Construction of Complementary Sets of Sequences with Low Aperiodic Correlation and Complementary Correlation

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Abstract

The construction of complementary sets of unimodular sequences of length *N*, with low correlation and complementary correlation coefficients is addressed. The design criterion is based on the minimization of a cost function that penalizes the integrated side lobe as well as the sum of the complementary correlations of the sequences in the set. Numerical solution to the proposed cost function is obtained using conventional optimization methods.

Introduction

In digital communications, as in other fields of applied technology, unimodular sequences with good correlation properties are of great interest. They found applications that range from radar waveform design, channel estimation, as well as medical imaging among others [1].

Considerable efforts have been devoted to generate unimodular sequences using different optimization techniques as reported in [1-5]. These methods make use of alternating projections, majorization-minimization techniques and Quasi-Newton methods. However, independently of the technique employed, the autocorrelation coefficient at lag *N* - 1 of a unimodular sequence of length *N*, does not vanish. This fact motivated the study of complementary sets of sequences [6].

A set of *M* sequences $S = \{x_1(n), x_2(n),\dots, x_M(n)\}$ is said to be complementary if the sum of the autocorrelation coefficients of the sequences in the set cancel for each of the lags $l = 1, 2,\dots, N - 1$. This is, if the autocorrelation coefficients of $x_m(n)$ are defined as,

$$r_m(l) = \sum_{n=l}^{N-1} x_m^*(n) x_m(n-l) \qquad (1)$$

for $l = 0, 1,\dots,N - 1$. Then, according to the previous definition, *S* is said to be complementary if

$$\mathrm{ISL} = \sum_{l=1}^{N-1} \left| \sum_{m=1}^{M} r_m(l) \right|^2 \qquad (2)$$

is zero. The minimization of of the integrated side lobe (ISL) for complementary sets of sequences, has been addressed in the literature [6-7]. Here, we consider the design of complex sets of sequences whose correlation and complementary correlation coefficients vanish when added for a given indices of lags, generalizing the work in [8].

Problem Statement

The problem addressed is to determine *M* complex unimodular sequences, each of length *N*, such that the sum of their correlation and complementary correlation coefficients cancel, for a given indices of lags.

This is, if we denote by $\{x_m(n)\}$ the set containing *M* complex sequences, for $m = 1, 2,\dots, M$. Then, we are interested in finding $x_m(n)$ for $n = 0, 1,\dots, N\text{-}1$ such that the functional,

$$C = \sum_{l=1}^{N_r-1} \left| \sum_{m=1}^{M} r_m(l) \right|^2 + \sum_{l=0}^{N_\gamma-1} \left| \sum_{m=1}^{M} \gamma_m(l) \right|^2 \qquad (3)$$

becomes arbitrarily small, with $r_m(l)$ and $\gamma_m(l)$ denoting the correlation and complementary correlation coefficients of $x_m(n)$ at lag *l*, respectively.

Numerical Solution

In order to find the elements of the sequences in the set $\{x_m(n)\}$, the following parameterization is employed $x_m(n) = \exp(i\phi_n^{(m)})$, for $n = 0, 1, \dots, N - 1$ and $m = 1, 2,\dots, M$.

Furthermore, if the following vectors are defined $\mathbf{x}_m = [x_m(0), x_m(1), \dots, x_m(N\text{-}1)]^T$ and $\mathbf{y}_m = [\mathbf{x}_m^T\ \mathbf{0}_{1xN}]^T$, which satisfy dim $\mathbf{x}_m = N \times 1$ and dim $\mathbf{y}_m = 2N \times 1$ respectively. Then, the correlation and complementary correlation coefficients can be concisely written by $\mathbf{r}_m = (2N)^{-1}\mathbf{F}_{2N}(\mathbf{F}_{2N}\mathbf{y}_m \circ (\mathbf{F}_{2N}\mathbf{y}_m)^*)$ and $\gamma_m = (2N)^{-1}\mathbf{F}_{2N}(\mathbf{F}_{2N}\mathbf{y}_m \circ (\mathbf{F}_{2N}\mathbf{y}_m)\_)$ where $\mathbf{F}_{2N}$ is the Fourier matrix with dim $\mathbf{F}_{2N} = 2N\text{x}2N$, ()_ denotes a time reversal operation and o is the Schur product of the involved vectors. Using these facts, the functional given by (3) can be written equivalently as,

$$f(\Phi) = \left(\sum_{m=1}^{M} \mathbf{r}_m(\boldsymbol{\Phi}) \right)^H \mathbf{M}_r \left(\sum_{m=1}^{M} \mathbf{r}_m(\boldsymbol{\Phi}) \right) + \left(\sum_{m=1}^{M} \boldsymbol{\gamma}_m(\boldsymbol{\Phi}) \right)^H \mathbf{M}_\gamma \left(\sum_{m=1}^{M} \boldsymbol{\gamma}_m(\boldsymbol{\Phi}) \right) \qquad (4)$$

where $\boldsymbol{\Phi} = [\boldsymbol{\Phi}_1^T\ \boldsymbol{\Phi}_2^T \dots \boldsymbol{\Phi}_M^T]^T$ and $\boldsymbol{\Phi}_m = [\phi_0^{(m)}, \phi_1^{(m)},\dots, \phi_{N-1}^{(m)}]^T$ for $m = 1, 2,\dots,M$. The matrices $\mathbf{M}_r$ and $\mathbf{M}_\gamma$ satisfy dim $\mathbf{M}_r$ = dim $\mathbf{M}\gamma = 2N \times 2N$ and have the following structure $\mathbf{M}_r = [\mathbf{0}_{2N\text{x}1}\ \hat{\mathbf{e}}_1\ \hat{\mathbf{e}}_2 \dots \hat{\mathbf{e}}_{Nr-1}\ \mathbf{0}_{2N\text{x}1} \dots \mathbf{0}_{2N\text{x}1}]$ and $\mathbf{M}\gamma = [\hat{\mathbf{e}}_0\ \hat{\mathbf{e}}_1\ \hat{\mathbf{e}}_2 \dots \hat{\mathbf{e}}_{N\gamma-1}\ \mathbf{0}_{2N\text{x}1} \dots \mathbf{0}_{2N\text{x}1}]$. The components of the gradient of the cost function given by (4) can be computed from,

$$\frac{\partial}{\partial \phi_n^{(m)}} f(\boldsymbol{\Phi}) = 2\mathrm{Re}\left\{ \mathbf{r}^H(\boldsymbol{\Phi})\mathbf{M}_r \frac{\partial}{\partial \phi_n^{(m)}} \mathbf{r}(\boldsymbol{\Phi}) \right\} + 2\mathrm{Re}\left\{ \boldsymbol{\gamma}^H(\boldsymbol{\Phi})\mathbf{M}_\gamma \frac{\partial}{\partial \phi_n^{(m)}} \boldsymbol{\gamma}(\boldsymbol{\Phi}) \right\} \qquad (5)$$

The cost function given by (4) can be minimized using Quasi-Newton methods by making use of the gradient (5).

Numerical Results

Example 1. Perfect Set of Complementary Sequences

For illustration purposes, the design of a complementary set of sequences with $M = 4$, $N = N_r = N\gamma = 64$ is considered. Fig. 1 shows the normalized sum of the correlation and the complementary correlation coefficients of the four sequences of length $N = 64$ designed.

The curves labeled as PCS in Fig. 1 were obtained by minimizing the cost function given by (4), which takes into account the integrated side lobe and the complementary correlation coefficients of the sequences in the set. For reference, curves obtained by minimizing only the ISL as given by (2) are also shown. Both complementary sets were computed using the same initialization vector $\boldsymbol{\Phi}_0$ and stopping criteria with the optimization method.

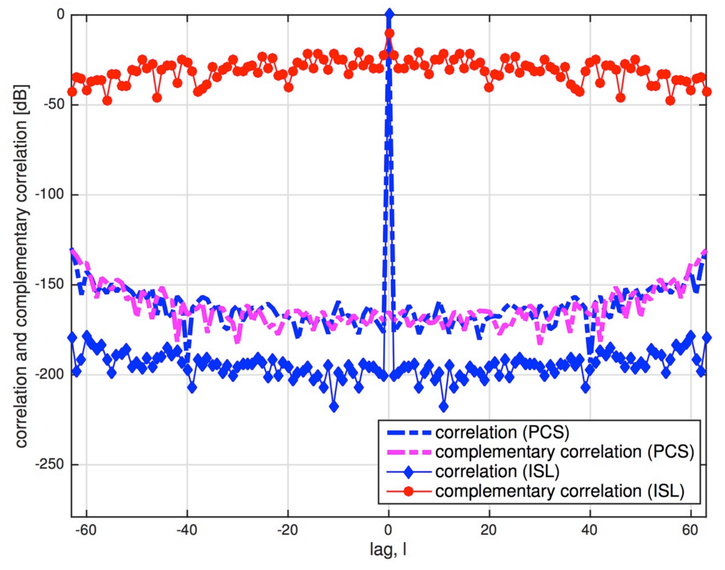

Fig. 1. Normalized sum of the correlation and complementary correlation for set of sequences

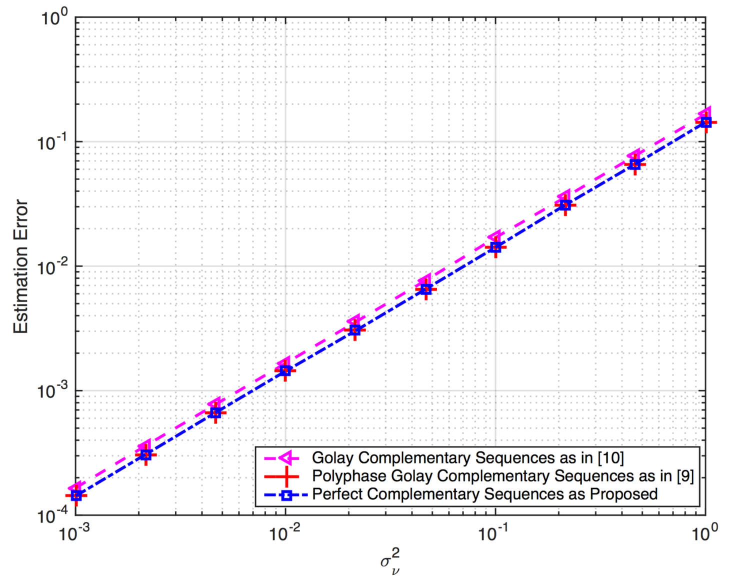

Fig. 2. Estimation error in the identification of widely linear systems.

Example 2. Widely linear system identification

The proposed sequences have direct applications in the estimation of widely linear systems. In Fig. 2, the estimation error of identifying such systems with $\{h1(n)\}_{n=0}^{7}$, $\{h_2(n)\}_{n=0}^{7}$, is depicted. For the case considered, sets with $M = 2$ and $N = 8$ were taken into account. The identification was done under the assumption that the output of the widely linear system was affected with additive white Gaussian noise of power σ_v^2. The performance achieved with the proposed sequences is compared with sequences reported in [9] and [10].

Example 3. Complementary Pair of sequences with $N_r + N\gamma = N$

As a final example, it is consider the design of a set with $M = 2$ and $N = 512$. The main difference with the previous examples is to consider only a region of interest where the sum of the correlation and complementary correlation coefficients cancel. Here, it is proposed the following set of parameters $N_r = N\gamma = 256$.

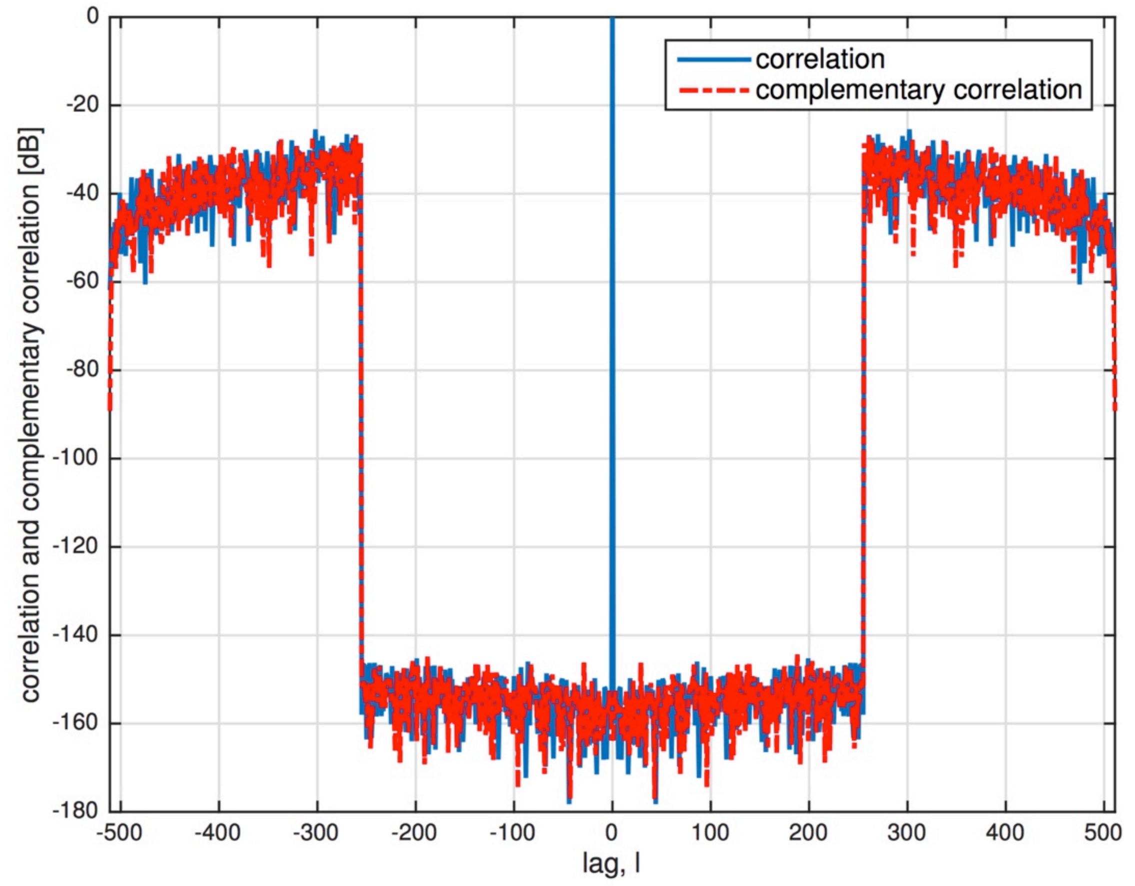

Fig. 3. Normalized sum of the correlation and complementary correlation coefficients for a pair of complementary sequences with $N_r = N\gamma = 256$ and $N = 512$.

Conclusions

In this paper, it has been proposed a cost function to generate complementary sets of sequences whose sum of autocorrelation and complementary correlation coefficients cancel for a given indices of lags.

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