PoGaIN: Supplementary Material

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I. Maximum Likelihood Derivation

A. Poisson-Noise Modeling

Let us denote the observed noisy image as y and the ground-truth noise-free image as x. Then, the Poisson-Gaussian model takes the form of the following equation

$$y = \frac{1}{a}\alpha + \beta, \quad \alpha \sim \mathcal{P}(ax), \quad \beta \sim \mathcal{N}(0, b^2). \tag{1}$$

Using the linearity property of expectation, we can compute the expected value

$$\mathbb{E}[y] = \frac{1}{a}\mathbb{E}[\alpha] = \frac{1}{a}ax = x. \tag{2}$$

Further, the variance has the following expression

$$\mathbb{V}[y] = \mathbb{E}\left[\left(\frac{1}{a}\alpha + \beta\right)^2\right] - x^2 = \frac{1}{a^2}\mathbb{E}[\alpha^2] + b^2 - x^2. \tag{3}$$

Given that $\mathbb{E}[\alpha^2] = ax + a^2x^2$, we have

$$\mathbb{V}[y] = \frac{x}{a} + x^2 + b^2 - x^2 = \frac{x}{a} + b^2. \tag{4}$$

B. Likelihood Function of Single-Pixel Image

From the definition of the probability mass function (PMF) of a Poisson random variable α, we get

$$\mathbb{P}[\alpha = k] = \frac{e^{-ax}(ax)^k}{k!}, \quad k \geq 0. \tag{5}$$

From the relation between the probability density function (PDF) and the PMF of discrete random variable established with the Dirac delta function, i.e. $f_X(t) = \sum_{k\in\mathbb{Z}} \mathbb{P}[X = k]\delta(t-k)$, we can derive that

$$f_\alpha(t|a,x) = \sum_{k=0}^{\infty} \frac{e^{-ax}(ax)^k}{k!}\delta(t-k). \tag{6}$$

Let us define $\alpha' = \frac{1}{a}\alpha$. Then, the cumulative distribution function (CDF) of this random variable α' has the following form

$$F_{\alpha'}(t) = \mathbb{P}[\alpha' \leq t] = \mathbb{P}[\alpha \leq at] = F_\alpha(at). \tag{7}$$

By taking the derivative of Equation (7), the PDF of α' can be found

$$f_{\alpha'}(t) = \frac{dF_{\alpha'}(t)}{dt} = \frac{dF_\alpha(at)}{dt} = af_\alpha(at). \tag{8}$$

Hence, by combining Equations (6) and (8), the likelihood function of α', which consists of the first part of the noise model, can be derived

$$\begin{aligned} f_{\alpha'}(t|a,x) &= a\sum_{k=0}^{\infty} \frac{e^{-ax}(ax)^k}{k!} \underbrace{\delta(at-k)}_{=\frac{1}{a}\delta(t-\frac{k}{a})} \\ &= \sum_{k=0}^{\infty} \frac{e^{-ax}(ax)^k}{k!}\delta(t-k/a). \end{aligned} \tag{9}$$

On the other hand, the likelihood function of a Gaussian random variable β with 0 mean is defined as

$$f_\beta(t|b) = \frac{1}{b\sqrt{2\pi}}e^{-t^2/2b^2}. \tag{10}$$

We then combine those equations and find the likelihood function of y. Since we know that α' and β are independent of each other, we have that

$$\begin{aligned} \mathcal{L}(y|a,b,x) &= (f_{\alpha'} * f_\beta)(y|a,b,x) \\ &= \sum_{k=0}^{\infty} \frac{(ax)^k}{k!b\sqrt{2\pi}} \exp\left(-ax - \frac{(y-k/a)^2}{2b^2}\right). \end{aligned} \tag{11}$$

C. Maximum Likelihood Solution for Single-Pixel Image

As derived, the maximum likelihood solution for a single-pixel image is the following

$$\begin{aligned} \hat{a}, \hat{b} &= \arg\max_{a,b} \mathcal{L}(y|a,b,x) \\ &= \arg\max_{a,b} \sum_{k=0}^{\infty} \frac{(ax)^k}{k!b\sqrt{2\pi}} \exp\left(-ax - \frac{(y-k/a)^2}{2b^2}\right). \end{aligned} \tag{12}$$

D. Likelihood Function of Multi-Pixel Image

We represent images as vectors of pixels, like y_n and x_n where $n \in \mathbb{N}$ is the index of single pixels. Hence, using this notation we obtain

$$\mathcal{L}(y_n|a,b,x_n) = \sum_{k=0}^{\infty} \frac{(ax_n)^k}{k!b\sqrt{2\pi}} \exp\left(-ax_n - \frac{(y_n-k/a)^2}{2b^2}\right). \tag{13}$$

Given x, i.e., the vector of all x_n, we can see that y_n and $y_{n'}$ are independent $\forall n \neq n'$. Therefore, we have

$$\begin{aligned} \mathcal{L}(y|a,b,x) = \prod_n \sum_{k=0}^{\infty} & \frac{(ax_n)^k}{k!b\sqrt{2\pi}} \\ & \exp\left(-ax_n - \frac{(y_n-k/a)^2}{2b^2}\right). \end{aligned} \tag{14}$$

E. Maximum Likelihood Solution for Multi-Pixel Image

Lastly, we get the following maximization problem

$$\hat{a}, \hat{b} = \arg\max_{a,b} \prod_{n} \sum_{k=0}^{\infty} \frac{(ax_n)^k}{k! b\sqrt{2\pi}} \exp\left(-ax_n - \frac{(y_n - k/a)^2}{2b^2}\right). \tag{15}$$

Using the strict monotonicity of the logarithm, we can simplify the optimization problem while not altering its results by using the log-likelihood $\mathcal{LL}$

$$\mathcal{LL}(y|a,b,x) = \sum_n \log\left(\sum_{k=0}^{\infty} \frac{(ax_n)^k}{k! b\sqrt{2\pi}} \exp\left(-ax_n - \frac{(y_n - k/a)^2}{2b^2}\right)\right). \tag{16}$$

Thus, the optimization problem becomes

$$\hat{a}, \hat{b} = \arg\max_{a,b} \mathcal{LL}(y|a,b,x). \tag{17}$$

In order to decrease the high computational complexity, we limit the range of k to a maximum value k_{max} which has to be chosen large enough to get a good approximation

$$\hat{a}, \hat{b} \approx \arg\max_{a,b} \sum_n \log\left(\sum_{k=0}^{k_{max}} \frac{(ax_n)^k}{k! b\sqrt{2\pi}} \exp\left(-ax_n - \frac{(y_n - k/a)^2}{2b^2}\right)\right). \tag{18}$$

With bigger values of k the log-likelihood starts to plateau and does not grow significantly anymore. Hence, by limiting the sum to a large enough k_{max}, the approximation of the log-likelihood is still good. Typically, we choose $k_{max} = 100$. We illustrate this property in the next Figure 1 where we can see how the log-likelihood is indeed reaching a plateau. We average over 25 pixels that we sample randomly, 25 linearly spaced values for $a \in [1, 100]$ and $b \in [0.01, 0.15]$. Additionally, we show the growing computation time needed to obtain those results.

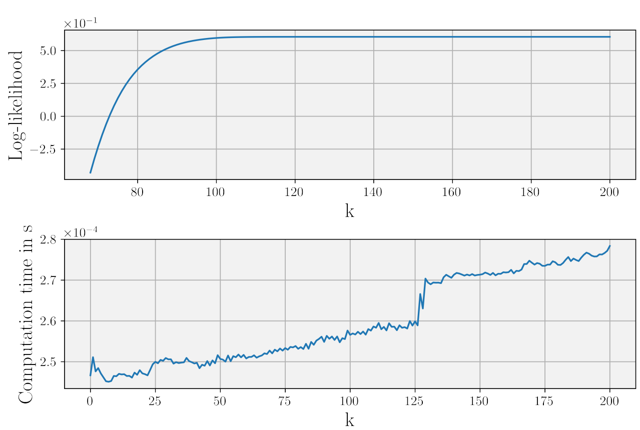

Fig. 1. The evolution of the log-likelihood with bigger k alongside the computation time.

II. Cumulants

A. The cumulant of a distribution

For a random variable X following the distribution $\mathcal{X}$, we consider the cumulant-generating function defined as

$$K_{\mathcal{X}}(t) = \log(\mathbb{E}[e^{Xt}]). \tag{19}$$

Then, we define $\kappa_r[\mathcal{X}]$, the r-th cumulant of $\mathcal{X}$, as

$$\kappa_r[\mathcal{X}] := K_{\mathcal{X}}^{(r)}(0), \tag{20}$$

with $K_{\mathcal{X}}^{(r)}(0)$ being the r-th derivative of $K_{\mathcal{X}}$ evaluated in 0.

B. Linearity

The cumulant-generating function of a sum of independent distributions is the sum of their cumulant-generating functions.

Proof.

$$\begin{aligned} K_{\mathcal{X}+\mathcal{Y}}(t) &= \log(\mathbb{E}(e^{(X+Y)t})) \\ &= \log(\mathbb{E}[e^{Xt+Yt}]) \\ &= \log(\mathbb{E}[e^{Xt}e^{Yt}]) \\ &= \log(\mathbb{E}[e^{Xt}]\mathbb{E}[e^{Yt}]) \\ &= \log(\mathbb{E}[e^{Xt}]) + \log(\mathbb{E}[e^{Yt}]) \\ &= K_{\mathcal{X}}(t) + K_{\mathcal{Y}}(t). \end{aligned} \tag{21}$$

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C. Homogeneity

The r-th cumulant is homogeneous of degree r.

Proof.

$$\kappa_r[a\mathcal{X}] = a^r \kappa_r[\mathcal{X}]. \tag{22}$$

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D. Unbiased estimator

For a vector x obtained by sampling independently n times from the distribution $\mathcal{X}$, the author of [1] describes an unbiased estimator of $\kappa_2[\mathcal{X}], \kappa_3[\mathcal{X}]$,

$$\kappa_2[\mathcal{X}] = \frac{n}{n-1} m_2(x), \quad \kappa_3[\mathcal{X}] = \frac{n^2}{(n-1)(n-2)} m_3(x), \tag{23}$$

with m_2 being the sample variance (2-rd sample central moment) and m_3 the 3-rd sample central moment, that can be calculated using the formulae taken from [2]

$$\begin{aligned} m_2(x) &= \frac{n-1}{n} \sum_i (x_i - \overline{x})^2 \\ m_3(x) &= \frac{(n-1)(n-2)}{n^2} \sum_i (x_i - \overline{x})^3. \end{aligned} \tag{24}$$

E. Cumulant of Poisson-Gaussian Noise Model

We have that $\mathcal{Y} = \frac{\mathcal{P}(a\mathcal{X})}{a} + \mathcal{N}(0, b^2)$ and we want to express $\kappa_2[\mathcal{Y}]$ and $\kappa_3[\mathcal{Y}]$ as a function of a and b. First, we use Equation (21), and get that, $\kappa_r[\mathcal{Y}] = \kappa_r\left[\frac{\mathcal{P}(a\mathcal{X})}{a}\right] + \kappa_r[\mathcal{N}(0, b^2)]$.

1) Gaussian noise component: The cumulants of $\mathcal{N}(0, b^2)$ are known to be

$$\begin{aligned} \kappa_2[\mathcal{N}(0, b^2)] &= b^2 \\ \kappa_3[\mathcal{N}(0, b^2)] &= 0. \end{aligned} \tag{25}$$

2) Poisson noise component: Instead of trying to find the cumulant of $\frac{\mathcal{P}(a\mathcal{X})}{a}$, we can use Equation (22), and find the cumulant of $Z \sim \mathcal{Z} = \mathcal{P}(a\mathcal{X})$

$$e^{K_{\mathcal{Z}}(t)} = \sum_k \mathbb{P}[Z = k] e^{tk}. \tag{26}$$

Moreover, we know that

$$\begin{aligned} \mathbb{P}[Z = k] &= \sum_i \mathbb{P}[X = x_i] \mathbb{P}[Z = k | X = i] \\ &= \sum_i n_i \frac{(ax_i)^k e^{-ax_i}}{k!}, \end{aligned} \tag{27}$$

where $n_i = \frac{|\{\, j : x_j = x_i \,\}|}{n}$ is the proportion of intensities that are equal to a given one x_i.

Thus, we have that

$$\begin{aligned} e^{K_{\mathcal{Z}}(t)} &= \sum_k \mathbb{P}[Z = k] e^{tk} \\ &= \sum_k \sum_i n_i \frac{(ax_i)^k e^{-ax_i}}{k!} \exp(t)^k \\ &= \sum_i n_i \frac{e^{-ax_i}}{exp(-ax_i e^t)} \sum_k \frac{(ax_i e^t)^k \exp(-ax_i e^t)}{k!} \\ &= \sum_i n_i \exp(ax_i(e^t - 1)). \end{aligned} \tag{28}$$

If we further note that, $f : t \mapsto \sum_i n_i \exp(ax_i(e^t - 1))$, then, we get that $K_{\mathcal{Z}}(t) = \log(f(t))$. Hence, we can now compute the different derivatives of $K_{\mathcal{Z}}(t)$

$$\begin{aligned} K_{\mathcal{Z}}(t) &= \log(f(t)) \\ K^1_{\mathcal{Z}}(t) &= \frac{f^{(1)}(t)}{f(t)} \\ K^2_{\mathcal{Z}}(t) &= \frac{f^{(2)}(t) f(t) - f^{(1)}(t)^2}{f(t)^2} \\ K^3_{\mathcal{Z}}(t) &= \frac{f(t)[f(t) f^{(3)}(t) - 3 f^{(2)}(t) f^{(1)}(t)] + 2 f^{(1)}(t)^3}{f(t)^3}. \end{aligned} \tag{29}$$

Further, by evaluating those at 0, we get

$$\begin{aligned} \kappa_0[\mathcal{Z}] &= 0 \\ \kappa_1[\mathcal{Z}] &= a\overline{x} \\ \kappa_2[\mathcal{Z}] &= a\overline{x} + a^2\overline{x^2} - a^2\overline{x}^2 \\ \kappa_3[\mathcal{Z}] &= a^3[\overline{x^3} - 3\overline{x^2}\,\overline{x} + 2\overline{x}^3] + a^2[3\overline{x^2} - 3\overline{x}^2] + a\overline{x}, \end{aligned} \tag{30}$$

using the properties that

$$\begin{aligned} f(0) &= 1 \\ f^{(1)}(0) &= a\overline{x} \\ f^{(2)}(0) &= a\overline{x} + a^2\overline{x^2} \\ f^{(3)}(0) &= a\overline{x} + 3a^2\overline{x^2} + 2a^3\overline{x^3}. \end{aligned} \tag{31}$$

Then, using Equation (22), we obtain

$$\begin{aligned} \kappa_2\left[\frac{\mathcal{P}(a\mathcal{X})}{a}\right] &= \frac{\overline{x}}{a} + \overline{x^2} - \overline{x}^2 \\ \kappa_3\left[\frac{\mathcal{P}(a\mathcal{X})}{a}\right] &= \overline{x^3} - 3\overline{x^2}\,\overline{x} + 2\overline{x}^3 + 3\frac{\overline{x^2}}{a} - 3\frac{\overline{x}^2}{a} + \frac{\overline{x}}{a^2}. \end{aligned} \tag{32}$$

3) Poisson-Gaussian Noise Model: By putting Equations (25) and (32) together, we obtain the complete expression of the cumulants

$$\begin{aligned} \kappa_2[\mathcal{Y}] &= \frac{\overline{x}}{a} + \overline{x^2} - \overline{x}^2 + b^2 \\ \kappa_3[\mathcal{Y}] &= \overline{x^3} - 3\overline{x^2}\,\overline{x} + 2\overline{x}^3 + 3\frac{\overline{x^2}}{a} - 3\frac{\overline{x}^2}{a} + \frac{\overline{x}}{a^2}. \end{aligned} \tag{33}$$

III. CNN Architecture

The detailed architecture of the CNN can be found in table I.

TABLE I
Architecture of the CNN

Layer	Out channels	Parameters
Input	1	-
Conv2D	16	kernel_size = $(3, 3)$, padding = same
ReLU	16	-
BatchNorm	16	over the channels
MaxPool2D	16	pool_size = $(2, 2)$
Conv2D	32	kernel_size = $(3, 3)$, padding = same
ReLU	32	-
BatchNorm	32	over the channels
MaxPool2D	32	pool_size = $(2, 2)$
Conv2D	64	kernel_size = $(3, 3)$, padding = same
ReLU	64	-
BatchNorm	64	over the channels
MaxPool2D	64	pool_size = $(2, 2)$
Dense	16	-
ReLU	16	-
BatchNorm	16	over the channels
Dropout	16	rate = 0.5
Dense	4	-
ReLU	4	-
Dense	2	-
Linear	2	-

References

[1] E. W. Weisstein, "k-statistic from mathworld–a wolfram web resource." [Online]. Available: https://mathworld.wolfram.com/k-Statistic.html

[2] ——, "Sample central moment. from mathworld–a wolfram web resource." [Online]. Available: https://mathworld.wolfram.com/SampleCentralMoment.html