Atomic Norm Minimization Based Range-Direction Indication For Frequency Diverse Array: A Matrix Completion Perspective

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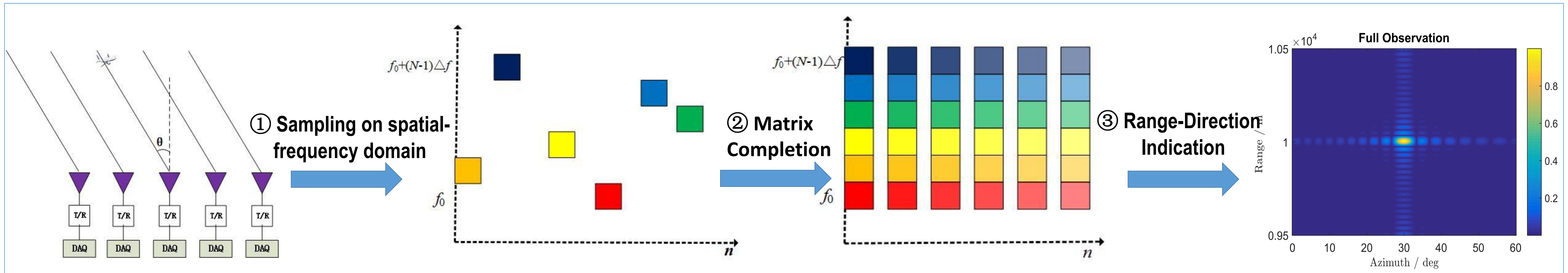
Contribution

- **Purpose**: Achieve range-direction indication in Frequency Diverse Array (FDA) without coupling and high sidelobe.
- **Key idea**:
  - Regard the FDA as a 2-D sampling on the spatial-frequency domain;
  - Use Atomic Norm Minimization (ANM) to complete the missing observation.
- **Performance**:
  - Achieve a 2-D-sinc-like structure beampattern;
  - Indicate targets successfully and accurately.

Method & Framework



✓ FDA transmit and receive system
✓ Received data arranged by sampling perspective
✓ Full observation matrix by MC
✓ The spatial-frequency spectrum

General FDA model & Problem

In the FDA, the carrier frequency of the transceiver is assigned as:

$$f_n = f_0 + m_n \Delta f$$

f_c: the center frequency, Δf: the frequency increment step, and m_n: the frequency increment sequence.

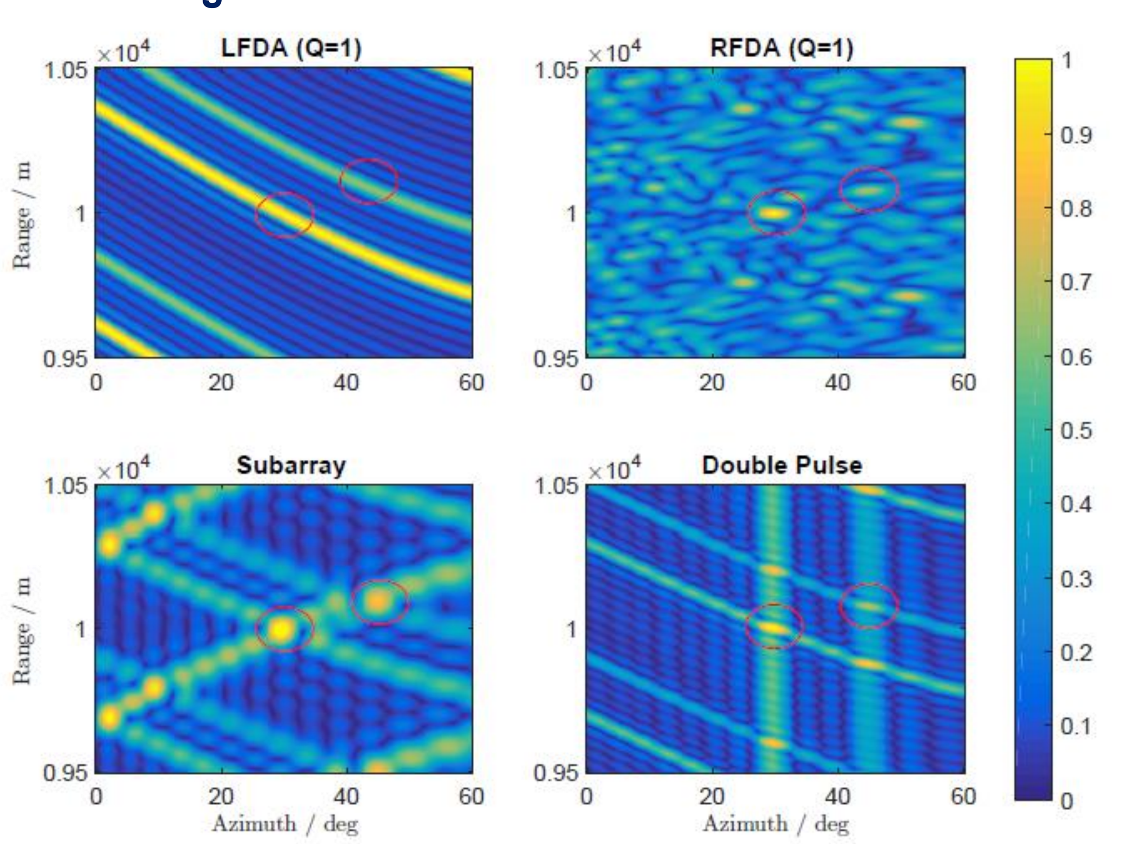
The transmit waveform of the nth antenna is

$$s_{tn} = \exp\{j2\pi(f_0 + m_n \Delta f\,)t\}.$$

By different arrangement of m_n, we can achieve the different existing FDA method.

- **LFDA** (Linear FDA): $m_n = [0,1,2,\dots N-1]$
- **RFDA** (Random FDA): m_n is a random variable.
- **Subarray FDA**: $m_n = [0,1,\dots\frac{N-1}{2},\frac{N-3}{2},\dots,1,0]$
- **Double Pulse** : $m_n = [0\&0,1\&0,\dots,N-1\&\,0]$

➤ **There are two targets and their indications are circled in the figure below.**



◆ **The existing decoupling methods suffer from coupling ridge residual or high sidelobe base problems.**

◆ **When there multiple targets and even strong and weak targets, all the existing methods fails.**

Progress & Algorithm

① Sampling on spatial-frequency domain

We regard FDA as the joint sampling of the targets' range-direction information on the 2-D spatial-frequency domain

$$\boldsymbol{y} = \boldsymbol{\mathcal{P}_{\mathcal{T}}}(\boldsymbol{X})$$

$\boldsymbol{y}$: FDA received data, $\boldsymbol{\mathcal{P}_{\mathcal{T}}}(\cdot)$: the orthogonal projection operator onto the subspace of vector supported on $\boldsymbol{\mathcal{T}}$. Define $\boldsymbol{\mathcal{T}}$ on the frequency diverse code m_n as

$$\boldsymbol{\mathcal{T}} = \{(n-1)M + m_n | n = 1,2,\dots,N\}$$

All the existing FDA methods can be regarded as sampling on the 2-D spatial-frequency domain just with different sampling type.

$\boldsymbol{X}$: the full observation matrix which can be formulated as each antenna transmits all the M monotones whose carrier frequency ranges from f_0 to $f_0+(M-1)\Delta f$, and afterwards receives all the reflected echoes with an orthogonal waveform design,

$$\boldsymbol{X} = \sum_{i=1}^{K} \boldsymbol{\alpha}_r(f_{ri})\boldsymbol{\alpha}_s(f_{si})^T$$

where

$$\boldsymbol{\alpha}_r(f_{ri}) = \left[1, e^{-j2\pi f_{ri}}, e^{-j2\pi 2 f_{ri}}, \dots, e^{-j2\pi(M-1)f_{ri}}\right]^T$$

and

$$\boldsymbol{\alpha}_s(f_{si}) = \left[1, e^{-j2\pi f_{si}}, e^{-j2\pi 2 f_{si}}, \dots, e^{-j2\pi(M-1)f_{si}}\right]^T$$

◆ **X → two-dimension harmonic structure .**

② Matrix Completion by Atomic Norm Minimization

Adopt Atomic Norm Minimization (ANM) to reconstruct the full matrix $\boldsymbol{X}$. The corresponding atomic set is defined as the collection of all the 2-D complex sinusoids :

$$\mathcal{A} \triangleq \{\boldsymbol{c}(f_r,f_s) = \boldsymbol{\alpha}_r(f_r)\otimes\boldsymbol{\alpha}_s(f_s) | f_r \in (0,1], f_s \in (0,1]\}$$

The atomic norm is defined as

$$\|\boldsymbol{x}\|_{\mathcal{A}} \triangleq \inf_{c(f_{ri},f_{si})\in\mathcal{A}} \left\{\sum_i |d_i| : \boldsymbol{x} = \sum_i d_i \boldsymbol{c}(f_{ri},f_{si})\right\}$$

So the reconstruction of the full matrix **X** can be arranged as the following optimization problem

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_{\mathcal{A}} \quad s.t. \quad \mathcal{P}_T(\boldsymbol{x}) = \boldsymbol{y}$$

where $\boldsymbol{x} = vec(\boldsymbol{X})$. This problem can be solved by a semi-definite programming [Chi, 2014]

$$\boldsymbol{x} = \arg\min_{\boldsymbol{T},x,t} \frac{1}{2}tr\big(S(\boldsymbol{T})\big) + \frac{1}{2}t$$

$$\text{subject to} \quad \begin{bmatrix} S(\boldsymbol{T}) & \boldsymbol{x} \\ \boldsymbol{x}^* & t \end{bmatrix} \succcurlyeq 0$$

$$\mathcal{P}_T(\boldsymbol{x}) = \boldsymbol{y}$$

$S(\boldsymbol{T})$: a two-fold Toeplitz structure.

③ Range-Direction Indication Using the Full Data

Then we use the full observation vector $\boldsymbol{x}$ we obtained to indicate the range and direction of the targets. For the single-snapshot case (termed the Single Measurement Vector (SMV) scenario), the maximum likelihood estimation can be obtained by a replica-correlator. So the spatial-frequency spectrum can be estimated as

$$\boldsymbol{p(r,\theta)} = \boldsymbol{c}(f_r,f_s)^H\boldsymbol{x}$$

If there are $L>1$ snapshots, we can formed all the full vector $\boldsymbol{x}(l)$ as a Multiple Measure Vector (MMV) scenario. And then we can use some methods such as MUSIC and Capon to achieve a super resolution

$$\boldsymbol{P(r,\theta)} = \frac{1}{\boldsymbol{c}(f_r,f_s)^H\boldsymbol{R}^{-1}\boldsymbol{c}(f_r,f_s)}$$

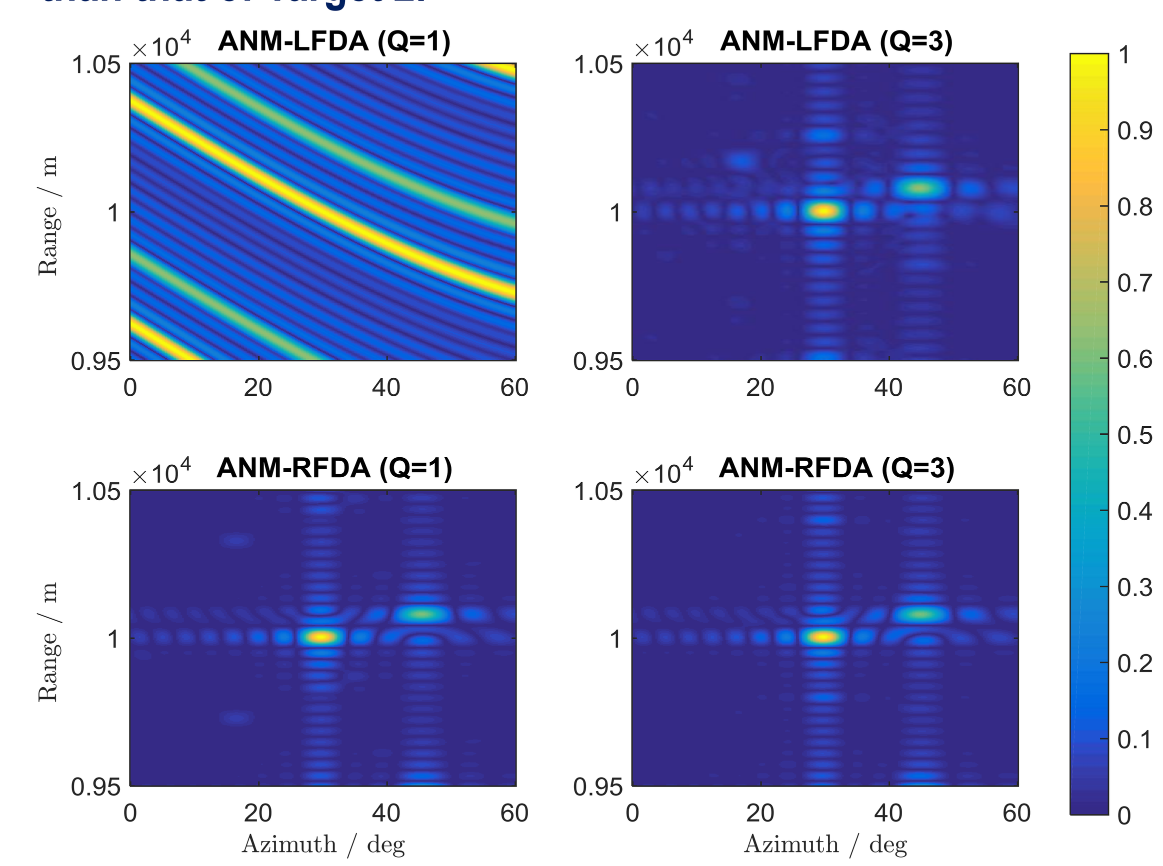
$\boldsymbol{R}$: the estimation of covariance matrix.

Both the replica-correlation method and the super resolution method avoid the off-grid problem with a more stable performance.

◆ **The corresponding beampattern based on the full observation data $\boldsymbol{x}$ has 2-D-sinc-like structure. The coupling and high sidelobe problems are avoided.**

Simulation Results

FDA consists of $N=16$ antennas, $f_0 = 9GHz$, Δf = 200kHz, Q = 1 for narrowband case and Q = 3 for broadband case. Two targets: $r_1 = 10km, \theta_1 = 30°$ and $r_2 = 10km + 75m, \theta_2 = 45°$. The power level of Target 1 is 5 dB larger than that of Target 2.



◆ **2-D-sinc-like beampattern; Accurately indication of targets.**

Table 1. Reconstruction NMSE of ANM in FDA.

	K=1	K=2	K=3	K=4	K=5
LFDA(Q=1)	0.9747	0.9741	0.9733	0.9718	0.9576
subarray	0.9151	0.9553	0.9487	1.0114	0.9711
double pulse	0.0044	1.4106	1.1124	1.0576	1.0157
RFDA(Q=1)	0.0034	0.0927	0.5180	0.8262	1.0026
LFDA(Q=3)	0.0027	0.0093	0.0081	0.6996	0.7783
RFDA(Q=3)	0.0018	0.0049	0.0058	0.0113	0.4515

◆ **Random diversity type and broad bandwidth benefit the observation completion.**

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