Linear Systems On Graphs

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4th Global Conference on Signal and Information Processing

Caltech

Outline

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1. Graph Signal Processing
2. Linear Systems in the Classical Domain
3. Linear Systems on Graphs
   - Definitions
   - Operator with repeated eigenvalues
   - Operators with distinct eigenvalues
   - Graph Laplacian v.s. Adjacency Matrix
4. Conclusions

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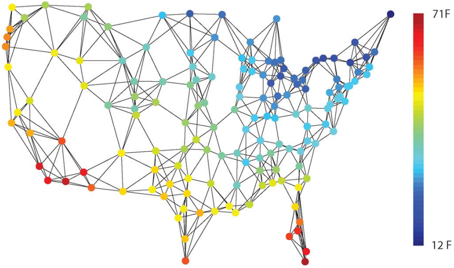
Preliminaries



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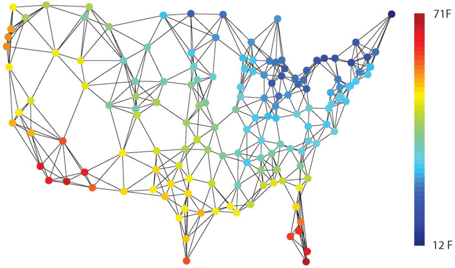

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Graph Laplacians[2]	:	$\boldsymbol{L}$, or $\mathcal{L}$
Other selections[3]		

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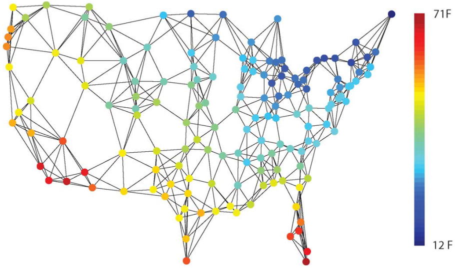

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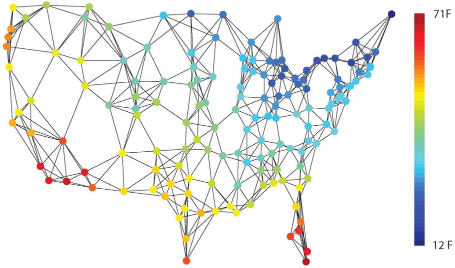

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- Graph Fourier Basis : $\boldsymbol{V}$
- Graph Fourier Transform : $\boldsymbol{F} = \boldsymbol{V}^{-1}$

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An *arbitrary* linear system

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Alias-Free $\Longleftarrow$

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$\boldsymbol{H}$ *is* polynomial $\iff$ $\boldsymbol{H} = H(\boldsymbol{S}) = \sum_{k=0}^{N-1} h_k\, \boldsymbol{S}^k$.

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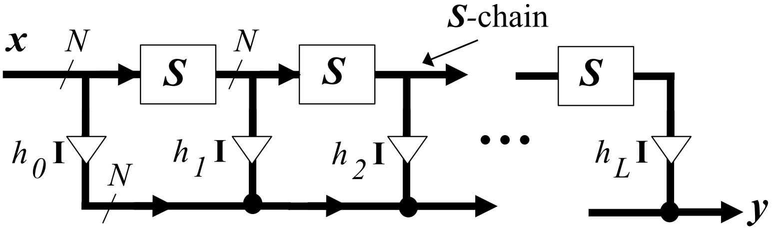

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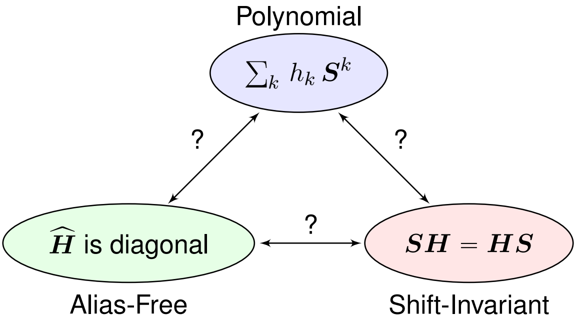
Interconnections when S is diagonalizable



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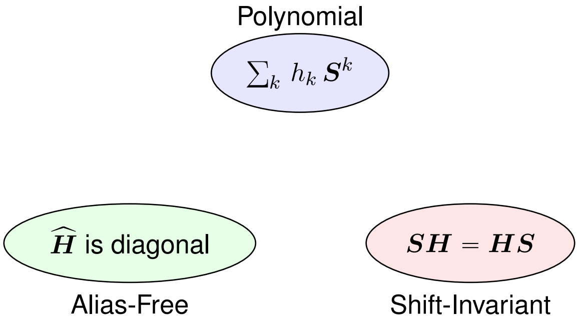

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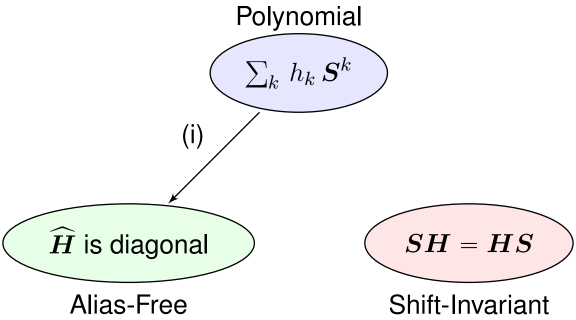

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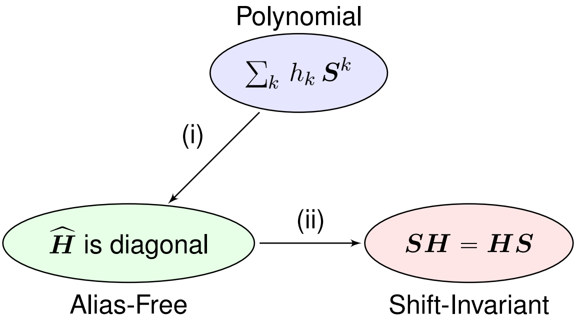

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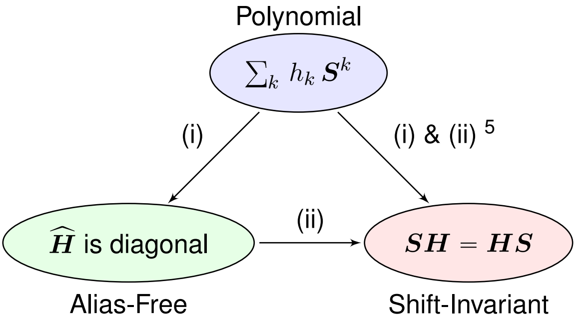

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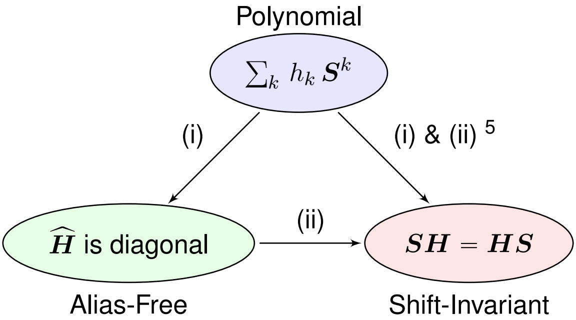

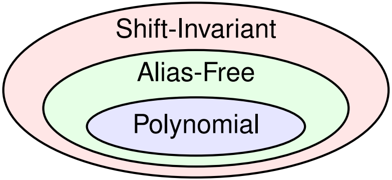

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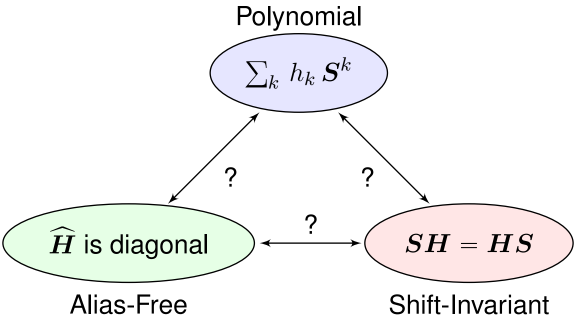

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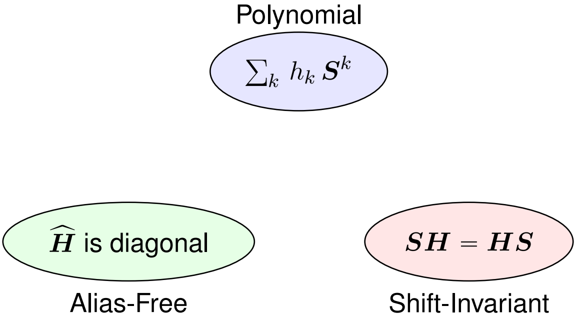

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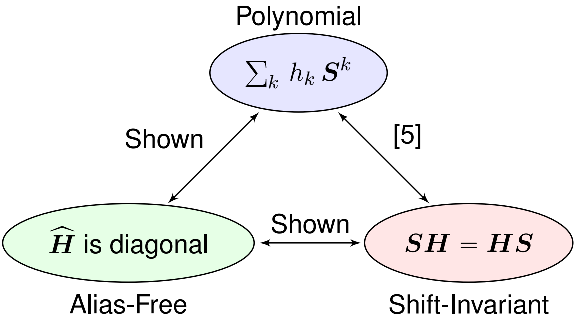

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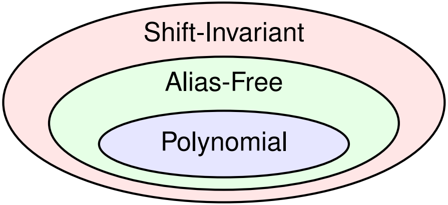

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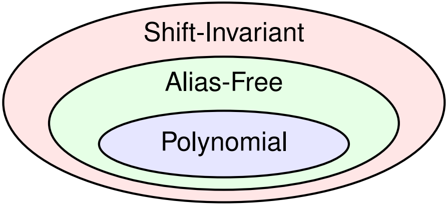

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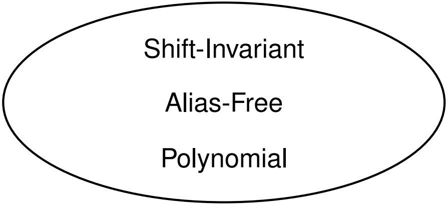

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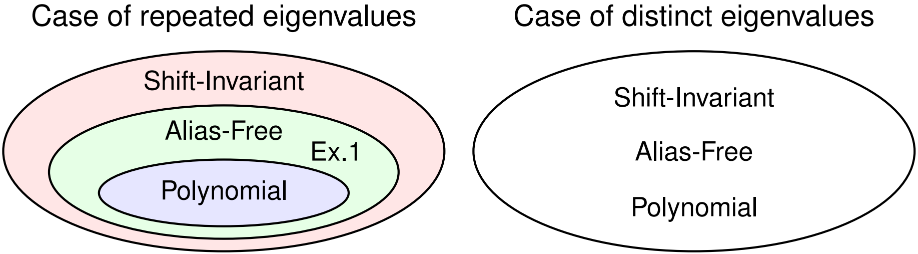

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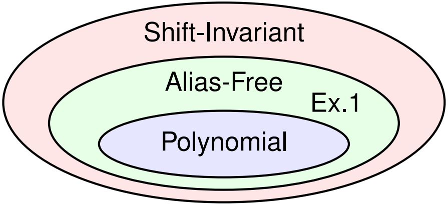

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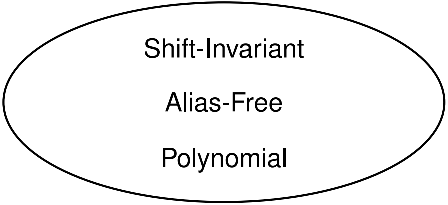

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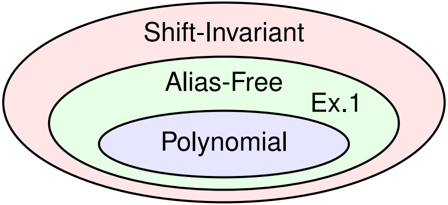

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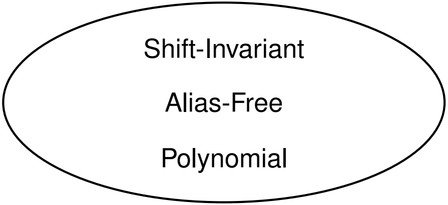

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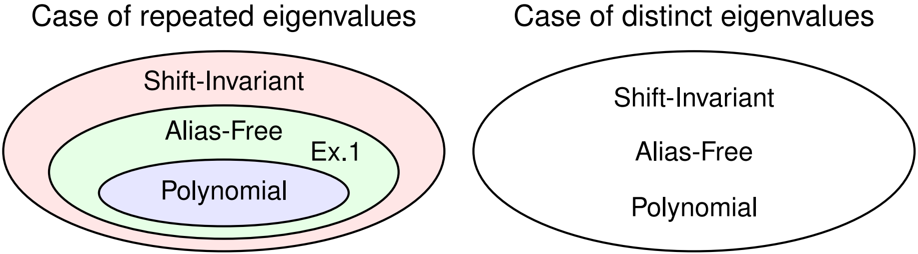

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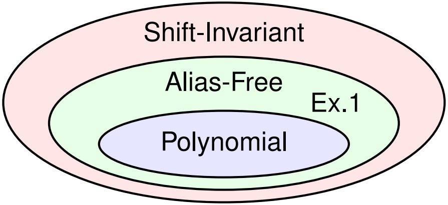

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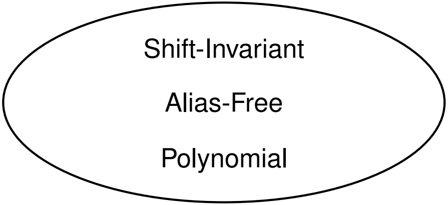

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Find $H(\cdot)$ s.t. $H(\lambda) = \widehat{h}_i$ and $H(\lambda) = \widehat{h}_j$

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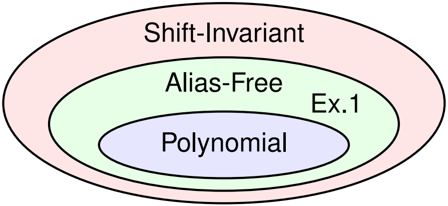

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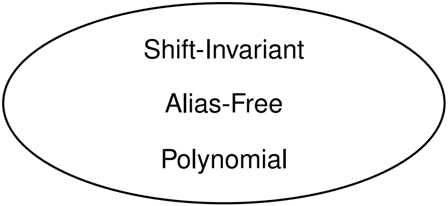

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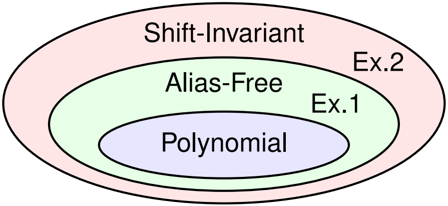
Diagonals of $\widehat{\boldsymbol{H}}$ are distinct $\implies$ $\widehat{h}_i \neq \widehat{h}_j$ for $i \neq j$

Find $H(\cdot)$ s.t. $H(\lambda) = \widehat{h}_i$ and $H(\lambda) = \widehat{h}_j$

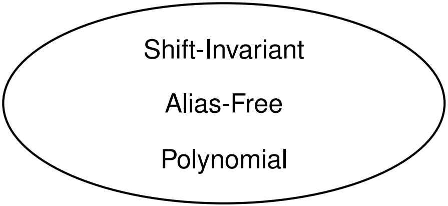
$\boldsymbol{H}$ is alias-free, but *not* polynomial

Examples

Case of repeated eigenvalues



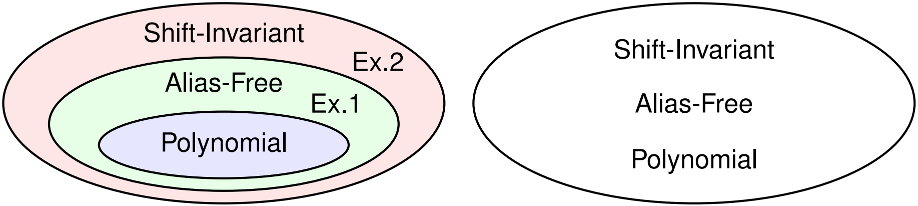
Case of distinct eigenvalues



Ex.2:

Examples

Case of repeated eigenvalues | Case of distinct eigenvalues



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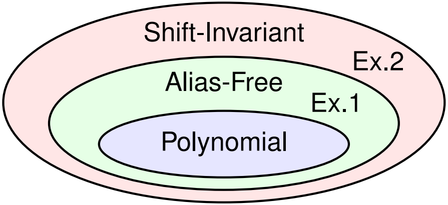

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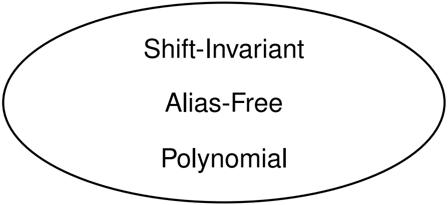

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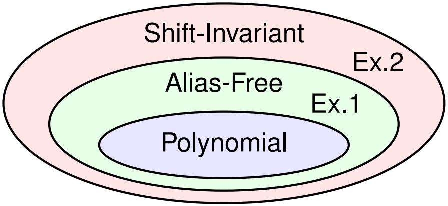

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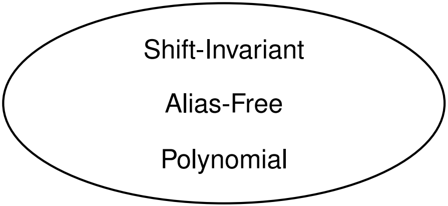

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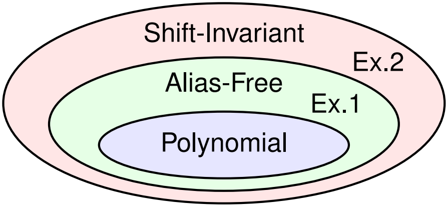

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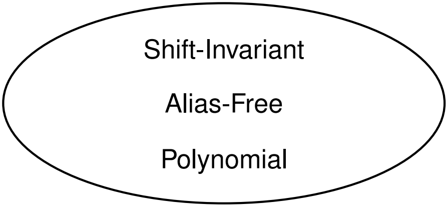

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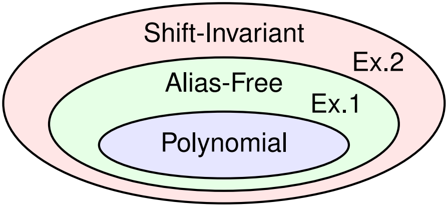

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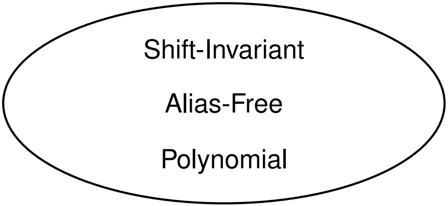

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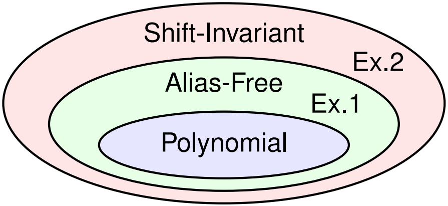

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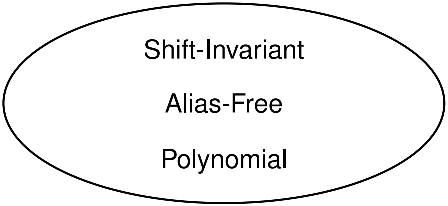

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  - How to select $\boldsymbol{V}$? [6]

[6] Teke & Vaidyanathan, "Uncertainty Principles and Sparse Eigenvectors of Graphs," *IEEE Trans. S. P., under review*