permutation Entropy	Univariate time series, simple, computationally fast
$\mathbf{PE_{2D}}$: PE for 2D data	Bidimensional data, valuable for texture analysis
$\mathbf{PE_G}$: **PE** for Graph Signals	Graph signals (including: time series and image) [1]
MPE: Multivariate **PE**	Multivariate data as a unique block (no interactions)
$\mathbf{MPE_G}$ Mult. graph **PE**	Multivariate data, including cross channel relationships [2]

3. Graph construction

Given a multivariate signal, we construct an underlying graph G as the Cartesian product of two graphs G_1 and G_2, where G_1 has the temporal information of each times series and G_2 models the relations between channels.

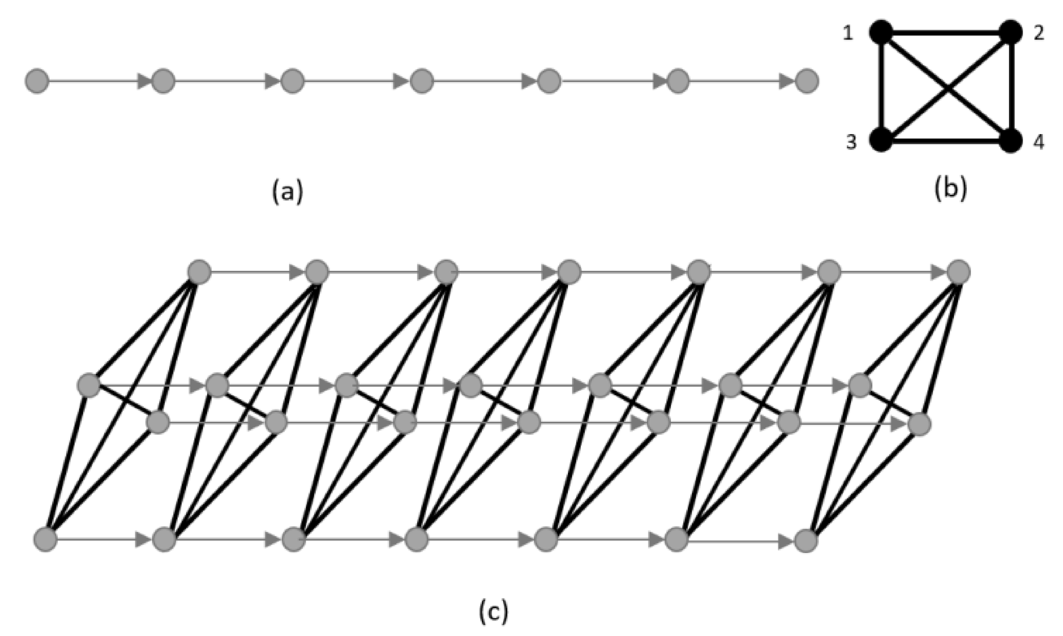


Fig. 1. (a) Directed path with seven vertices, denoted by $\overrightarrow{P_7}$. (b) Interactions between the four channels are encoded with the complete graph on four vertices, denoted by K_4. (c) The Cartesian product $\overrightarrow{P_7}\square K_4$.

4. Multivariate permutation entropy

Let $\mathbf{U} = \{u_{t,s}\}_{t=1,2,\ldots,n}^{s=1,2,\ldots,p}$ be a multivariate time series with interaction graph I_p between channels:

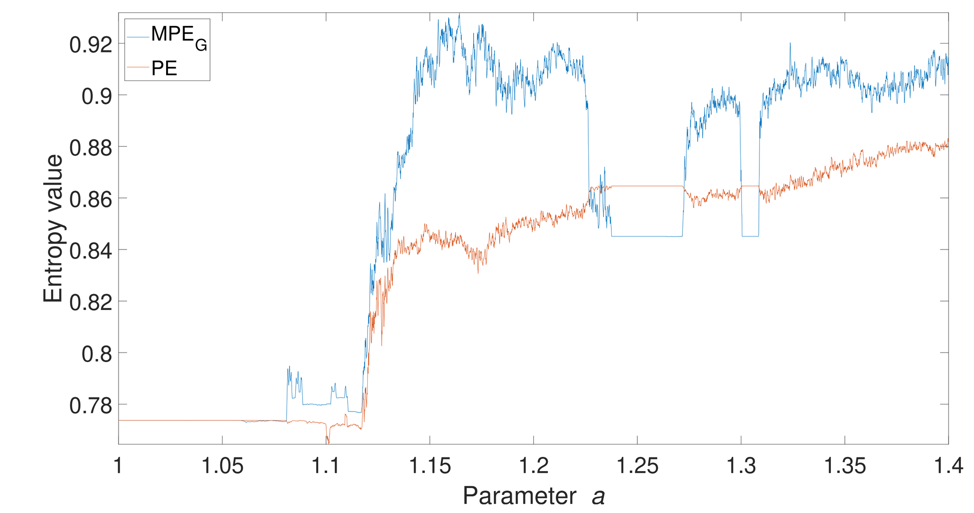
1. **Graph construction** of $\mathcal{G}_\mathbf{U}$ as described in Sec. 3:
$$\mathcal{G}_\mathbf{U} := \overrightarrow{P_n}\square I_p \, .$$
2. **Graph signal U** is defined on the graph $\mathcal{G}_\mathbf{U}$ as in Sec. 1:
$$\mathbf{U} \colon \mathcal{V}(\mathcal{G}_\mathbf{U}) \longrightarrow \mathbb{R} \, .$$
3. $\mathbf{PE_G}$ **for graph signals** is used to define the *multivariate permutation entropy* ($\mathrm{MPE_G}$) for the signal $\mathbf{U}$ on the graph $\mathcal{G}_\mathbf{U}$:
$$\mathrm{MPE_G} = \mathrm{PE_G}(\mathbf{U}) \, .$$

5. The algorithm: $\mathrm{PE_G}$

Construct the embedding vector $\mathbf{y}_i^{m,L} \in \mathbb{R}^m$ given by $\mathbf{y}_i^{m,L} = \left(y_i^0, y_i^L, \ldots y_i^{(m-1)L}\right)$, where
$$y_i^{kL} = \frac{1}{|\mathcal{N}_{kL}(i)|}(\mathbf{A}^{kL}\mathbf{X})_i \, .$$
The vector $\mathbf{y}_i^{m,L}$ is associated with integer numbers (1 to m, a *permutation pattern*) and arranged in increasing order. Shannon entropy for the relative frequency for the distinct permutations $\pi_1, \pi_2, \ldots, \pi_k$
$$\mathrm{PE_G}(m, L) = -\sum_{i=1}^{k} p(\pi_i) \ln p(\pi_i) \, .$$

6. Applications

Examples include the detection of dynamical changes in nonlinear systems such as, **the Hénon map**
$$x_{n+1} = 1 - ax_n^2 + y_n \quad \text{and} \quad y_{n+1} = bx_n \, .$$



$$\textbf{Lorenz system} \begin{cases} x' &= \sigma(y - x) \\ y' &= x(\rho - z) - y \\ z' &= xy - \beta z \end{cases}$$

For $\rho < 1$ all orbits converge to a unique equilibrium point.

	$m = 3$	$m = 4$	$m = 5$	$m = 6$
$\rho = 0.8$	0.452	0.286	0.198	0.147
$\rho = 0.9$	0.453	0.287	0.198	0.148
$\rho = 1.2$	0.725	0.667	0.556	0.447
$\rho = 1.3$	0.722	0.687	0.590	0.481

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7. Conclusions

1. For the first time, the concept of a nonlinear entropy metric – permutation entropy, **PE** – is extended, from unidimensional time series and two-dimensional images to data residing on the vertices of graphs: $\mathbf{PE_G}$.
2. By considering G as a path (1D), $\mathbf{PE_G}$ reduces to the original **PE**.
3. $\mathbf{PE_G}$ enables the extension of similar nonlinear dynamics analysis to data acquired over networks.
4. MATLAB code is freely available at https://github.com/JohnFabila/PEG.

References

[1] J.S. Fabila-Carrasco, C. Tan, and J. Escudero, "Permutation Entropy for Graph Signal", *IEEE Trans. Signal and Inf. Process. over Networks*, 2022, 8, pp. 288-300.

[2] J.S. Fabila-Carrasco, C. Tan, and J. Escudero, "Multivariate permutation entropy, a Cartesian graph product approach", 30th EUSIPCO, Serbia, 2022, pp. 2081-2085.