Decentralized Sparsity Pattern Recovery using 1-bit Compressed Sensing

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Outline

- Motivation
- Problem Formulation
- Binary Iterative Hard Thresholding (BIHT) Algorithm
- Centralized BIHT (C-BIHT) Algorithm
- Decentralized Algorithms
- Conclusion

Motivation

- Compressed Sensing (CS) allows us to reconstruct high dimensional sparse signals from low dimensional measurements
- CS is promising in resource constrained communication networks
- Signal reconstruction is not required in many applications
- Spectrum Sensing in Cognitive radios
  - Assumption: Only a few PUs are present
  - Usage of frequency slots: sparse vector
  - Multiple SUs try to detect the presence or absence of PUs
  - Decide the zero and non-zero locations of the sparse vector
- Sparsity Pattern Recovery Problem (SPRP)

- Questions:
  - How to approach the problem if the network has additional restrictions on the resources?
  - Can we further compress the compressed measurements? (1-bit quantization)
  - How does the performance of decentralized algorithms compare to centralized algorithms?
  - Can performance be improved with collaboration?
- **Goal**: Decentralized Algorithms for SPRP using 1-bit CS

Problem Formulation

- Consider a distributed network with P nodes
- At node p, for $p = 1, \cdots, P$,

$$\mathbf{y}_p = \mathbf{\Phi}_p(\mathbf{s}_p + \mathbf{n}_p).$$

  where $\mathbf{\Phi}_p \in \mathbb{R}^{M \times N}, \boldsymbol{s}_p \in \mathbb{R}^N, \boldsymbol{n}_p \in \mathbb{R}^N$ is i.i.d. Gaussian noise with covariance matrix $\sigma_n^2\ \mathbf{I}_N$
- $\boldsymbol{s}_p$ is K-sparse and is assumed to have same sparse support for $p = 1, \cdots, P$ with possibly different signal amplitudes
- Element wise quantization of $\mathbf{y}_p$

$$q_{ip} = \text{sign}(y_{ip}) = \begin{cases} -1, & \text{if } -\infty < y_{ip} < 0 \\ 1, & \text{if } 0 \le y_{ip} < \infty. \end{cases}$$

- The matrix of quantized measurement

$$\mathbf{Q} = [\mathbf{q}_1 | \mathbf{q}_2 | ... | \mathbf{q}_P]$$

Binary Iterative Hard Thresholding (BIHT) [1]

- BIHT reconstructs the signal from the element-wise quantized vector
- BIHT aims to decrease the cost function

$$\mathcal{J}(s) = \|[\mathbf{q} \odot (\boldsymbol{\Phi}\mathbf{s})]_-\|_1$$

where $[\cdot]_-$ denote negative function, i.e., $[z]_- = z$ if $z < 0$ and 0 else.

- k-th iteration of BIHT algorithm

$$\mathbf{s}^k = \boldsymbol{\Theta}_{\mathbf{K}}\left(\mathbf{s}^{k-1} - \tau \underbrace{\boldsymbol{\Phi}^T(\text{sign}(\boldsymbol{\Phi}\mathbf{s}^{k-1}) - \mathbf{q})}_{\in \text{ sub-differential of } \|[\mathbf{q}\odot(\boldsymbol{\Phi}\mathbf{s})]_-\|_1}\right)$$

where $\boldsymbol{\Theta}_{\mathbf{K}}$ is the hard-thresholding operator

- Susceptible to noise
- Motivates the use of more measurement vectors

[1] "Robust 1-Bit Compressive Sensing via Binary Stable Embeddings of Sparse Vectors", L. Jacques et al., IEEE trans. Inf. Theory, 2013

Centralized-BIHT (C-BIHT)

- All the measurement vectors are available at the fusion center (FC)

$$\mathbf{Y} = \mathbf{\Phi}(\mathbf{S} + \mathbf{N}), \qquad \mathbf{Q} = \text{sign}(\mathbf{Y}).$$

where $\mathbf{S} = [\mathbf{s}_1, \cdots, \mathbf{s}_P]$ and $\mathbf{N} = [\mathbf{n}_1, \cdots, \mathbf{n}_P]$

- Decision on sparsity pattern is made at the FC
- Algorithm

Algorithm 1 Centralized Binary- IHT algorithm (C-BIHT)

Inputs : $\mathbf{\Phi}, K, \mathbf{Q}, \tau$

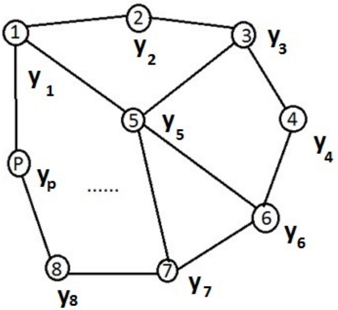
1. Initialize $\mathbf{S}^0$
2. For iteration j until the stopping criteria
3. $\mathbf{S}^j = \mathbf{S}^{j-1} + \tau \mathbf{\Phi}^T (Q - \text{sign}(\mathbf{\Phi}\mathbf{S}^{j-1}))$
4. $\mathbf{T}^j$ = DetectSupport($\mathbf{S}^j, K$)
5. $\mathbf{S}^j$ = Threshold ($\mathbf{S}^j, \mathbf{T}^j$)
6. End For
7. $\hat{\mathbf{S}} = \mathbf{S}^{j^*}$ and $\hat{\mathbf{T}} = \mathbf{T}^{j^*}$ when stopping at iteration j^*

- Vulnerable to the failure of FC.

Decentralized Algorithms

- Avoids the use of FC
- Measurements are sent to and received from one hop neighbors only
- Embed collaboration and fusion among nodes
- Algorithms can be structured involving two distinct stages
  - Information Fusion
  - Index Fusion

Decentralized Algorithms (contd...)



- Network topology is represented as an undirected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$
- $neigh(i) = \{j \mid (i, j) \in \mathbf{E}\}$ the set of neighboring nodes of node i
- Node p has access to the measurement matrix $\mathbf{Q}_p = [\mathbf{Q}_{neigh(p)},\ \mathbf{q}_p]$
- Let $\hat{\mathbf{T}}_p$ be local support estimate at node p

Decentralized BIHT 1 (D-BIHT 1)

Algorithm 2 Decentralized BIHT 1 (D-BIHT 1)

Inputs : $\mathbf{\Phi}, K, \tau, neigh(p)$ for all $p \in \mathbf{V}$

1. Initialize $\mathbf{S}_p^0$ for all $p \in \mathbf{V}$
2. Local Communication at node p , for all $p \in \mathbf{V}$
   Transmit $\mathbf{q}_p$ to its one-hop neighborhood
   Receive $\mathbf{q}_r$ where $r \in neigh(p)$ and form $\mathbf{Q}_p$
3. For iteration j until the stopping criteria
4. $\mathbf{S}_p^j = \mathbf{S}_p^{j-1} + \tau\mathbf{\Phi}^T(\mathbf{Q}_p - \text{sign}(\mathbf{\Phi}\mathbf{S}_p^{j-1}))$
5. $\mathbf{T}_p^j$ = DetectSupport($\mathbf{S}_p^j, K$)
6. $\mathbf{S}_p^j$ = Threshold ($\mathbf{S}_p^j, \mathbf{T}_p^j$)
7. End For
8. $\hat{\mathbf{S}_p} = \mathbf{S}^{j^*}$ and $\hat{\mathbf{T}}_p = \mathbf{T}^{j^*}$ when stopping at iteration j^*
9. Global Communication
   For all $p \in \mathbf{V}$, transmit $\hat{\mathbf{T}}_p$ to $\mathbf{V}$
   Receive $\hat{\mathbf{T}}_i$ for $\forall i \neq p$
10. $\hat{\mathbf{T}}$ = Majority($\hat{\mathbf{T}}_1$,$\hat{\mathbf{T}}_2, .., \hat{\mathbf{T}}_P$)

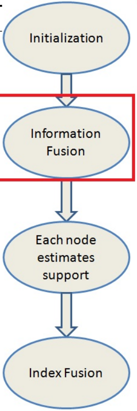
D-BIHT 1

Information Fusion

Algorithm 2 Decentralized BIHT 1 (D-BIHT 1)

Inputs : $\mathbf{\Phi}, K, \tau, neigh(p)$ for all $p \in \mathbf{V}$

1. Initialize $\mathbf{S}_p^0$ for all $p \in \mathbf{V}$
2. Local Communication at node p , for all $p \in \mathbf{V}$
   - Transmit $\mathbf{q}_p$ to its one-hop neighborhood
   - Receive $\mathbf{q}_r$ where $r \in neigh(p)$ and form $\mathbf{Q}_p$
3. For iteration j until the stopping criteria
4. $\mathbf{S}_p^j = \mathbf{S}_p^{j-1} + \tau \mathbf{\Phi}^T(\mathbf{Q}_p - \text{sign}(\mathbf{\Phi}\mathbf{S}_p^{j-1}))$
5. $\mathbf{T}_p^j$ = DetectSupport($\mathbf{S}_p^j, K$)
6. $\mathbf{S}_p^j$ = Threshold ($\mathbf{S}_p^j, \mathbf{T}_p^j$)
7. End For
8. $\hat{\mathbf{S}}_p = \mathbf{S}^{j^*}$ and $\hat{\mathbf{T}}_p = \mathbf{T}^{j^*}$ when stopping at iteration j^*
9. Global Communication
   - For all $p \in \mathbf{V}$, transmit $\hat{\mathbf{T}}_p$ to $\mathbf{V}$
   - Receive $\hat{\mathbf{T}}_i$ for $\forall i \neq p$
10. $\hat{\mathbf{T}}$ = Majority($\hat{\mathbf{T}}_1, \hat{\mathbf{T}}_2, .., \hat{\mathbf{T}}_P$)



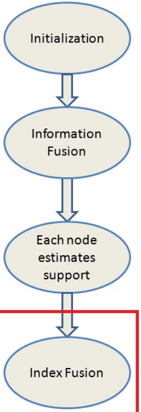
D-BIHT 1

Index Fusion

Algorithm 2 Decentralized BIHT 1 (D-BIHT 1)

Inputs : $\mathbf{\Phi}, K, \tau, neigh(p)$ for all $p \in \mathbf{V}$

1. Initialize $\mathbf{S}_p^0$ for all $p \in \mathbf{V}$
2. Local Communication at node p , for all $p \in \mathbf{V}$
   Transmit $\mathbf{q}_p$ to its one-hop neighborhood
   Receive $\mathbf{q}_r$ where $r \in neigh(p)$ and form $\mathbf{Q}_p$
3. For iteration j until the stopping criteria
4. $\mathbf{S}_p^j = \mathbf{S}_p^{j-1} + \tau \mathbf{\Phi}^T (\mathbf{Q}_p - \text{sign}(\mathbf{\Phi} \mathbf{S}_p^{j-1}))$
5. $\mathbf{T}_p^j$ = DetectSupport($\mathbf{S}_p^j, K$)
6. $\mathbf{S}_p^j$ = Threshold ($\mathbf{S}_p^j, \mathbf{T}_p^j$)
7. End For
8. $\hat{\mathbf{S}}_p = \mathbf{S}^{j^*}$ and $\hat{\mathbf{T}}_p = \mathbf{T}^{j^*}$ when stopping at iteration j^*
9. Global Communication
   For all $p \in \mathbf{V}$, transmit $\hat{\mathbf{T}}_p$ to $\mathbf{V}$
   Receive $\hat{\mathbf{T}}_i$ for $\forall i \neq p$
10. $\hat{\mathbf{T}}$ = Majority($\hat{\mathbf{T}}_1, \hat{\mathbf{T}}_2, .., \hat{\mathbf{T}}_P$)



Modified Decentralized BIHT (D-BIHTm)

- Modified form of D-BIHT 1 algorithm
- For networks with severe restrictions on bandwidth usage
- Information Fusion stage is omitted
- Each node makes decisions based on its own measurements (Self Decision Stage)
- Followed by Index fusion stage

Decentralized BIHT 2 (D-BIHT 2)

Algorithm 3 Decentralized BIHT: Algorithm 2 (D-BIHT 2)

Inputs : $\mathbf{\Phi}, K, \tau, neigh(p)$ for all $p \in \mathbf{V}$

1. Initialize $\mathbf{S}_p^0$ for all $p \in \mathbf{V}$
2. Local Communication at node p , for all $p \in \mathbf{V}$
   Transmit $\mathbf{q}_p$ to its one-hop neighborhood
   Receive $\mathbf{q}_r$ where $r \in neigh(p)$ and form $\mathbf{Q}_p$
3. For iteration j until the stopping criteria
4. $\mathbf{S}_p^j = \mathbf{S}_p^{j-1} + \tau \mathbf{\Phi}^T(\mathbf{Q}_p - \text{sign}(\mathbf{\Phi}\mathbf{S}_p^{j-1}))$
5. $\mathbf{T}_p^j$ = DetectSupport($\mathbf{S}_p^j, K$)
6. Global Communication
   For all $p \in \mathbf{V}$, transmit $\hat{\mathbf{T}}_p$ to $\mathbf{V}$
   Receive $\hat{\mathbf{T}}_i$ for $\forall i \neq p$
7. $\hat{\mathbf{T}}^j$ = Majority($\hat{\mathbf{T}}_1^j, \hat{\mathbf{T}}_2^j, .., \hat{\mathbf{T}}_P^j$)
8. $\mathbf{S}_p^j$ = Threshold ($\mathbf{S}_p^j, \mathbf{T}^j$) , for all $p \in \mathbf{V}$
9. EndFor
10. $\hat{\mathbf{T}} = \hat{\mathbf{T}}^{j^*}$ when stopping at iteration j^*

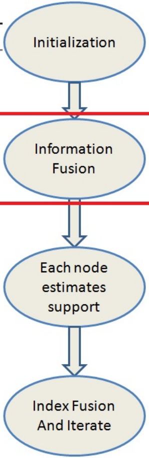
D-BIHT 2

Information Fusion

Algorithm 3 Decentralized BIHT: Algorithm 2 (D-BIHT 2)

Inputs : $\mathbf{\Phi}, K, \tau, neigh(p)$ for all $p \in \mathbf{V}$

1. Initialize $\mathbf{S}_p^0$ for all $p \in \mathbf{V}$
2. Local Communication at node p , for all $p \in \mathbf{V}$
   Transmit $\mathbf{q}_p$ to its one-hop neighborhood
   Receive $\mathbf{q}_r$ where $r \in neigh(p)$ and form $\mathbf{Q}_p$
3. For iteration j until the stopping criteria
4. $\mathbf{S}_p^j = \mathbf{S}_p^{j-1} + \tau \mathbf{\Phi}^T(\mathbf{Q}_p - \text{sign}(\mathbf{\Phi}\mathbf{S}_p^{j-1}))$
5. $\mathbf{T}_p^j = \text{DetectSupport}(\mathbf{S}_p^j, K)$
6. Global Communication
   For all $p \in \mathbf{V}$, transmit $\hat{\mathbf{T}}_p$ to $\mathbf{V}$
   Receive $\hat{\mathbf{T}}_i$ for $\forall i \neq p$
7. $\hat{\mathbf{T}}^j = \text{Majority}(\hat{\mathbf{T}}_1^j, \hat{\mathbf{T}}_2^j, .., \hat{\mathbf{T}}_P^j)$
8. $\mathbf{S}_p^j = \text{Threshold}\ (\mathbf{S}_p^j, \mathbf{T}^j)$, for all $p \in \mathbf{V}$
9. EndFor
10. $\hat{\mathbf{T}} = \hat{\mathbf{T}}^{j^*}$ when stopping at iteration j^*



D-BIHT 2

Index Fusion

Algorithm 3 Decentralized BIHT: Algorithm 2 (D-BIHT 2)

Inputs : $\mathbf{\Phi}, K, \tau, neigh(p)$ for all $p \in \mathbf{V}$

1. Initialize $\mathbf{S}_p^0$ for all $p \in \mathbf{V}$
2. Local Communication at node p , for all $p \in \mathbf{V}$
   Transmit $\mathbf{q}_p$ to its one-hop neighborhood
   Receive $\mathbf{q}_r$ where $r \in neigh(p)$ and form $\mathbf{Q}_p$
3. For iteration j until the stopping criteria
4. $\mathbf{S}_p^j = \mathbf{S}_p^{j-1} + \tau\mathbf{\Phi}^T(\mathbf{Q}_p - \text{sign}(\mathbf{\Phi}\mathbf{S}_p^{j-1}))$
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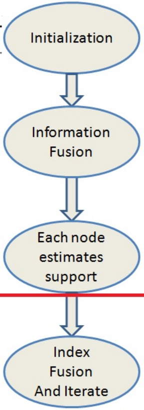
7. $\hat{\mathbf{T}}^j$ = Majority($\hat{\mathbf{T}}_1^j, \hat{\mathbf{T}}_2^j, .., \hat{\mathbf{T}}_P^j$)
8. $\mathbf{S}_p^j$ = Threshold ($\mathbf{S}_p^j, \mathbf{T}^j$) , for all $p \in \mathbf{V}$
9. EndFor
10. $\hat{\mathbf{T}} = \hat{\mathbf{T}}^{j^*}$ when stopping at iteration j^*



Simulation Results

- Performance Metrics:

$$\text{Percentage of Sparsity Pattern Recovery} = \frac{\#\ \text{Correct Support Recovered}}{\#\ \text{Correct Support}}$$

$$\text{Probability of Exact Sparsity Pattern Recovery(PESPR)} = \frac{\#\ \text{all the support estimated}}{\#\ \text{of run of experiments}}$$

Simulation Results (Contd...)

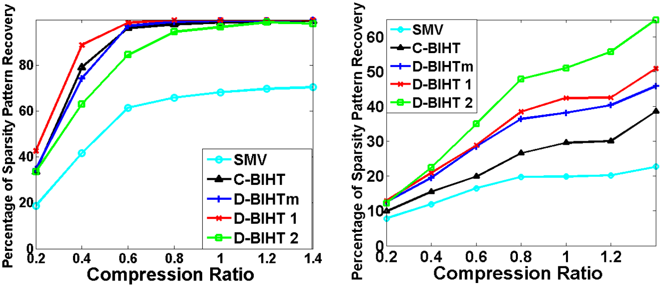
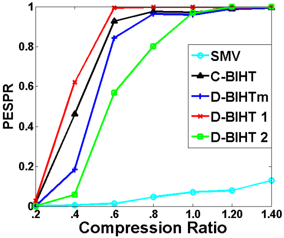


Figure 1: Percentage of Sparsity Pattern Recovery for D-BIHT 1, D-BIHT 2, D-BIHTm, C-BIHT, BIHT (SMV) algorithms in a network of Node degree 3 for high SNR (noise variance =0.0004) and low SNR (noise variance = .0625) respectively.

Simulation Results (Contd...)



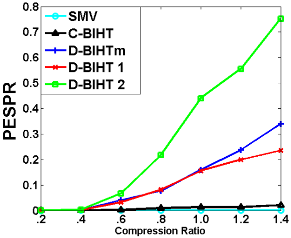


Figure 2: Probability of Exact Sparsity Pattern Recovery (PESPR) for D-BIHT 1, D-BIHT 2, D-BIHTm, C-BIHT, BIHT (SMV) algorithms in a network of Node degree 6 for high SNR (noise variance =.002025) and low SNR (noise variance =.01) respectively.

Simulation Results (Contd...)

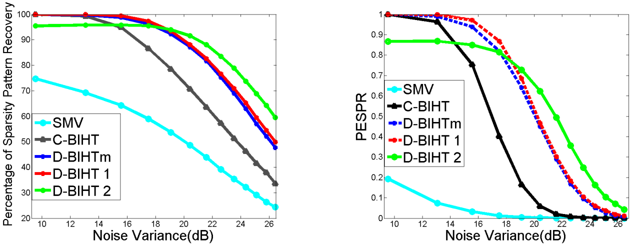


Figure 3: Percentage of Sparsity Pattern Recovery and Probability of Exact Sparsity Pattern Recovery (PESPR) for D-BIHT 1, D-BIHT 2, D-BIHTm, C-BIHT, BIHT (SMV) in a network of Node degree 4 for M = 40.

Conclusion

- Proposed two decentralized algorithms using 1-bit CS
- Strategic collaboration and fusion can lead to improved results
- D-BIHT 1 performs better in the high SNR regime
- D-BIHT 2 performs better in the low SNR regime
- **Future work**: Theoretical results on performance bounds and guarantees

Thank you