Fast Computations for Approximation and Compression in Slepian Spaces

Santhosh Karnik[1], Zhihui Zhu[2], Michael B. Wakin[2], Justin Romberg[1], Mark A. Davenport[1]

[1]Georgia Institute of Technology
[2]Colorado School of Mines

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Outline

- Spectral Leakage
- Slepian Basis
- Fast Computations

Spectral Leakage

Suppose we take N equispaced samples of a bandlimited signal.

$$x_c(t) = \int_{-W/T}^{W/T} X_c(F) e^{j2\pi Ft}\, dF, \ \ t \in \mathbb{R}$$

$$\boldsymbol{x}[n] = x_c(nT), \ \ n = 0, 1, \ldots, N-1$$

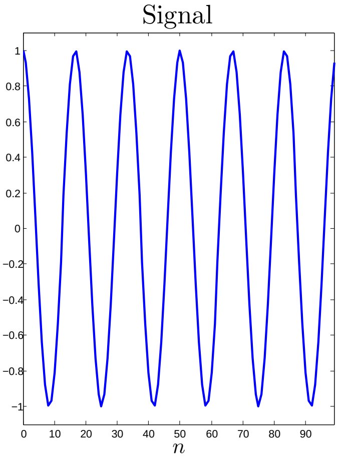
If $W = \frac{1}{2}$, we are sampling at the Nyquist rate. We expect N degrees of freedom in $\boldsymbol{x}$.

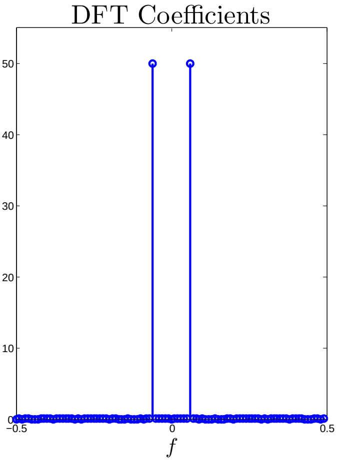
If $0 < W < \frac{1}{2}$, we are sampling faster than the Nyquist rate. We expect $\approx 2NW$ degrees of freedom in $\boldsymbol{x}$.

What basis yields a sparse representation for $\boldsymbol{x}$?

Spectral Leakage

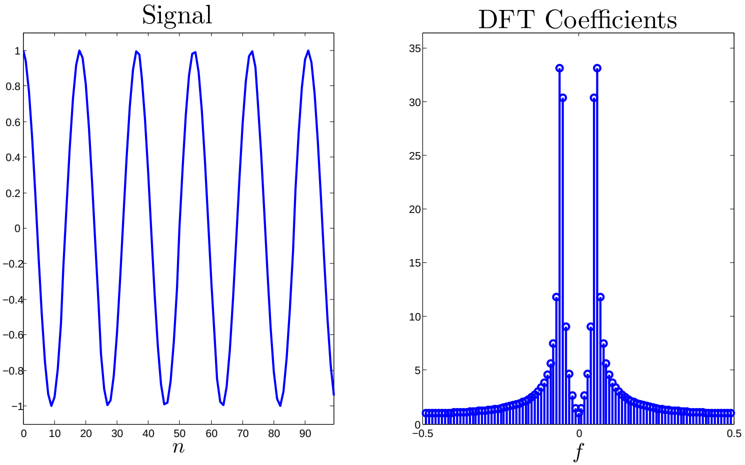
$$x[n] = \cos\left(\frac{2\pi \cdot 6}{100} n\right), \quad n = 0, 1, \ldots, 99$$





Spectral Leakage

$$\boldsymbol{x}[n] = \cos\left(\frac{2\pi \cdot 5.5}{100} n\right), \quad n = 0, 1, \ldots, 99$$



Motivation

The relationship between a discrete signal $x \in \ell_2(\mathbb{Z})$ and its discrete time Fourier transform (DTFT) $\widetilde{x} \in L_2([-\frac{1}{2}, \frac{1}{2}])$ is given by

$$\widetilde{x}(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn},$$

$$x[n] = \int_{-1/2}^{1/2} \widetilde{x}(f) e^{j2\pi fn}\, df.$$

For a given positive integer N and bandlimit $0 < W < \frac{1}{2}$, the only signal which is both bandlimited to $f \in [-W, W]$ and timelimited to $n \in \{0, 1, \ldots, N-1\}$ is the zero signal.

Can we find timelimited signals whose energy is maximally concentrated in the frequency interval $[-W, W]$?

Definition

For any length N signal, $\boldsymbol{x} \in \mathbb{C}^N$,

$$\frac{\int_{-W}^{W} |\tilde{\boldsymbol{x}}(f)|^2 \, df}{\int_{-1/2}^{1/2} |\tilde{\boldsymbol{x}}(f)|^2 \, df} = \frac{\boldsymbol{x}^* \boldsymbol{B}_{N,W} \boldsymbol{x}}{\boldsymbol{x}^* \boldsymbol{x}},$$

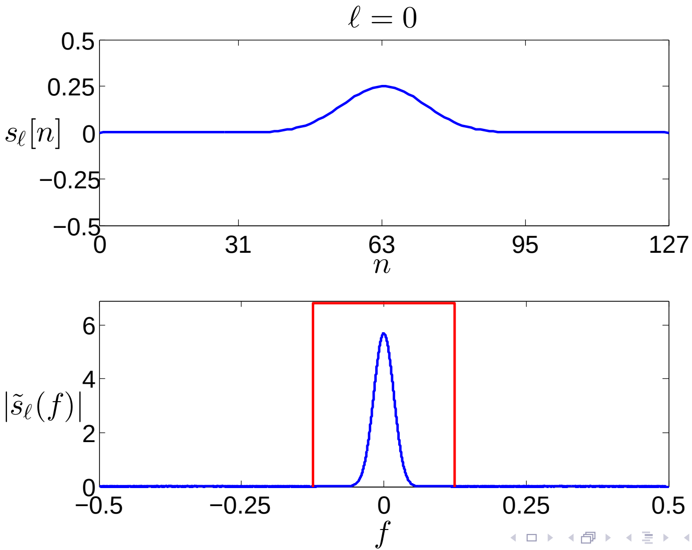
where $\boldsymbol{B}_{N,W}$ is an $N \times N$ matrix with entries

$$\boldsymbol{B}_{N,W}[m,n] = \begin{cases} \dfrac{\sin(2\pi W(m-n))}{\pi(m-n)} & \text{if } m \neq n \\ 2W & \text{if } m = n \end{cases}.$$

The Slepian basis vectors $\boldsymbol{s}_{N,W}^{(0)}, \boldsymbol{s}_{N,W}^{(1)}, \ldots, \boldsymbol{s}_{N,W}^{(N-1)}$ are defined as the eigenvectors of $\boldsymbol{B}_{N,W}$, where the respective eigenvalues $\lambda_{N,W}^{(0)}, \lambda_{N,W}^{(1)}, \ldots, \lambda_{N,W}^{(N-1)}$ are sorted in decreasing order [1].

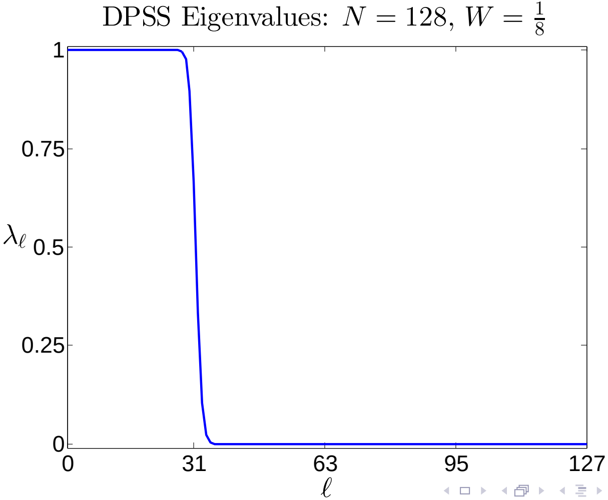
Slepian Basis Vector Plots

Signal Length: $N = 128$, Bandlimit: $W = \frac{1}{8}$



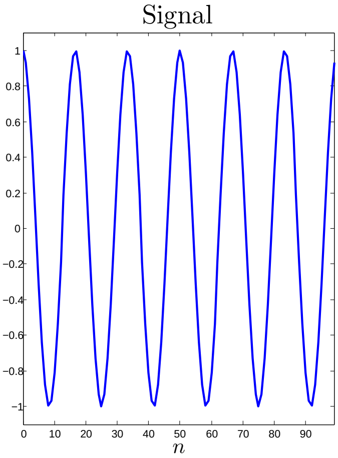
Eigenvalue Behavior

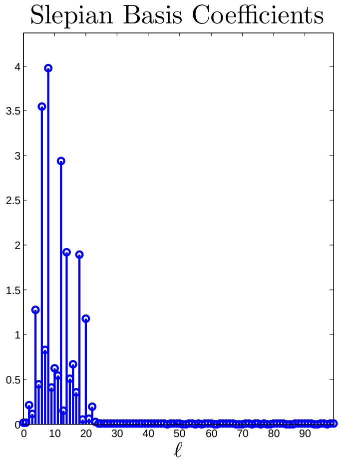
Slightly less than $2NW$ eigenvalues are close to 1, and slightly less than $N - 2NW$ eigenvalues are close to 0. For a fixed $\epsilon > 0$, roughly $O(\log N)$ eigenvalues are between ϵ and $1 - \epsilon$.



Slepian Transform

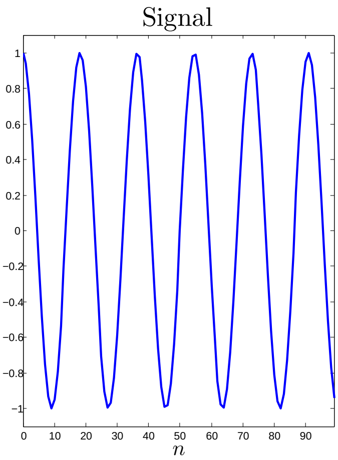
$$x[n] = \cos\left(\frac{2\pi \cdot 6}{100} n\right), \quad n = 0, 1, \ldots, 99, \;\; W = \frac{1}{10}$$

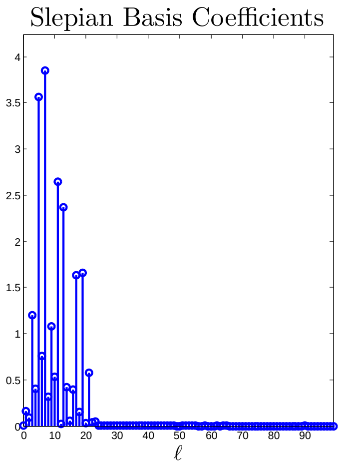




Slepian Transform

$$\boldsymbol{x}[n] = \cos\left(\frac{2\pi \cdot 5.5}{100} n\right), \quad n = 0, 1, \ldots, 99, \ W = \frac{1}{10}$$





Goals

- A fast method to compress N dimensional data into $K \approx 2NW$ dimensional data, and still be able to recover the original data if it lies in the span of the first K Slepian basis vectors.
- A fast method to project a vector in $\mathbb{C}^N$ onto the span of the first $K \approx 2NW$ Slepian basis vectors.
- A fast method to apply the rank $K \approx 2NW$ pseudoinverse of $\boldsymbol{B}_{N,W}$ to a vector in $\mathbb{C}^N$.

Prolate Matrix vs. Partial Fourier Matrix

Let $\boldsymbol{F}_{N,W}$ be an $N \times 2NW'$ matrix whose columns are the lowest $2NW'$ frequency DFT vectors of length N.

Theorem 1

For any $N \in \mathbb{N}$, $W \in (0, \frac{1}{2})$, and $\epsilon \in (0, \frac{1}{2})$, there exist $N \times R_L$ matrices $\boldsymbol{L}_1, \boldsymbol{L}_2$ and an $N \times N$ matrix $\boldsymbol{E}_F$ such that

$$\boldsymbol{B}_{N,W} = \boldsymbol{F}_{N,W}\boldsymbol{F}_{N,W}^* + \boldsymbol{L}_1\boldsymbol{L}_2^* + \boldsymbol{E}_F$$

where

$$R_L \le \left(\frac{4}{\pi^2}\log(8N) + 6\right)\log\left(\frac{15}{\epsilon}\right) \text{ and } \|\boldsymbol{E}_F\| \le \epsilon.$$

Due to the length limitations, we have included the proof in Section 2 of "The Fast Slepian Transform". A preprint available at https://arxiv.org/abs/1611.04950.

Number of Eigenvalues in Transition Region

Slepian [1] showed that for a fixed W and ϵ,

$$\#\{\ell : \epsilon \leq \lambda_{N,W}^{(\ell)} \leq 1-\epsilon\} \sim \frac{2}{\pi^2} \log N \log\left(\frac{1}{\epsilon} - 1\right).$$

Zhu and Wakin [2] showed that for any N, W, and ϵ,

$$\#\{\ell : \epsilon \leq \lambda_{N,W}^{(\ell)} \leq 1-\epsilon\} \leq \frac{\frac{2}{\pi^2}\log(N-1) + \frac{2}{\pi^2}\frac{2N-1}{N-1}}{\epsilon(1-\epsilon)}.$$

Using the previous theorem, and the Courant-Fischer-Weyl min-max principle, we can improve this bound.

Theorem 2

For any $N \in \mathbb{N}$, $W \in (0, \frac{1}{2})$, and $\epsilon \in (0, \frac{1}{2})$,

$$\#\{\ell : \epsilon < \lambda_{N,W}^{(\ell)} < 1-\epsilon\} \leq \left(\frac{8}{\pi^2}\log(8N) + 12\right)\log\left(\frac{15}{\epsilon}\right).$$

Define $\boldsymbol{S}_{[K]} = \begin{bmatrix} \boldsymbol{s}_{N,W}^{(0)} & \boldsymbol{s}_{N,W}^{(1)} & \cdots & \boldsymbol{s}_{N,W}^{(K-1)} \end{bmatrix}$.

Theorem 3

For any $N \in \mathbb{N}$, $W \in (0, \frac{1}{2})$, *and* $\epsilon \in (0, \frac{1}{2})$, *pick* K *such that* $\lambda_{N,W}^{(K-1)} > \epsilon$ *and* $\lambda_{N,W}^{(K)} < 1 - \epsilon$. *Then, there exist* $N \times R_U$ *matrices* $\boldsymbol{U}_1, \boldsymbol{U}_2$ *and an* $N \times N$ *matrix* $\boldsymbol{E}_S$ *such that*

$$\boldsymbol{S}_{[K]}\boldsymbol{S}_{[K]}^* = \boldsymbol{B}_{N,W} + \boldsymbol{U}_1\boldsymbol{U}_2^* + \boldsymbol{E}_S$$

where

$$R_U \leq \left(\frac{8}{\pi^2}\log(8N) + 12\right)\log\left(\frac{15}{\epsilon}\right) \text{ and } \|\boldsymbol{E}_S\| \leq \epsilon.$$

Idea: $\boldsymbol{B}_{N,W}$ and $\boldsymbol{S}_{[K]}\boldsymbol{S}_{[K]}^*$ have the same eigenvectors, and most of the eigenvalues are almost the same.

Fast Projection

Define

$$\mathcal{I}_1 = \{\ell < K : 1 - \epsilon \le \lambda_{N,W}^{(\ell)} \le 1\},$$

$$\mathcal{I}_2 = \{\ell < K : \epsilon < \lambda_{N,W}^{(\ell)} < 1 - \epsilon\},$$

$$\mathcal{I}_3 = \{\ell \ge K : \epsilon < \lambda_{N,W}^{(\ell)} < 1 - \epsilon\},$$

$$\mathcal{I}_4 = \{\ell \ge K : 0 \le \lambda_{N,W}^{(\ell)} \le \epsilon\},$$

and accordingly partition

$$\boldsymbol{S}_{N,W} := \begin{bmatrix} \boldsymbol{s}_{N,W}^{(0)} & \cdots & \boldsymbol{s}_{N,W}^{(N-1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_2 & \boldsymbol{S}_3 & \boldsymbol{S}_4 \end{bmatrix}$$

$$\boldsymbol{\Lambda}_{N,W} := \mathrm{diag}(\lambda_{N,W}^{(0)}, \ldots, \lambda_{N,W}^{(N-1)}) = \mathrm{blockdiag}(\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \boldsymbol{\Lambda}_3, \boldsymbol{\Lambda}_4).$$

Fast Projection

So, $\boldsymbol{B}_{N,W} = \boldsymbol{S}_1\boldsymbol{\Lambda}_1\boldsymbol{S}_1^* + \boldsymbol{S}_2\boldsymbol{\Lambda}_2\boldsymbol{S}_2^* + \boldsymbol{S}_3\boldsymbol{\Lambda}_3\boldsymbol{S}_3^* + \boldsymbol{S}_4\boldsymbol{\Lambda}_4\boldsymbol{S}_4^*$,
and $\boldsymbol{S}_{[K]}\boldsymbol{S}_{[K]}^* = \boldsymbol{S}_1\boldsymbol{S}_1^* + \boldsymbol{S}_2\boldsymbol{S}_2^*$.

Then, if we define

$$\boldsymbol{U}_1 = \begin{bmatrix} \boldsymbol{S}_2(\boldsymbol{I} - \boldsymbol{\Lambda}_2)^{1/2} & \boldsymbol{S}_3\boldsymbol{\Lambda}_3^{1/2} \end{bmatrix},$$

$$\boldsymbol{U}_2 = \begin{bmatrix} \boldsymbol{S}_2(\boldsymbol{I} - \boldsymbol{\Lambda}_2)^{1/2} & -\boldsymbol{S}_3\boldsymbol{\Lambda}_3^{1/2} \end{bmatrix},$$

$$\boldsymbol{E}_S = \boldsymbol{S}_1(\boldsymbol{I} - \boldsymbol{\Lambda}_1)\boldsymbol{S}_1^* - \boldsymbol{S}_4\boldsymbol{\Lambda}_4\boldsymbol{S}_4^*,$$

we have

$$\boxed{\boldsymbol{S}_{[K]}\boldsymbol{S}_{[K]}^* = \underbrace{\boldsymbol{B}_{N,W}}_{\text{Toeplitz}} + \underbrace{\boldsymbol{U}_1\boldsymbol{U}_2^*}_{\text{Low Rank}} + \underbrace{\boldsymbol{E}_S}_{\text{Small}}}.$$

So, $\boldsymbol{B}_{N,W} + \boldsymbol{U}_1\boldsymbol{U}_2^*$ is an approximation for $\boldsymbol{S}_{[K]}\boldsymbol{S}_{[K]}^*$ which can be applied to a vector in $O(N \log N \log \frac{1}{\epsilon})$ operations.

Fast Compression

Theorem 4

For any $N \in \mathbb{N}$, $W \in (0, \frac{1}{2})$, and $\epsilon \in (0, \frac{1}{2})$, pick K such that $\lambda_{N,W}^{(K-1)} > \epsilon$ and $\lambda_{N,W}^{(K)} < 1 - \epsilon$. Then, there exist $N \times K'$ matrices $\boldsymbol{T}_1, \boldsymbol{T}_2$ where

$$K' \leq \lceil 2NW \rceil + \left(\frac{12}{\pi^2} \log(8N) + 18 \right) \log \left(\frac{15}{\epsilon} \right)$$

such that $\| \boldsymbol{T}_1 \boldsymbol{T}_2^* - \boldsymbol{S}_{[K]} \boldsymbol{S}_{[K]}^* \| \leq 2\epsilon$, and $\boldsymbol{T}_1$ and $\boldsymbol{T}_2^*$ can be applied to a vector in $O(N \log N \log \frac{1}{\epsilon})$ operations.

Idea: Combine Theorem 1 and Theorem 3 to get

$$\boldsymbol{S}_{[K]} \boldsymbol{S}_{[K]}^* = \boldsymbol{F}_{N,W} \boldsymbol{F}_{N,W}^* + \boldsymbol{L}_1 \boldsymbol{L}_2^* + \boldsymbol{U}_1 \boldsymbol{U}_2^* + \boldsymbol{E}_F + \boldsymbol{E}_S.$$

Then, set $\boldsymbol{T}_1 = \begin{bmatrix} \boldsymbol{F}_{N,W} & \boldsymbol{L}_1 & \boldsymbol{U}_1 \end{bmatrix}$ and $\boldsymbol{T}_2 = \begin{bmatrix} \boldsymbol{F}_{N,W} & \boldsymbol{L}_2 & \boldsymbol{U}_2 \end{bmatrix}$.

Fast Pseudoinverse

Let $\boldsymbol{B}_{N,W}^{\dagger}$ be the rank-K truncated pseudoinverse of $\boldsymbol{B}_{N,W}$.

Theorem 5

For any $N \in \mathbb{N}$, $W \in (0, \frac{1}{2})$, *and* $\epsilon \in (0, \frac{1}{2})$, *pick* K *such that* $\lambda_{N,W}^{(K-1)} > \epsilon$ *and* $\lambda_{N,W}^{(K)} < 1 - \epsilon$. *Then, there exist* $N \times R_V$ *matrices* $\boldsymbol{V}_1, \boldsymbol{V}_2$ *and an* $N \times N$ *matrix* $\boldsymbol{E}_B$ *such that*

$$\boldsymbol{B}_{N,W}^{\dagger} = \boldsymbol{B}_{N,W} + \boldsymbol{V}_1 \boldsymbol{V}_2^* + \boldsymbol{E}_B$$

where

$$R_V \leq \left(\frac{8}{\pi^2} \log(8N) + 12 \right) \log \left(\frac{15}{\epsilon} \right) \text{ and } \|\boldsymbol{E}_B\| \leq 3\epsilon.$$

Idea: $\boldsymbol{B}_{N,W}^{\dagger}$ and $\boldsymbol{B}_{N,W}$ have the same eigenvectors, and most of the eigenvalues are almost the same.

Fast Pseudoinverse

So, $\boldsymbol{B}_{N,W} = \boldsymbol{S}_1\boldsymbol{\Lambda}_1\boldsymbol{S}_1^* + \boldsymbol{S}_2\boldsymbol{\Lambda}_2\boldsymbol{S}_2^* + \boldsymbol{S}_3\boldsymbol{\Lambda}_3\boldsymbol{S}_3^* + \boldsymbol{S}_4\boldsymbol{\Lambda}_4\boldsymbol{S}_4^*$,
and $\boldsymbol{B}_{N,W}^\dagger = \boldsymbol{S}_1\boldsymbol{\Lambda}_1^{-1}\boldsymbol{S}_1^* + \boldsymbol{S}_2\boldsymbol{\Lambda}_2^{-1}\boldsymbol{S}_2^*$.

Then, if we set

$$\boldsymbol{V}_1 = \begin{bmatrix} \boldsymbol{S}_2(\boldsymbol{\Lambda}_2^{-1} - \boldsymbol{\Lambda}_2)^{1/2} & \boldsymbol{S}_3\boldsymbol{\Lambda}_3^{1/2} \end{bmatrix},$$

$$\boldsymbol{V}_2 = \begin{bmatrix} \boldsymbol{S}_2(\boldsymbol{\Lambda}_2^{-1} - \boldsymbol{\Lambda}_2)^{1/2} & -\boldsymbol{S}_3\boldsymbol{\Lambda}_3^{1/2} \end{bmatrix},$$

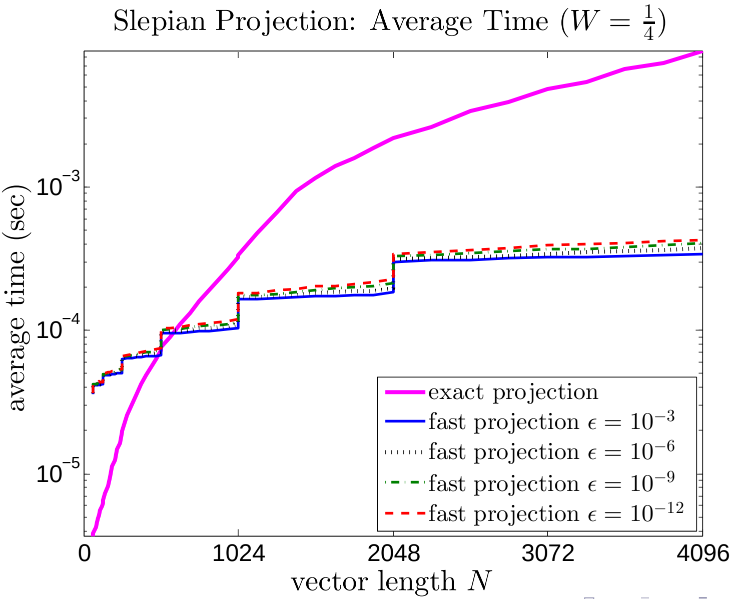
$$\boldsymbol{E}_B = \boldsymbol{S}_1(\boldsymbol{\Lambda}_1^{-1} - \boldsymbol{\Lambda}_1)\boldsymbol{S}_1^* - \boldsymbol{S}_4\boldsymbol{\Lambda}_4\boldsymbol{S}_4^*,$$

we have

$$\boxed{\boldsymbol{B}_{N,W}^\dagger = \underbrace{\boldsymbol{B}_{N,W}}_{\text{Toeplitz}} + \underbrace{\boldsymbol{V}_1\boldsymbol{V}_2^*}_{\text{Low Rank}} + \underbrace{\boldsymbol{E}_B}_{\text{Small}}}.$$

So, $\boldsymbol{B}_{N,W} + \boldsymbol{V}_1\boldsymbol{V}_2^*$ is an approximation for $\boldsymbol{B}_{N,W}^\dagger$ which can be applied to a vector in $O(N \log N \log \frac{1}{\epsilon})$ operations.

Simulations



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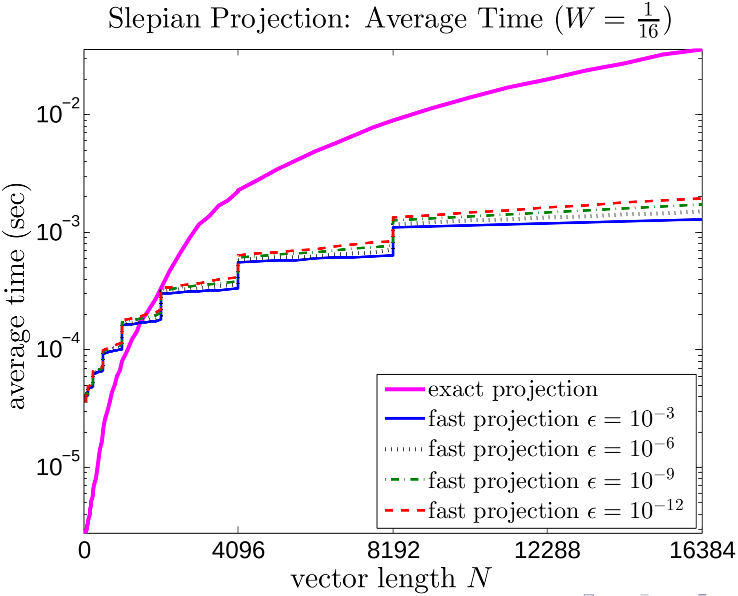

Conclusions

- The Slepian basis is a good choice of basis for working with samples of a bandlimited signal.
- We now have fast (approximate) algorithms for working with the Slepian basis, whose complexity scales comparably to the FFT.

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Acknowledgements

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