Simple and Accurate Algorithms for Sinusoidal Frequency Estimation

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Introduction

What is sinusoidal frequency estimation? [1]-[3]

Determine the frequency of a sinusoidal signal

- Consider a sinusoid $s(t) = A\cos(\omega t + \theta)$, the frequency is ω in radian or $\omega/(2\pi)$ in Hz
- The problem of sinusoidal frequency estimation is to estimate ω given a noisy version of $s(t)$ and the major difficulty is that the frequency is nonlinear in the signal
- Once the frequency is known, the amplitude and phase parameters are easily obtained as they can be transformed as linear parameters

Similar terminologies include spectral analysis, spectral line estimation, harmonic retrieval

Application Areas

- Wireless communications

e.g., frequency shift keying (FSK) signal demodulation:

$$s(t) = \cos(\omega_1 t) \quad \text{or} \quad s(t) = \cos(\omega_2 t)?$$

- Audio and speech signal processing

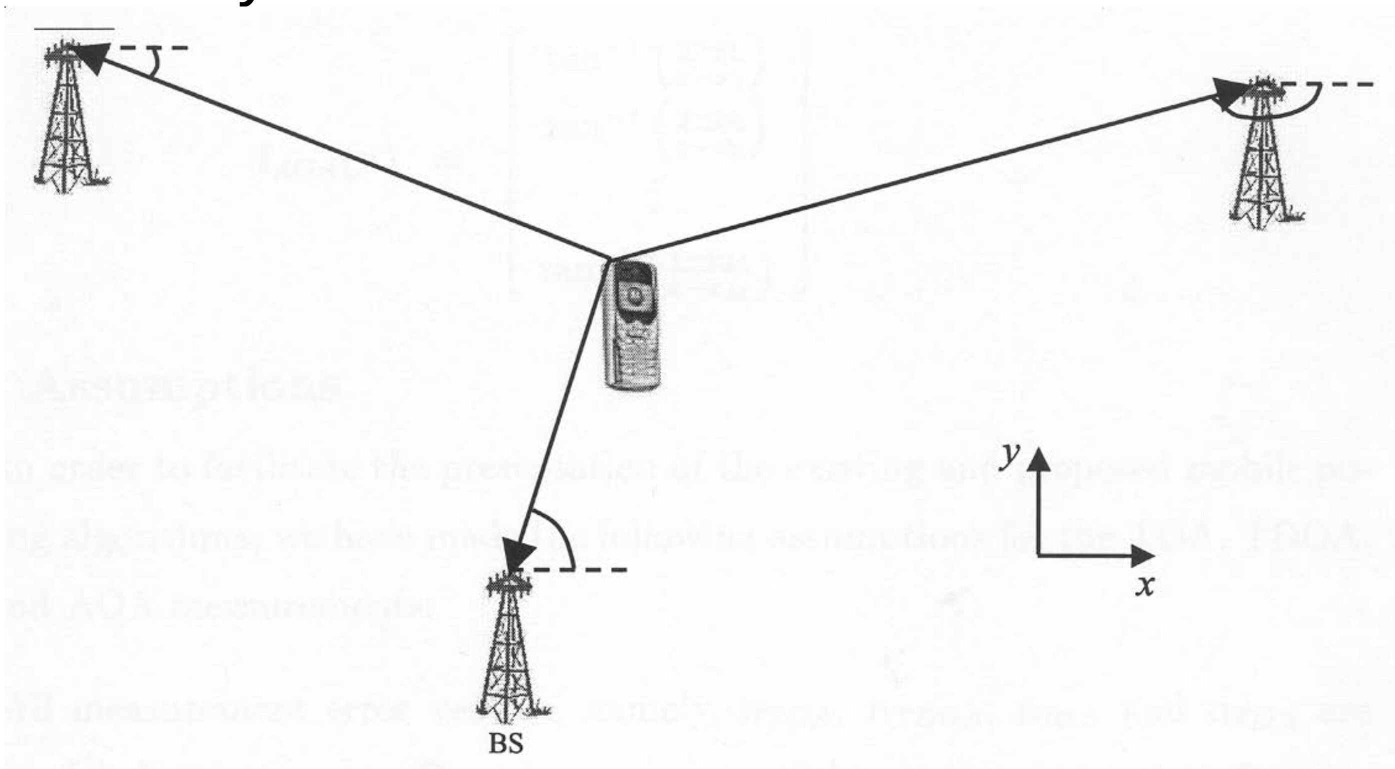
  - e.g., speech and music analysis using harmonic model:

$$x(t) = a(t) \sum_{m=1}^{M} c_m \cos(m\omega_0 t + \phi_m)$$

where ω_0 is the fundamental frequency or pitch

Source localization

- Position of a target can be obtained via direction-of-arrival (DOA) estimation from signals received at an antenna array



- DOA estimation model can be converted to the problem of frequency estimation

- Biomedical engineering

  e.g., nuclear magnetic resonance (NMR) or magnetic resonance spectroscopy (MRS) signal analysis

$$y(t) = \sum_{m=1}^{M} A_m e^{j\phi_m} e^{(-\lambda_m + j\omega_m)t} + w(t)$$

- Power electronics

  e.g., reliable frequency measurement in a power system is important for effective power control, load restoration and generator protection, and smart grid [4]

- Instrumentation and measurement

  e.g., IEEE Standard for Digitalizing Waveform Recorder (IEEE Std. 1057-1994) [5]

Common 1D Signal Models

- Complex tone model:

$$x_n = \sum_{m=1}^{M} A_m e^{j\phi_m} e^{(-\lambda_m + j\omega_m)n} + q_n, \quad n = 0,1,\cdots, N-1$$

where $\{A_m\}$, $\{\phi_m\}$, $\{\lambda_m\}$ and $\{\omega_m\}$ are constants while q_n is a zero-mean white noise

Simplest case: $x_n = Ae^{j(\omega n+\phi)} + q_n$

Using nonlinear least squares (NLS), optimum frequency estimation is achieved from:

$$(\hat{A},\hat{\omega},\hat{\phi}) = \arg\min_{\tilde{A},\tilde{\omega},\tilde{\phi}} \sum_{n=0}^{N-1} \left| x_n - \tilde{A}e^{j(\tilde{\omega}n+\tilde{\phi})} \right|^2$$

- Real tone model:

$$x_n = \sum_{m=1}^{M} A_m \cos(\omega_m n + \phi_m) + q_n, \qquad n = 0,1,\cdots, N-1$$

Simplest case: $x_n = A\cos(\omega n + \phi) + q_n$

Using NLS, optimum frequency estimation is achieved from:

$$(\hat{A}, \hat{\omega}, \hat{\phi}) = \arg \min_{\tilde{A}, \tilde{\omega}, \tilde{\phi}} \sum_{n=0}^{N-1} \left(x_n - \tilde{A}\cos(\tilde{\omega} n + \tilde{\phi})\right)^2$$

As the cost functions are multi-modal, global solution is not guaranteed

Key Ideas in Algorithm Development

- Linear prediction (LP) property of sinusoids

  - M (damped) complex sinusoid: $s_n = -\sum_{i=1}^{M} a_i s_{n-i}$

    where $\{a_i\}$ are LP parameters characterized by frequencies

    e.g., for $s_n = Ae^{j(\omega n+\phi)}$:

    $$s_n = e^{j\omega} \cdot s_{n-1}, \qquad a_1 = -e^{j\omega}$$

  - M (damped) real sinusoid: $s_n = -\sum_{i=1}^{2M} a_i s_{n-i}$ with $a_i = a_{2M-i}$ and $a_{2M} = 1$

e.g., for $s_n = A\cos(\omega n + \phi)$

$$s_n = 2\cos(\omega) \cdot s_{n-1} - s_{n-2}, \quad a_1 = -2\cos(\omega),\ a_2 = 1$$

Two advantages of LP:

- Nonlinear frequency parameters are transformed into linear $\{a_i\}$ which simplifies the estimation process
- Amplitude and phase parameters do not appear in the LP signal model which means that less parameters are needed for estimation

- Least squares (LS) or weighted least squares (WLS)

e.g., given $x_1 = A + q_1$ and $x_2 = A + q_2$

LS estimate for A is:

$$\hat{A} = \arg\min_{\tilde{A}} \sum_{i=1}^{2} (x_i - \tilde{A})^2$$

$$= \arg\min_{\tilde{A}} \left\{ \begin{bmatrix} x_1 - \tilde{A} & x_2 - \tilde{A} \end{bmatrix} \begin{bmatrix} x_1 - \tilde{A} \\ x_2 - \tilde{A} \end{bmatrix} \right\} = \frac{x_1 + x_2}{2}$$

Two advantages of LS:

- Low computational complexity
- No prior noise information is needed

If the noise characteristics are known, i.e., $E\{q_1^2\}$, $E\{q_1 q_2\}$ and $E\{q_2^2\}$ are available, an optimum estimate is the WLS solution:

$$\hat{A} = \arg\min_{\tilde{A}} \left\{ [x_1 - \tilde{A} \quad x_2 - \tilde{A}] \cdot \mathbf{W} \cdot \begin{bmatrix} x_1 - \tilde{A} \\ x_2 - \tilde{A} \end{bmatrix} \right\}$$

where

$$\mathbf{W} = \begin{bmatrix} E\{q_1^2\} & E\{q_1 q_2\} \\ E\{q_1 q_2\} & E\{q_1^2\} \end{bmatrix}^{-1}$$

The main advantage of WLS is high estimation accuracy while the increase in computational complexity is small

- Constrained optimization

$$\hat{\boldsymbol{\rho}} = \arg\min_{\tilde{\boldsymbol{\rho}}} \tilde{\boldsymbol{\rho}}^T \mathbf{Y}^T \mathbf{Y} \tilde{\boldsymbol{\rho}} \quad \text{subject to} \quad \tilde{\boldsymbol{\rho}}^T \boldsymbol{\Sigma} \tilde{\boldsymbol{\rho}} = 1$$

is equal to unconstrained optimization:

$$\hat{\boldsymbol{\rho}} = \arg\min_{\tilde{\boldsymbol{\rho}}} \frac{\tilde{\boldsymbol{\rho}}^T \mathbf{Y}^T \mathbf{Y} \tilde{\boldsymbol{\rho}}}{\tilde{\boldsymbol{\rho}}^T \boldsymbol{\Sigma} \tilde{\boldsymbol{\rho}}}$$

where $\mathbf{Y}$ is data matrix and $\tilde{\boldsymbol{\rho}}^T \mathbf{Y}^T \mathbf{Y} \tilde{\boldsymbol{\rho}}$ is a LS cost function

For the former, it can be solved by the method of Lagrange multipliers:

$$L(\tilde{\boldsymbol{\rho}}, \lambda) = \tilde{\boldsymbol{\rho}}^T \mathbf{Y}^T \mathbf{Y} \tilde{\boldsymbol{\rho}} + \lambda \left(1 - \tilde{\boldsymbol{\rho}}^T \boldsymbol{\Sigma} \tilde{\boldsymbol{\rho}}\right)$$

$$\frac{\partial L(\tilde{\boldsymbol{\rho}},\lambda)}{\partial \tilde{\boldsymbol{\rho}}} = 0$$

$$\Rightarrow \mathbf{Y}^T \mathbf{Y}\hat{\boldsymbol{\rho}} = \lambda \boldsymbol{\Sigma}\hat{\boldsymbol{\rho}}$$

$\Rightarrow$ $\hat{\boldsymbol{\rho}}$ is generalized eigenvector corresponding to the smallest generalized eigenvalue of the pair $\left(\mathbf{Y}^T \mathbf{Y}, \boldsymbol{\Sigma}\right)$

The main advantage of using constraints is to achieve unbiased frequency estimation

Proposed Algorithms

1. Single real-tone estimation via LP, LS and constraint [6]

- Recall signal model is:

$$x_n = s_n + q_n, \qquad n = 1,2,\cdots,N$$

- Recall $s_n = A\cos(\omega n + \phi)$ obeys

$$s_n = \rho \cdot s_{n-1} - s_{n-2}, \qquad \rho = 2\cos(\omega)$$

- Construct LP error function:

$$e_n = x_n - \tilde{\rho} x_{n-1} + x_{n-2}$$

- The LS or modified covariance (MC) estimate is simply:

$$\hat{\rho} = \arg\min_{\tilde{\rho}} \left\{ \sum_{n=3}^{N} e_n^2 \right\} = \left(\frac{\sum_{n=3}^{N} x_{n-1}(x_n + x_{n-2})}{\sum_{n=3}^{N} x_{n-1}^2} \right)$$

and

$$\hat{\omega} = \cos^{-1}\left(\frac{\hat{\rho}}{2}\right)$$

which is known to be a biased estimator

- The biasedness can be examined from the expected value of the LS cost function:

$$E\left\{\sum_{n=3}^{N} e_n^2\right\} = \sum_{n=3}^{N} (s_n - \tilde{\rho} s_{n-1} + s_{n-2})^2 + (N-2)(2+\tilde{\rho}^2)\sigma^2$$

because its noise component is also a function of $\tilde{\rho}$:

- Unbiased frequency estimation is attained by minimizing

$$\sum_{n=3}^{N} e_n^2 \quad \text{subject to} \quad (N-2)(2+\tilde{\rho}^2)\sigma^2 = 1$$

or

$$\hat{\rho} = \arg\min_{\tilde{\rho}} \left\{ \frac{\sum_{n=3}^{N} e_n^2}{2+\tilde{\rho}^2} \right\}$$

- Direct minimization on the unconstrained optimization formulation will lead to a cubic equation so we use the transformation:

$$\tilde{\rho} = 2\cos(\tilde{\omega})$$

to convert it as:

$$\frac{\sum_{n=3}^{N} e_n^2}{2 + 4\cos^2(\tilde{\omega})}$$

with

$$e_n = x_n - 2\cos(\tilde{\omega})x_{n-1} + x_{n-2}$$

- Differentiating with respect to $\tilde{\omega}$ and setting the resultant expression to zero:

$$\sum_{n=3}^{N} e_n\left((x_n + x_{n-2})\cos(\hat{\omega}) + x_{n-1}\right) = 0$$

$$\Rightarrow 2A_N \cos^2(\hat{\omega}) - B_N \cos(\hat{\omega}) - A_N = 0$$

where

$$A_N = \sum_{n=3}^{N} (x_n + x_{n-2}) x_{n-1}$$

and

$$B_N = x_N^2 - x_{N-1}^2 - x_2^2 + x_1^2 + 2 \sum_{n=3}^{N} x_n x_{n-2}$$

$$\Rightarrow \hat{\omega} = \cos^{-1}\left(\frac{B_N + \sqrt{B_N^2 + 8A_N^2}}{4A_N} \right)$$

- The frequency estimate is similar to the Pisarenko harmonic decomposition (PHD) method:

$$\hat{\omega}^{PHD} = \cos^{-1}\left(\frac{r_2 + \sqrt{r_2^2 + 8r_1^2}}{4r_1} \right)$$

which is obtained by finding the eigenvector corresponding to the smallest eigenvalue of:

$$\mathbf{R}=\begin{bmatrix} r_0 & r_1 & r_2 \\ r_1 & r_0 & r_1 \\ r_2 & r_1 & r_0 \end{bmatrix}$$

where

$$r_k=\frac{1}{N-k}\sum_{n=1}^{N-k} x_n x_{n+k}, \qquad k=0,1,2$$

$$A_N=2(N-2)\left(r_1+\frac{2r_1-x_1x_2-x_{N-1}x_N}{2(N-2)}\right)$$

and

$$B_N = 2(N-2)\left(r_2 + \frac{x_1^2 - x_2^2 - x_{N-1}^2 + x_N^2}{2(N-2)}\right)$$

$\Rightarrow$ two estimators are identical at $N \to \infty$

- On-line implementation:

$$A_N = A_{N-1} + x_{N-2}\left(x_{N-3} + x_{N-1}\right)$$

and

$$B_N = B_{N-1} + x_{N-3}^2 - 2x_{N-2}^2 + x_{N-1}^2 + 2x_{N-3}x_{N-1}$$

$\Rightarrow$ 8 additions, 7 multiplications, 1 division, 1 root operation and 1 $\cos^{-1}$ operation per sampling interval

➢ Variance analysis

$$
\begin{aligned}
\mathrm{var}(\hat{\omega}) &= E\{(\hat{\omega}-\omega)^2\} \\
&\approx \frac{\cos^2(2\omega)+\cos^2(\omega)}{\mathrm{SNR}^2(N-2)(2+\cos(2\omega))^2\sin^2(\omega)} \\
&\quad + \frac{1}{\mathrm{SNR}(N-2)^2\sin^2(\omega)} \\
&\quad + \frac{3+4\cos(2\omega)-\cos(4\omega)}{4\mathrm{SNR}^2(N-2)^2(2+\cos(2\omega))^2\sin^2(\omega)}, \qquad \mathrm{SNR}=\frac{A^2}{2\sigma^2}
\end{aligned}
$$

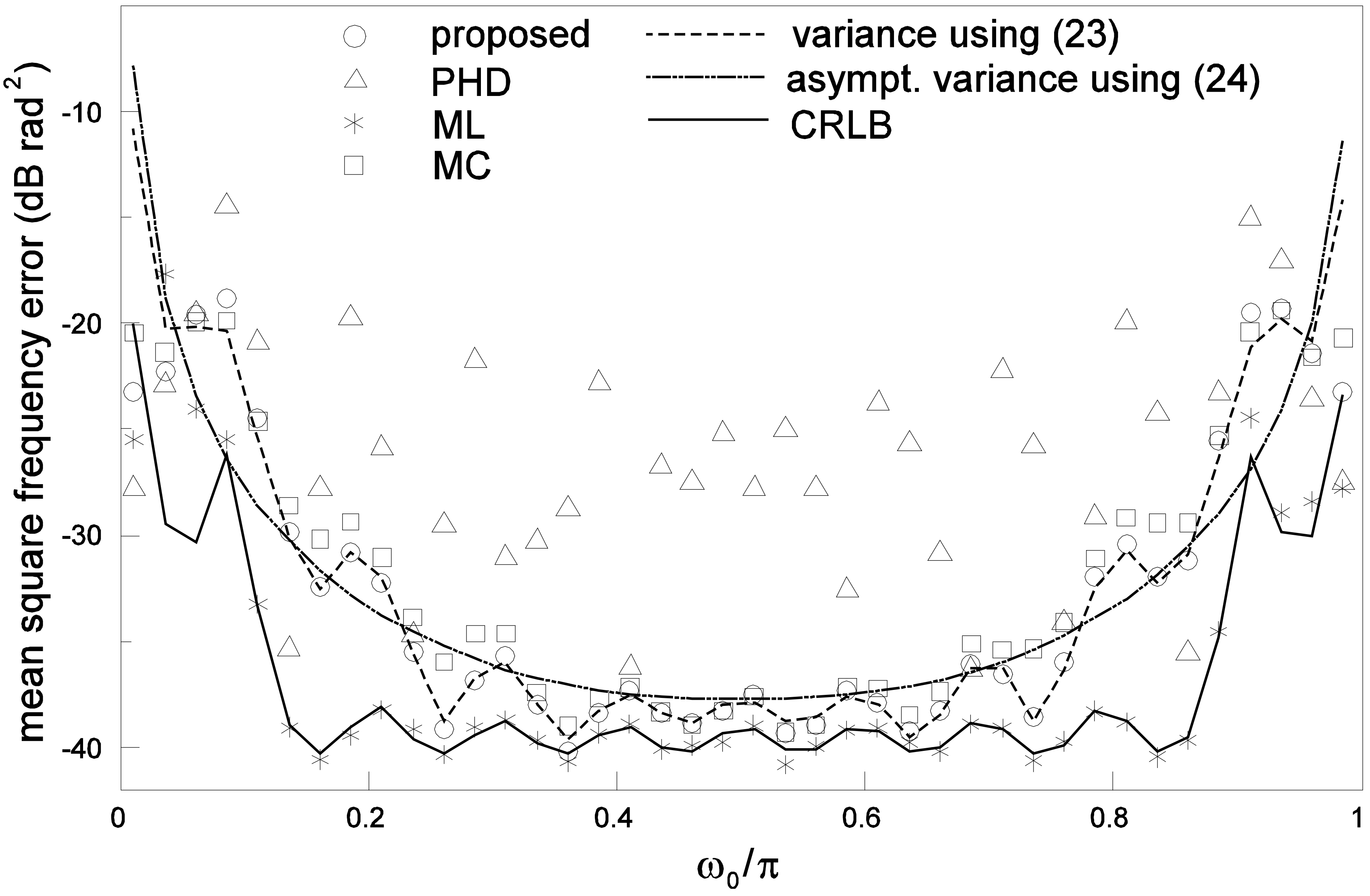
for $N >> 1$

Further approximation at high SNR:

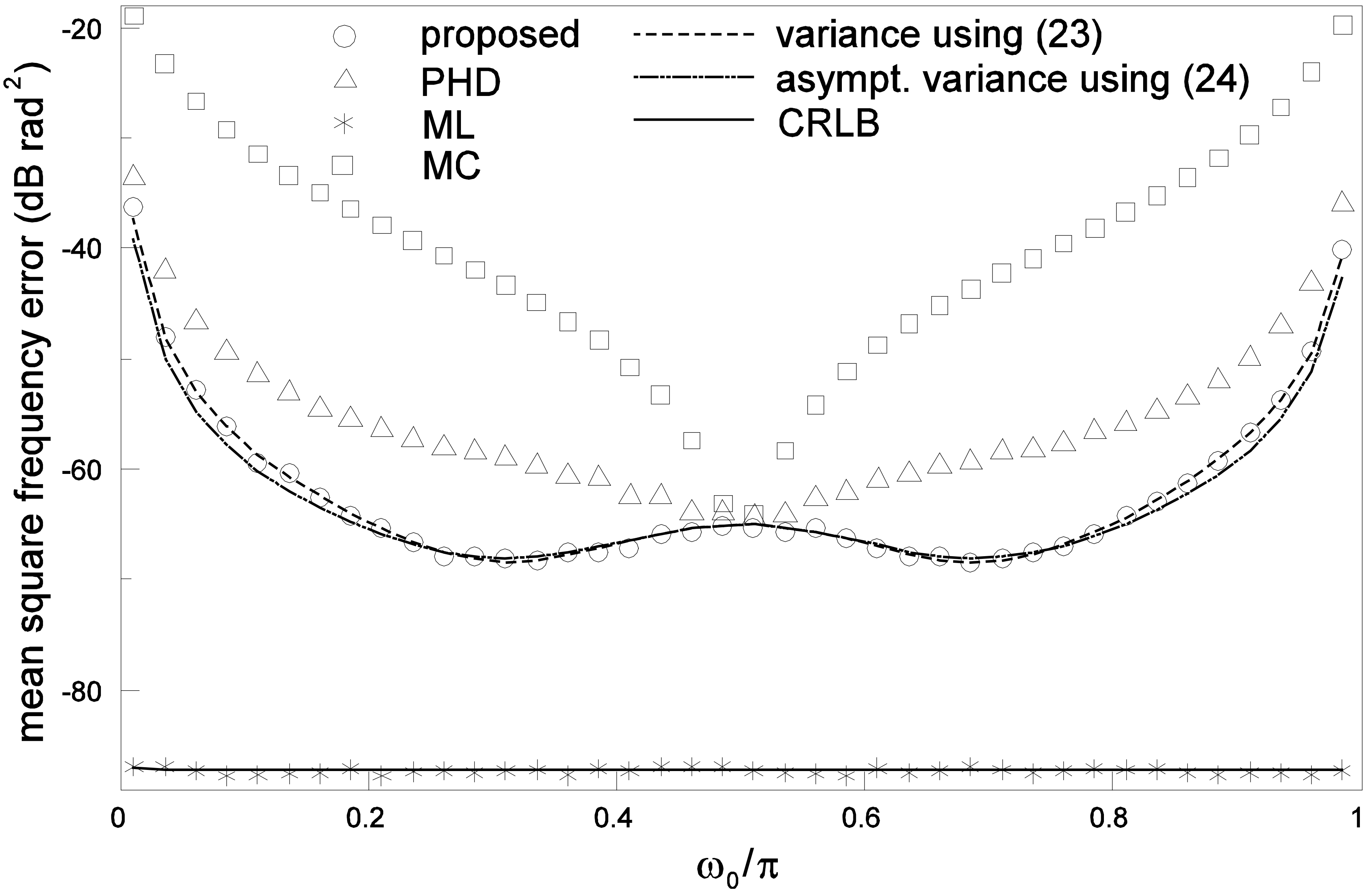
$$\mathrm{var}(\hat{\omega}) \approx \frac{1}{\mathrm{SNR}(N-2)^2 \sin^2(\omega)}$$

Nevertheless, $\mathrm{var}(\hat{\omega})$ is frequency dependent and the frequency estimator is suboptimal as its performance cannot attain the Cramer-Rao lower bound (CRLB) at $N >> 1$:

$$\mathrm{CRLB}(\omega) \approx \frac{12}{\mathrm{SNR} \cdot N(N^2-1)}$$



Mean square frequency errors versus ω at SNR = 20 dB & N=10



Mean square frequency errors versus ω at SNR = 20 dB & N=400

2. Single real-tone estimation via LP, WLS and constraint [7]

- Recall the LP error function:

$$e_n = x_n - \tilde{\rho} x_{n-1} + x_{n-2}, \qquad \rho = 2\cos(\omega)$$

- An alternate form is

$$e_n = \tilde{a}_0 \left(x_n + x_{n-2} \right) + \tilde{a}_1 x_{n-1}, \qquad \tilde{a}_1 / \tilde{a}_0 = -\tilde{\rho}$$

- In vector form of $\mathbf{e} = [e_N, e_{N-1}, \cdots, e_3]^T$:

$$\mathbf{e} = \mathbf{X}\tilde{\mathbf{a}} = \mathbf{S}\tilde{\mathbf{a}} + \mathbf{Q}\tilde{\mathbf{a}}$$

where

$$\tilde{\mathbf{a}} = [\tilde{a}_0, \tilde{a}_1]^T$$

$$\mathbf{X} = \begin{bmatrix} x_N + x_{N-2} & x_{N-1} \\ x_{N-1} + x_{N-3} & x_{N-2} \\ \vdots & \vdots \\ x_3 + x_1 & x_2 \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} s_N + s_{N-2} & s_{N-1} \\ s_{N-1} + s_{N-3} & s_{N-2} \\ \vdots & \vdots \\ s_3 + s_1 & s_2 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} q_N + q_{N-2} & q_{N-1} \\ q_{N-1} + q_{N-3} & q_{N-2} \\ \vdots & \vdots \\ q_3 + q_1 & q_2 \end{bmatrix}$$

- The WLS cost function is then:

$$\mathbf{e}^T \mathbf{W} \mathbf{e} = \tilde{\mathbf{a}}^T \mathbf{X}^T \mathbf{W} \mathbf{X} \tilde{\mathbf{a}}$$

where

$$\mathbf{W} = \left(E\left\{ \boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T \right\} \right)^{-1}$$

$$\boldsymbol{\varepsilon} = \left[\varepsilon_{N-1}, \varepsilon_{N-2}, \cdots, \varepsilon_2\right]^T$$

$$\varepsilon_n = a_0\left(q_n + q_{n-1}\right) + a_1 q_{n-2}$$

Taking expected value yields:

$$E\{\mathbf{e}^T \mathbf{W} \mathbf{e}\} = (\mathbf{S}\tilde{\mathbf{a}})^T \mathbf{W} \mathbf{S} \tilde{\mathbf{a}} + \tilde{\mathbf{a}}^T \boldsymbol{\gamma} \tilde{\mathbf{a}}, \qquad \boldsymbol{\gamma} = E\{\mathbf{Q}^T \mathbf{W} \mathbf{Q}\}$$

As a result, unbiased WLS estimate is

$$\hat{\mathbf{a}} = \arg\min_{\tilde{\mathbf{a}}} \tilde{\mathbf{a}}^T \mathbf{X}^T \mathbf{W} \mathbf{X} \tilde{\mathbf{a}} \quad \text{subject to} \quad \tilde{\mathbf{a}}^T \boldsymbol{\gamma} \tilde{\mathbf{a}} = 1$$

where

$$\boldsymbol{\gamma} = \begin{bmatrix} 2(D_0 + D_2) & 2D_1 \\ 2D_1 & D_0 \end{bmatrix}, \qquad D_j = \sum_{i=1}^{N-L-j} [\mathbf{W}]_{i,i+j}$$

- By the method of Lagrange multipliers:

$$L(\tilde{\mathbf{a}},\lambda) = \tilde{\mathbf{a}}^T \mathbf{X}^T \mathbf{W}\mathbf{X}\tilde{\mathbf{a}} + \lambda\left(1 - \tilde{\mathbf{a}}^T \boldsymbol{\gamma}\tilde{\mathbf{a}}\right)$$

$$\frac{\partial L(\tilde{\mathbf{a}},\lambda)}{\partial \tilde{\mathbf{a}}} = 0 \Rightarrow \mathbf{X}^T \mathbf{W}\mathbf{X}\hat{\mathbf{a}} = \lambda\boldsymbol{\gamma}\hat{\mathbf{a}}$$

$\Rightarrow$ $\hat{\mathbf{a}}$ is generalized eigenvector corresponding to the smallest generalized eigenvalue of the pair $\left(\mathbf{X}^T \mathbf{W}\mathbf{X}, \boldsymbol{\gamma}\right)$

- Since $\mathbf{W}$ is unknown, the constrained WLS solution is determined using an iterative procedure:

  (i) Find initial estimate of $\mathbf{a}$ from generalized eigenvalue decomposition of $\left(\mathbf{X}^T\mathbf{W}\mathbf{X},\boldsymbol{\gamma}\right)$ with $\mathbf{W}=\mathbf{I}$

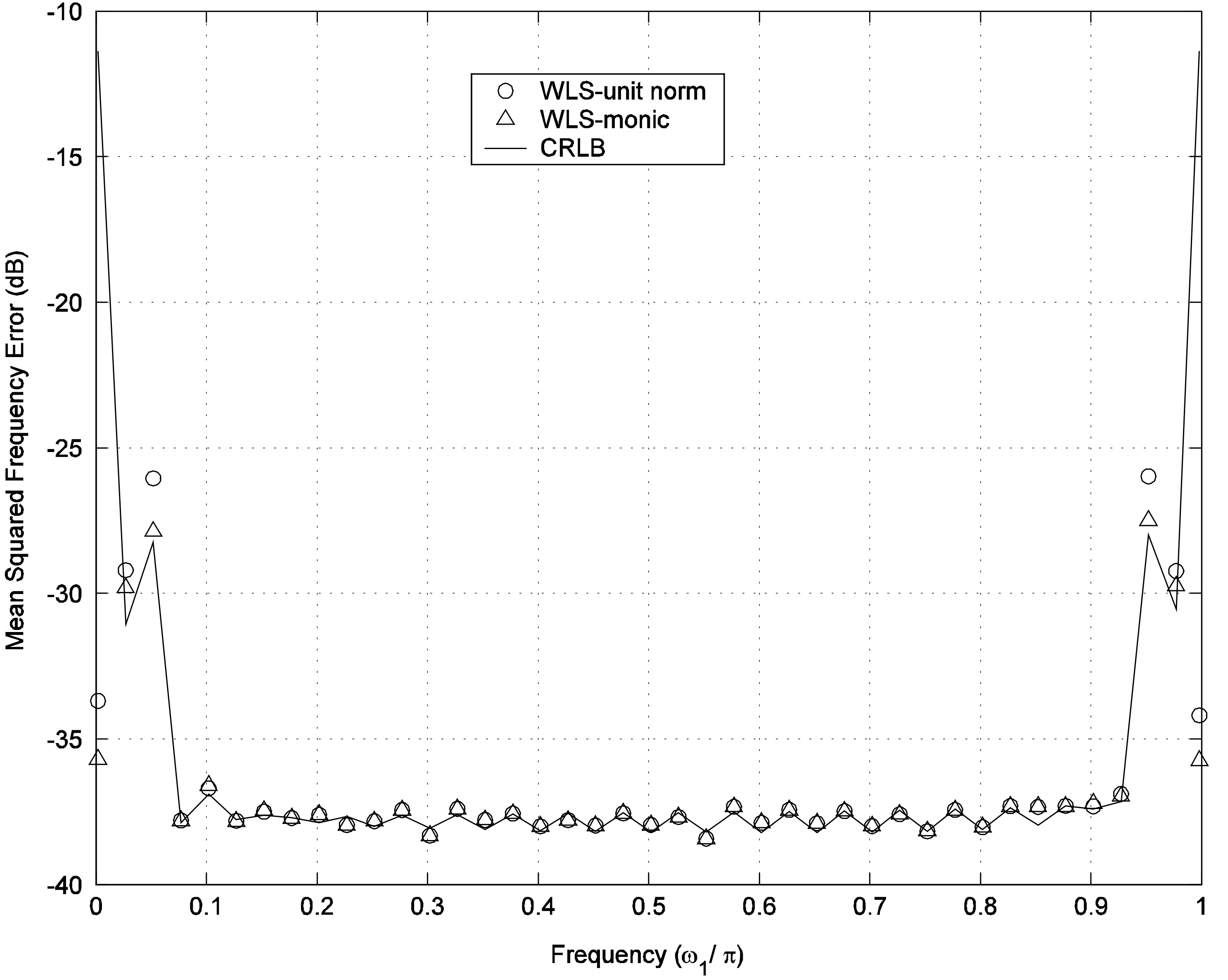
  (ii) Use $\hat{\mathbf{a}}$ to construct $\mathbf{W}$ and $\boldsymbol{\gamma}$

  (iii) Determine an updated estimate from generalized eigenvalue decomposition of $\left(\mathbf{X}^T\mathbf{W}\mathbf{X},\boldsymbol{\gamma}\right)$

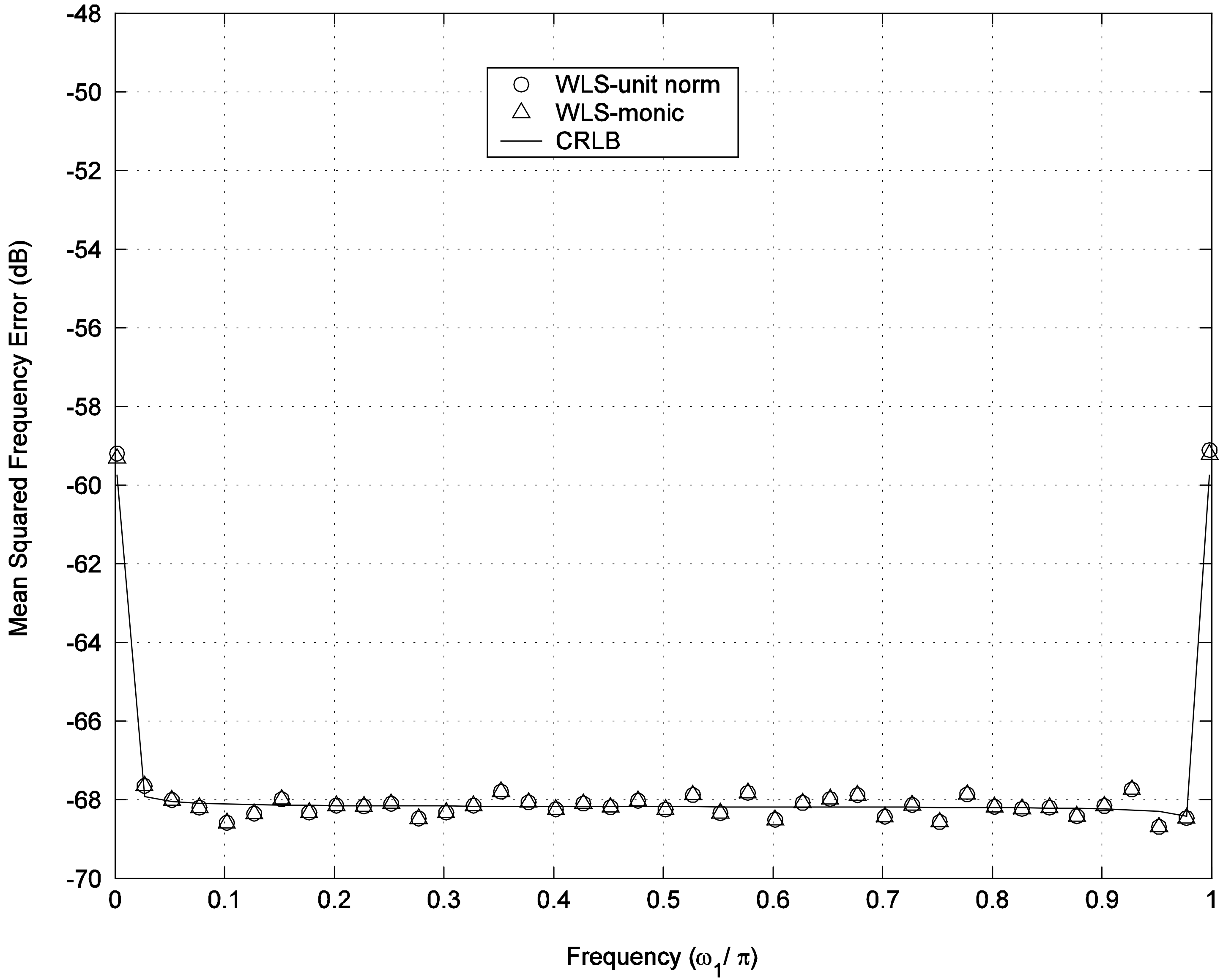
  (iv) Repeat (ii)-(iii) until a stopping criterion is reached

  (v) The frequency estimate is computed as:

$$\hat{\omega} = \cos^{-1}\left(-\frac{\hat{a}_1}{2\hat{a}_0}\right)$$



Mean square frequency errors versus ω at SNR = 10 dB & N=20



Mean square frequency errors versus ω at SNR = 10 dB & N=200

3. Single complex-tone estimation via LP and WLS [8]

- Recall $s_n = Ae^{j(\omega n + \phi)}$ obeys

$$s_n = \rho \cdot s_{n-1}, \qquad \rho = e^{j\omega}$$

- Construct a LP error function:

$$e_n = x_n - \tilde{\rho} x_{n-1}$$

- In matrix form:

$$\mathbf{e} = \mathbf{X_1} - \tilde{\rho}\mathbf{X_2}$$

where

$$\mathbf{e} = [e_{N-1}, e_{N-2}, \cdots, e_1]^T$$

$$\mathbf{X_1} = [x_{N-1}, x_{N-2}, \cdots, x_1]^T, \ \ \mathbf{X_2} = [x_{N-2}, x_{N-3}, \cdots, x_0]^T$$

- The WLS cost function is

$$J(\tilde{\rho}) = \mathbf{e}^H \mathbf{W} \mathbf{e} = (\mathbf{X_1} - \tilde{\rho} \mathbf{X_2})^H \mathbf{W} (\mathbf{X_1} - \tilde{\rho} \mathbf{X_2})$$

$$\Rightarrow \qquad \hat{\rho} = \frac{\mathbf{X_2}^H \mathbf{W} \mathbf{X_1}}{\mathbf{X_2}^H \mathbf{W} \mathbf{X_2}}$$

- The optimum weighting matrix is

$$\mathbf{W} = \begin{bmatrix} 1+|\rho|^2 & -\rho & 0 & 0 & \cdots & 0 \\ -\rho^* & 1+|\rho|^2 & -\rho & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -\rho^* & 1+|\rho|^2 & -\rho \\ 0 & 0 & \cdots & 0 & -\rho^* & 1+|\rho|^2 \end{bmatrix}^{-1}$$

- $\mathbf{W}$ can be simplified by putting $\rho = e^{j\omega}$:

$$[\mathbf{W}]_{m,n} = \frac{N \min(m,n) - mn}{N} e^{j(n-m)\omega}, \quad 1 \leq m \leq N-1, \quad 1 \leq n \leq N-1$$

The frequency estimate is now simplified as

$$\hat{\omega} = \angle\left(\frac{\mathbf{X_2}^H \mathbf{W} \mathbf{X_1}}{\mathbf{X_2}^H \mathbf{W} \mathbf{X_2}}\right) = \angle\left(\mathbf{X_2}^H \mathbf{W} \mathbf{X_1}\right)$$

as $\mathbf{X_2}^H \mathbf{W} \mathbf{X_2}$ is real and positive

- Since $\mathbf{W}$ is unknown, the WLS solution is determined using an iterative procedure:

(i) Find an initial frequency estimate, e.g., i.e.,

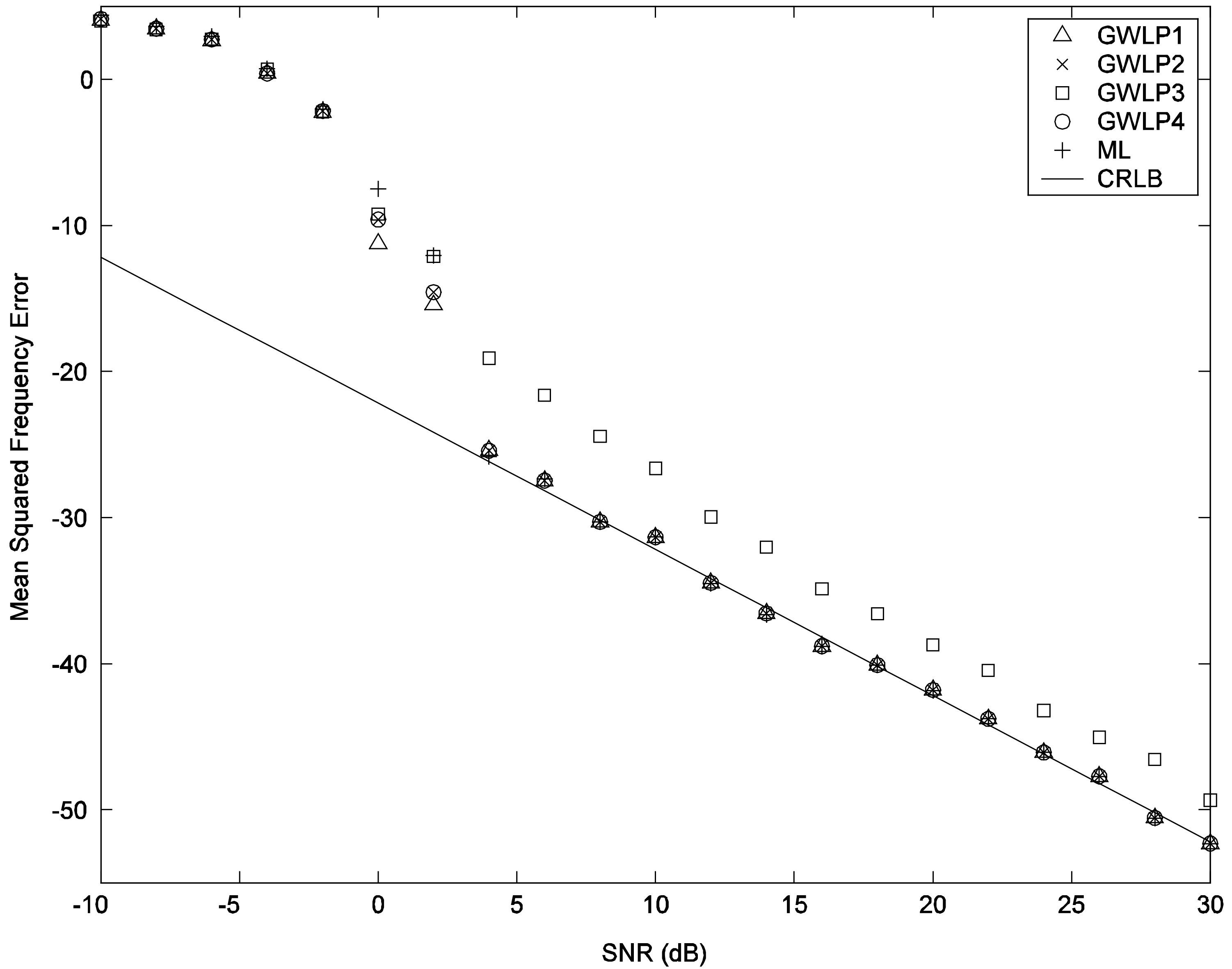
$$\hat{\omega} = \angle\left(\mathbf{X_2}^H \mathbf{X_1}\right)$$

(ii) Use $\hat{\omega}$ to construct $\mathbf{W}$

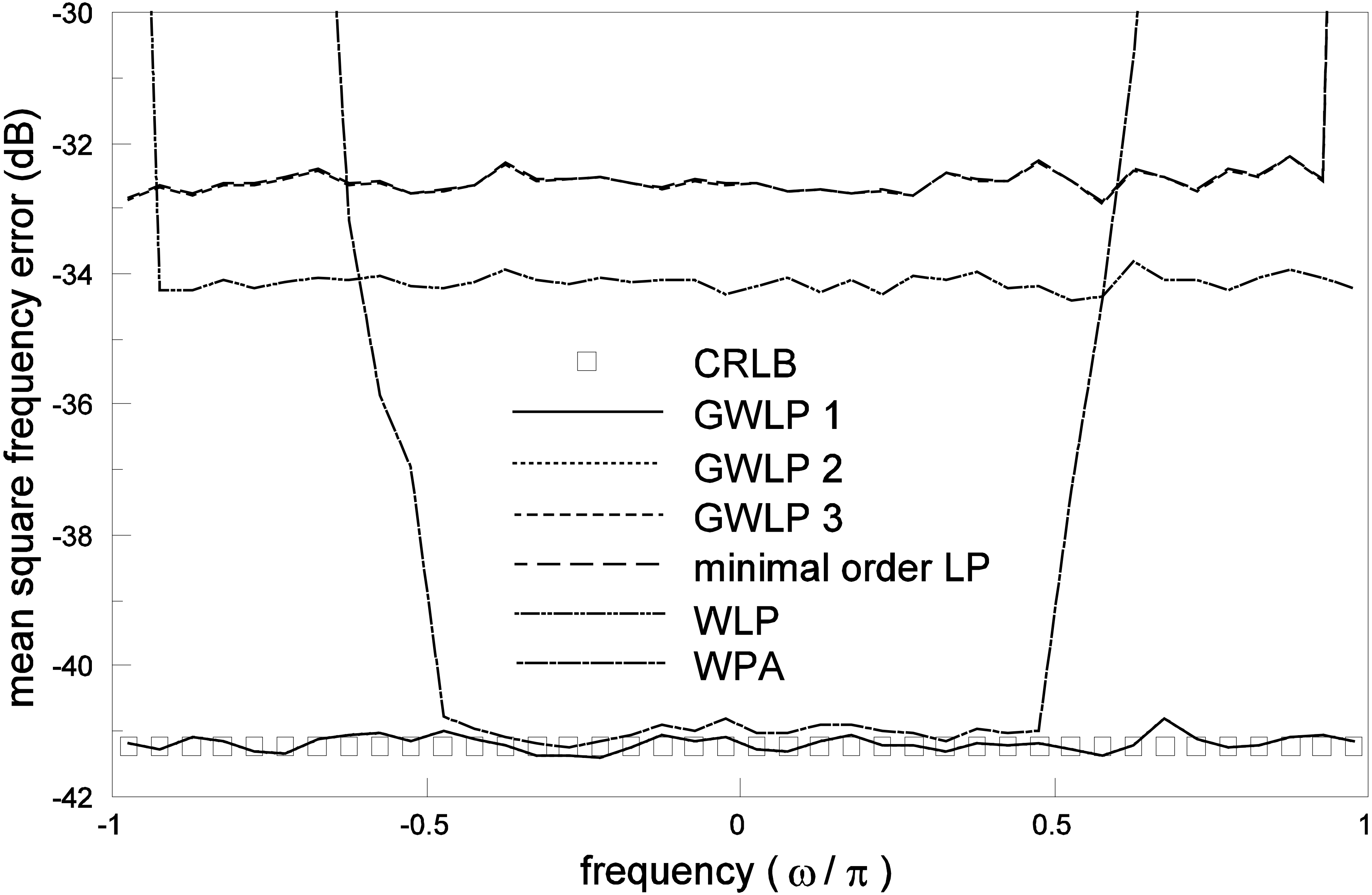
(iii) Determine an updated frequency estimate using

$$\hat{\omega} = \angle\left(\mathbf{X_2}^H \mathbf{W} \mathbf{X_1}\right)$$

(iv) Repeat (ii)-(iii) until a stopping criterion is reached



Mean square frequency errors versus SNR at $\omega = 0.1\pi$ & N=10



Mean square frequency errors versus ω at SNR=10dB & N=20

Common 2D Signal Models

- Complex 2D single-tone model:

$$r_{m,n} = \gamma \exp\{j(\mu m + \nu n)\} + q_{m,n}$$

where γ is the unknown complex amplitude, $\mu \in (-\pi, \pi)$ and $\nu \in (-\pi, \pi)$ are the unknown 2D frequencies while $q_{m,n}$ is a zero-mean white Gaussian noise with unknown variance σ^2, $m = 1, 2, \cdots, M, \quad n = 1, 2, \cdots, N$

- Complex 2D damped single-tone model:

$$r_{m,n} = \gamma \alpha^m \beta^n \exp\{j(\mu m + \nu n)\} + q_{m,n}$$

where the additional unknown parameters are α and β, which are the associated damping factors for μ and ν

- Complex 2D damped multiple-tone model:

$$r_{m,n} = \sum_{k=1}^{K} \gamma_k \alpha_k^m \beta_k^n \exp\{j(\mu_k m + \nu_k n)\} + q_{m,n}$$

where $\{\gamma_k\}$ are the unknown complex amplitude, $\{\mu_k\}$ and $\{\nu_k\}$ are the unknown 2D frequencies and the number of tones, K, is assumed known

- Real 2D single-tone model:

$$r_{m,n} = \gamma \cos(\mu m + \phi) \cos(\nu n + \theta) + q_{m,n}$$

where $\gamma > 0$, $\mu \in (0, \pi)$, $\nu \in (0, \pi)$ and the additional unknown parameters are $\phi \in [0, 2\pi)$ and $\theta \in [0, 2\pi)$ which are the associated phases for μ and ν

- Real 2D damped single-tone model:

$$r_{m,n} = \gamma \alpha^m \beta^n \cos(\mu m + \phi) \cos(\nu n + \theta) + q_{m,n}$$

- Real 2D damped multiple-tone model:

$$r_{m,n} = \sum_{k=1}^{K} \gamma_k \alpha_k^m \beta_k^n \cos(\mu_k m + \phi_k) \cos(\nu_k n + \theta_k) + q_{m,n}$$

Key Ideas in Algorithm Development

- Utilizing principal singular vectors of $\{r_{m,n}\}$

  Frequency estimation is performed using principal singular vectors of $\{r_{m,n}\}$ whose sizes are $M \times 1$ and $N \times 1$, instead of raw data with size of $M \times N$

  Expressing the 2D data as $\mathbf{R}$ where $[\mathbf{R}]_{m,n} = r_{m,n}$ and let its singular vector decomposition (SVD) be

$$\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H$$

  where $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, $\mathbf{U} = [\mathbf{u}_1 \cdots \ \mathbf{u}_N]$ and $\mathbf{V} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_N]$.

In case of a complex single-tone, we use $\mathbf{u}_1$ and $\mathbf{v}_1$ to find frequencies

- Applying generalized weighted linear predictor (GWLP)

Recall the GWLP approach [8] which utilizes WLS and sinusoidal LP property for 1D frequency estimation

Proposed Algorithms

- Complex 2D single-tone model [9]:

  Recall signal model is:

$$r_{m,n} = \gamma \exp\{j(\mu m + \nu n)\} + q_{m,n}$$

  Let its matrix representation be:

$$\mathbf{R} = \mathbf{S} + \mathbf{Q}$$

  where $\mathbf{S}$ and $\mathbf{Q}$ are the signal and noise components, respectively

  First, $\mathbf{S}$ can be factorized as:

$$\mathbf{S} = \gamma \begin{bmatrix} e^{j(\mu+\nu)} & e^{j(\mu+2\nu)} & \cdots & e^{j(\mu+N\nu)} \\ e^{j(2\mu+\nu)} & e^{j(2\mu+2\nu)} & \cdots & e^{j(2\mu+N\nu)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j(M\mu+\nu)} & e^{j(M\mu+2\nu)} & \cdots & e^{j(M\mu+N\nu)} \end{bmatrix} = \gamma \mathbf{g}\mathbf{h}^T$$

where

$$\mathbf{g} = [e^{j\mu} \quad e^{j2\mu} \quad \cdots \quad e^{jM\mu}]^T$$

$$\mathbf{h} = [e^{j\nu} \quad e^{j2\nu} \quad \cdots \quad e^{jN\nu}]^T$$

Thus $\mathbf{g}$ and $\mathbf{h}$ satisfy the LP property:

$$[\mathbf{g}]_m = e^{j\mu}[\mathbf{g}]_{m-1}, \quad m = 2, 3, \cdots, M$$

$$[\mathbf{h}]_n = e^{j\nu}[\mathbf{h}]_{n-1}, \quad n = 2, 3, \cdots, N$$

where $[\ ]_m$ represents the mth element in the vector

However, it is not straightforward to estimate $\mathbf{g}$ and $\mathbf{h}$

On the other hand, noting the rank-1 property of $\mathbf{S}$ and assuming that $M \geq N$, its SVD is

$$\mathbf{S} = \bar{\mathbf{U}}\bar{\mathbf{\Lambda}}\bar{\mathbf{V}}^H = \bar{\lambda}_1 \bar{\mathbf{u}}_1 \bar{\mathbf{v}}_1^H$$

where

$$\bar{\mathbf{\Lambda}} = \mathrm{diag}(\bar{\lambda}_1, \ldots, \bar{\lambda}_N), \qquad \bar{\lambda}_1 \geq 0, \, \bar{\lambda}_2 = \cdots = \bar{\lambda}_N = 0$$

$$\bar{\mathbf{U}} = [\bar{\mathbf{u}}_1 \cdots \ \bar{\mathbf{u}}_N]$$

$$\bar{\mathbf{V}} = [\bar{\mathbf{v}}_1 \ \cdots \ \bar{\mathbf{v}}_N]$$

It can be shown that

$$\bar{\lambda}_1 = \sqrt{MN}|\gamma|$$

$$\bar{\mathbf{u}}_1 = \mathbf{g}e^{-j\varphi_g}/\sqrt{M}$$

$$\bar{\mathbf{v}}_1^* = \mathbf{h}e^{-j\varphi_h}/\sqrt{N}$$

with unknown φ_g and φ_h

That is, we can utilize $\bar{\mathbf{u}}_1$ to find μ and $\bar{\mathbf{v}}_1$ to find ν

In practice, the best rank-1 estimate of $\mathbf{S}$ is obtained from SVD of $\mathbf{R}$:

$$\hat{\mathbf{S}} = \lambda_1 \mathbf{u}_1 \mathbf{v}_1^H$$

Let

$$a = e^{j\mu}$$

$$\mathbf{x}_1 = [[\mathbf{u}_1]_1 \quad [\mathbf{u}_1]_2 \quad \cdots \quad [\mathbf{u}_1]_{M-1}]^T$$

$$\mathbf{x}_2 = [[\mathbf{u}_1]_2 \quad [\mathbf{u}_1]_3 \quad \cdots \quad [\mathbf{u}_1]_M]^T$$

According to LP property, we have:

$$\mathbf{x}_1 a \approx \mathbf{x}_2$$

The WLS solution for a is:

$$\hat{a} = \arg\min_{\tilde{a}} (\mathbf{x}_1 \tilde{a} - \mathbf{x}_2)^H \mathbf{W}_M(a)(\mathbf{x}_1 \tilde{a} - \mathbf{x}_2) = \frac{\mathbf{x}_1^H \mathbf{W}_M(a)\mathbf{x}_2}{\mathbf{x}_1^H \mathbf{W}_M(a)\mathbf{x}_1}$$

The optimum weighting matrix is obtained from:

$$\begin{aligned}\mathbf{W}_M^{-1}(a) &= E\left\{(\mathbf{\Delta x}_1 a - \mathbf{\Delta x}_2)(\mathbf{\Delta x}_1 a - \mathbf{\Delta x}_2)^H\right\} \\ &= \mathbf{A} E\left\{\mathbf{\Delta u}_1 \mathbf{\Delta u}_1^H\right\} \mathbf{A}^H\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & a & 0 & \cdots & 0 \\ 0 & 1 & a & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & 1 & a \end{bmatrix}$$

Making use of the first-order approximation [10]:

$$\mathbf{\Delta u}_1 \approx \bar{\lambda}_1^{-1}\bar{\mathbf{U}}_n\bar{\mathbf{U}}_n^H\mathbf{Q}\bar{\mathbf{v}}_1, \qquad \bar{\mathbf{U}}_n = [\bar{\mathbf{u}}_2 \cdots \ \bar{\mathbf{u}}_N]$$

We get:

$$E\left\{\mathbf{\Delta u}_1\mathbf{\Delta u}_1^H\right\} \approx \bar{\lambda}_1^{-2}\sigma^2\bar{\mathbf{U}}_n\bar{\mathbf{U}}_n^H$$

Utilizing $\mathbf{A}\bar{\mathbf{u}}_1 = \mathbf{0}$ and $\bar{\mathbf{U}}_n\bar{\mathbf{U}}_n^H = \mathbf{I} - \bar{\mathbf{u}}_1\bar{\mathbf{u}}_1^H$, $\mathbf{W}_M(a)$ is simplified as:

$$\mathbf{W}_M(a) = (\mathbf{A}\bar{\mathbf{U}}_n\bar{\mathbf{U}}_n^H\mathbf{A}^H)^{-1} = (\mathbf{A}\mathbf{A}^H)^{-1}$$

Changing the variable from a to μ yields closed-form computation for $\mathbf{W}_M(\mu)$:

$$[\mathbf{W}_M(\mu)]_{m,n} = \frac{M\min(m,n) - mn}{M}e^{j(n-m)\mu}$$

We finally have:

$$\hat{\mu} = \angle\left(\mathbf{x}_1^H \mathbf{W}_M(\mu)\mathbf{x}_2\right)$$

We follow the GWLP procedure to find μ:

(i) Obtain an initial frequency estimate using $\mathbf{W}_M(\mu)$ with $[\mathbf{W}_M(\mu)]_{m,n} = 0$ for $m \neq n$

(ii) Use $\hat{\mu} = \mu$ to construct $\mathbf{W}_M(\mu)$

(iii) Compute an updated $\hat{\mu}$

(iv) Repeat Steps (ii)-(iii) until a stopping criterion is reached

In a similar manner, ν is estimated from $\mathbf{v}_1$ and its conceptual solution is:

$$\hat{\nu} = -\angle\left(\mathbf{y}_1^H \mathbf{W}_N(\nu)\mathbf{y}_2\right)$$

where

$$\mathbf{y}_1 = [[\mathbf{v}_1]_1 \quad [\mathbf{v}_1]_2 \quad \cdots \quad [\mathbf{v}_1]_{N-1}]^T$$

$$\mathbf{y}_2 = [[\mathbf{v}_1]_2 \quad [\mathbf{v}_1]_3 \quad \cdots \quad [\mathbf{v}_1]_N]^T$$

It is noteworthy that μ and ν are independently estimated from the left and right principal singular vectors

At sufficiently high SNRs, it is proved that:

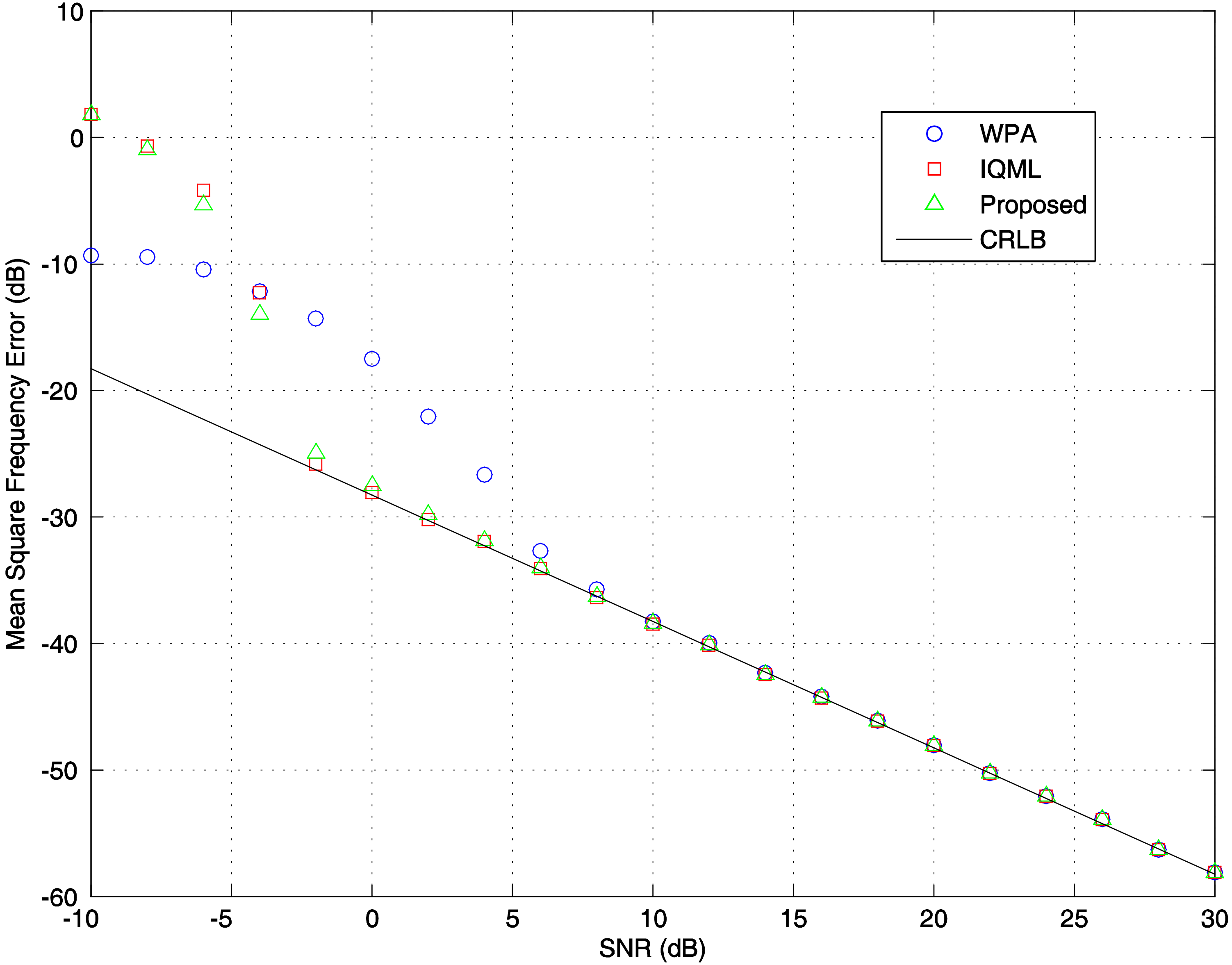
$$E\{\hat{\mu}\} \approx \mu$$

$$\mathrm{var}(\hat{\mu}) \approx \frac{\sigma^2}{2\bar{\lambda}_1^2 \bar{\mathbf{x}}_1^H \mathbf{W}_M(\mu)\bar{\mathbf{x}}_1} = \frac{6}{\mathrm{SNR} NM \left(M^2 - 1\right)}$$

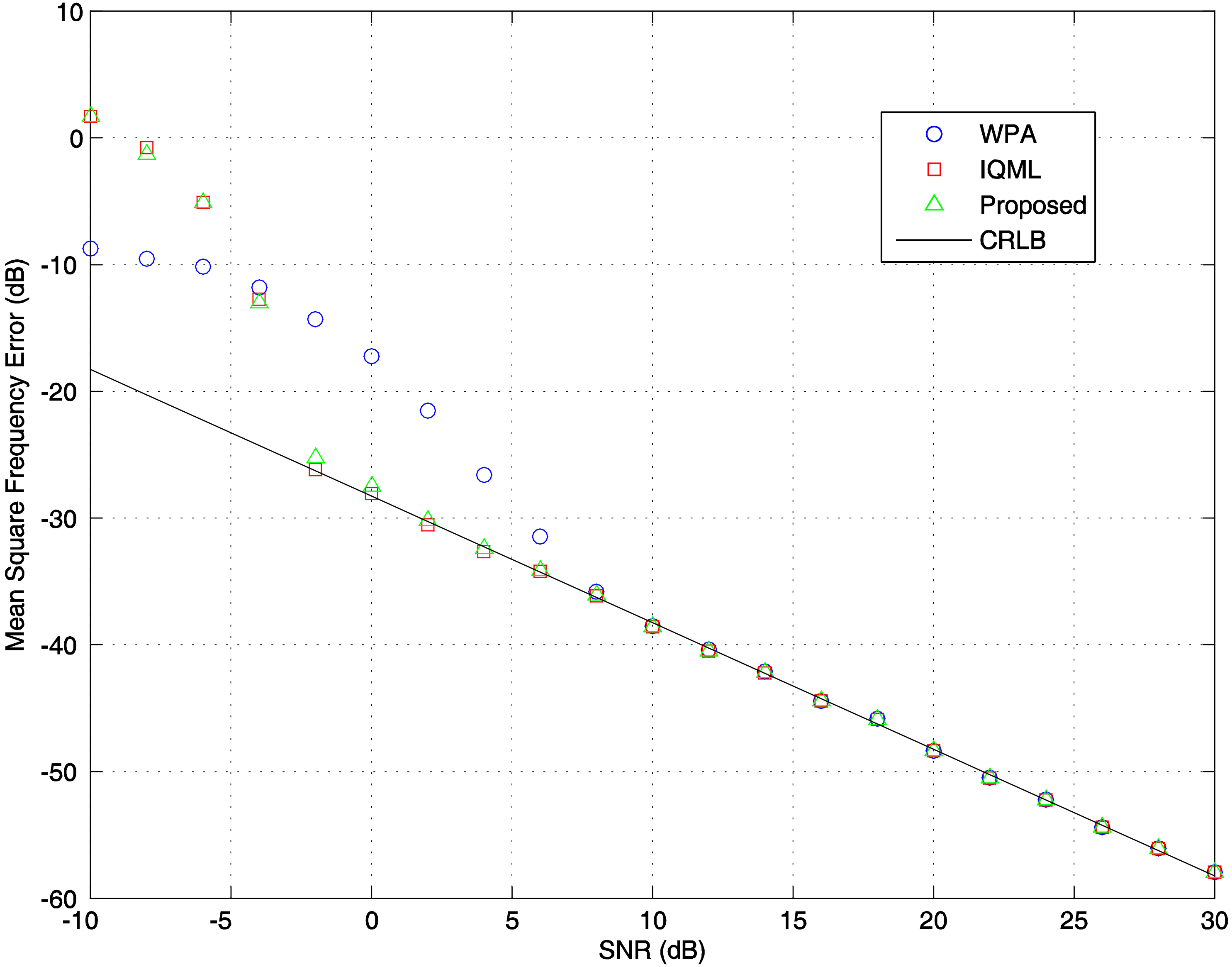
with $\mathrm{SNR} = |\gamma|^2/\sigma^2$, which is CRLB. Similarly, we have:

$$E\{\hat{\nu}\} \approx \nu$$

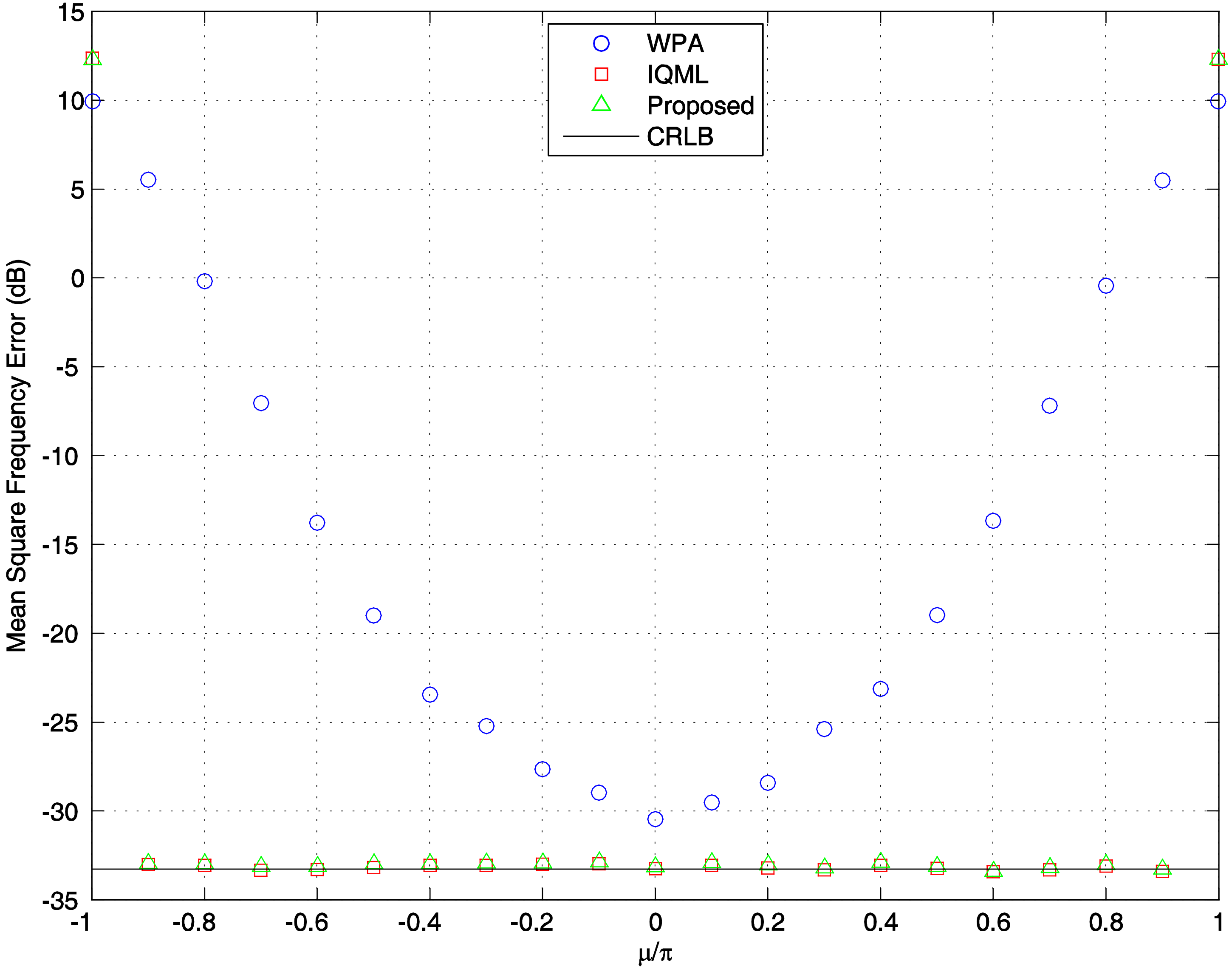
$$\mathrm{var}(\hat{\nu}) \approx \frac{6}{\mathrm{SNR} MN \left(N^2 - 1\right)}$$



Mean square error of μ versus SNR



Mean square error of ν versus SNR



Mean square error of μ versus μ at SNR = 5 dB

- Complex 2D damped single-tone model [9]:

$$r_{m,n} = \gamma \alpha^m \beta^n \exp\{j(\mu m + \nu n)\} + q_{m,n}$$

In a similar manner, $\mathbf{S}$ can be factorized as:

$$\mathbf{S} = \gamma \begin{bmatrix} \alpha e^{j(\mu+\nu)} & e^{j(\mu+2\nu)} & \cdots & \beta^N e^{j(\mu+N\nu)} \\ \alpha^2 e^{j(2\mu+\nu)} & \alpha^2\beta^2 e^{j(2\mu+2\nu)} & \cdots & \alpha^2\beta^N e^{j(2\mu+N\nu)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^M e^{j(M\mu+\nu)} & \alpha^M \beta^2 e^{j(M\mu+2\nu)} & \cdots & \alpha^M \beta^N e^{j(M\mu+N\nu)} \end{bmatrix} = \gamma \mathbf{g}\mathbf{h}^T$$

where

$$\mathbf{g} = [\alpha e^{j\mu} \quad (\alpha e^{j\mu})^2 \quad \cdots \quad (\alpha e^{j\mu})^M]^T$$

$$\mathbf{h} = [\beta e^{j\nu} \quad (\beta e^{j\nu})^2 \quad \cdots \quad (\beta e^{j\nu})^N]^T$$

Now $\mathbf{g}$ and $\mathbf{h}$ satisfy the LP property:

$$[\mathbf{g}]_m = a[\mathbf{g}]_{m-1}, \quad a = \alpha \exp\{j\mu\}, \quad m = 2, 3, \cdots, M$$

$$[\mathbf{h}]_n = b[\mathbf{h}]_{n-1}, \quad b = \beta \exp\{j\nu\}, \quad n = 2, 3, \cdots, N$$

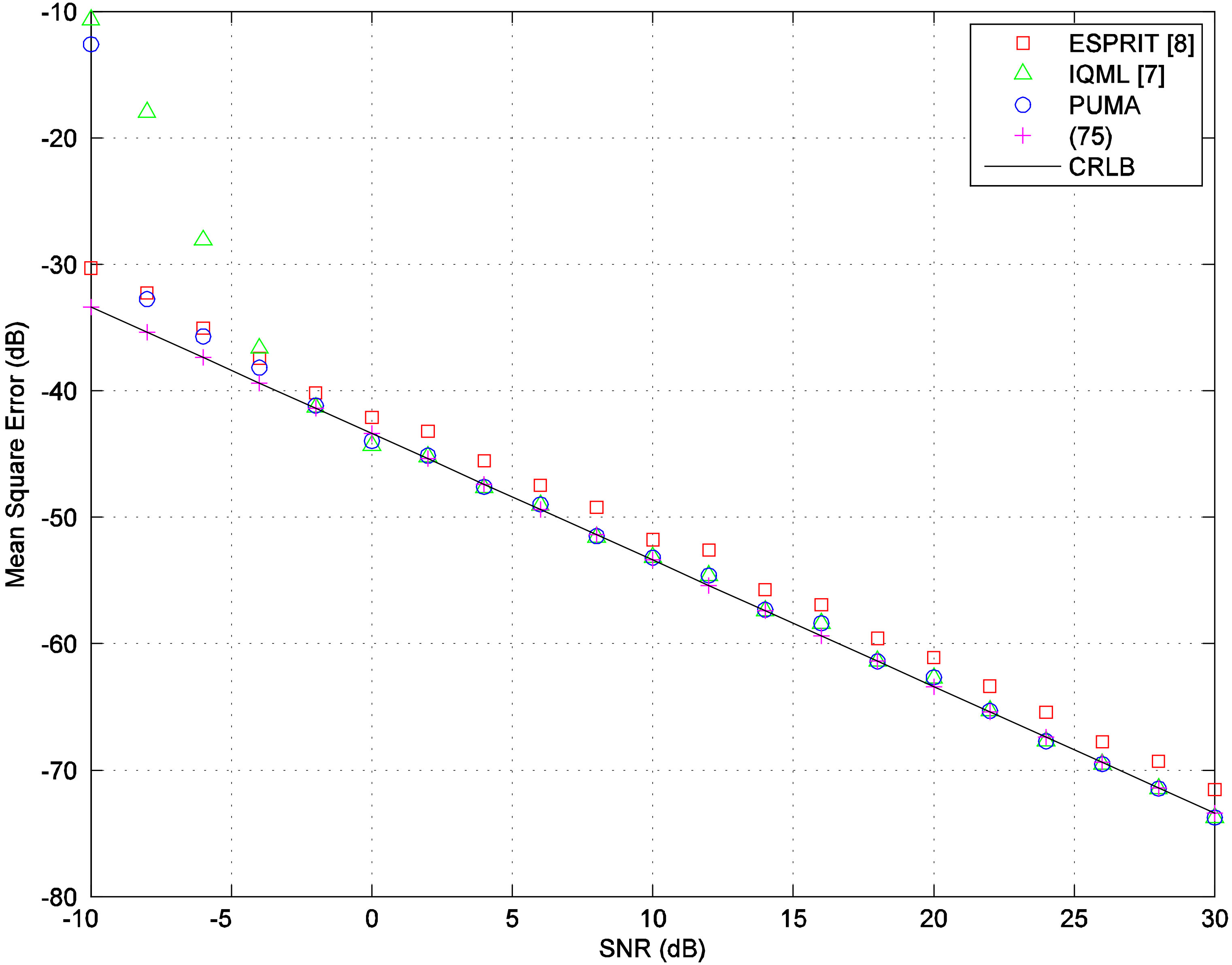
The conceptual WLS solution for a is:

$$\hat{a} = \arg\min_{\tilde{a}} (\mathbf{x}_1\tilde{a} - \mathbf{x}_2)^H \mathbf{W}_M(a)(\mathbf{x}_1\tilde{a} - \mathbf{x}_2) = \frac{\mathbf{x}_1^H \mathbf{W}_M(a)\mathbf{x}_2}{\mathbf{x}_1^H \mathbf{W}_M(a)\mathbf{x}_1}$$

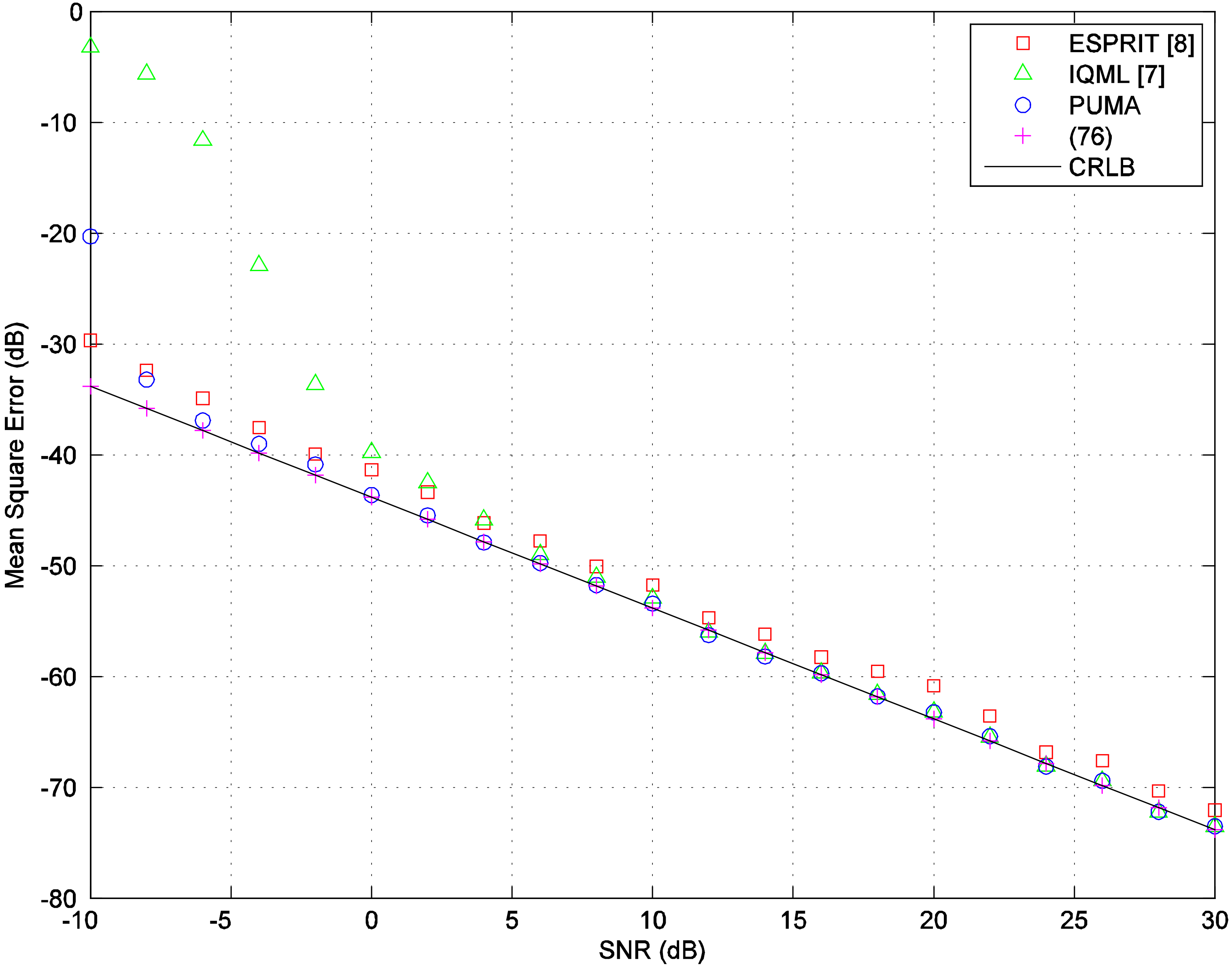
We follow the GWLP procedure and finally:

$$\hat{\alpha} = |\hat{a}|$$

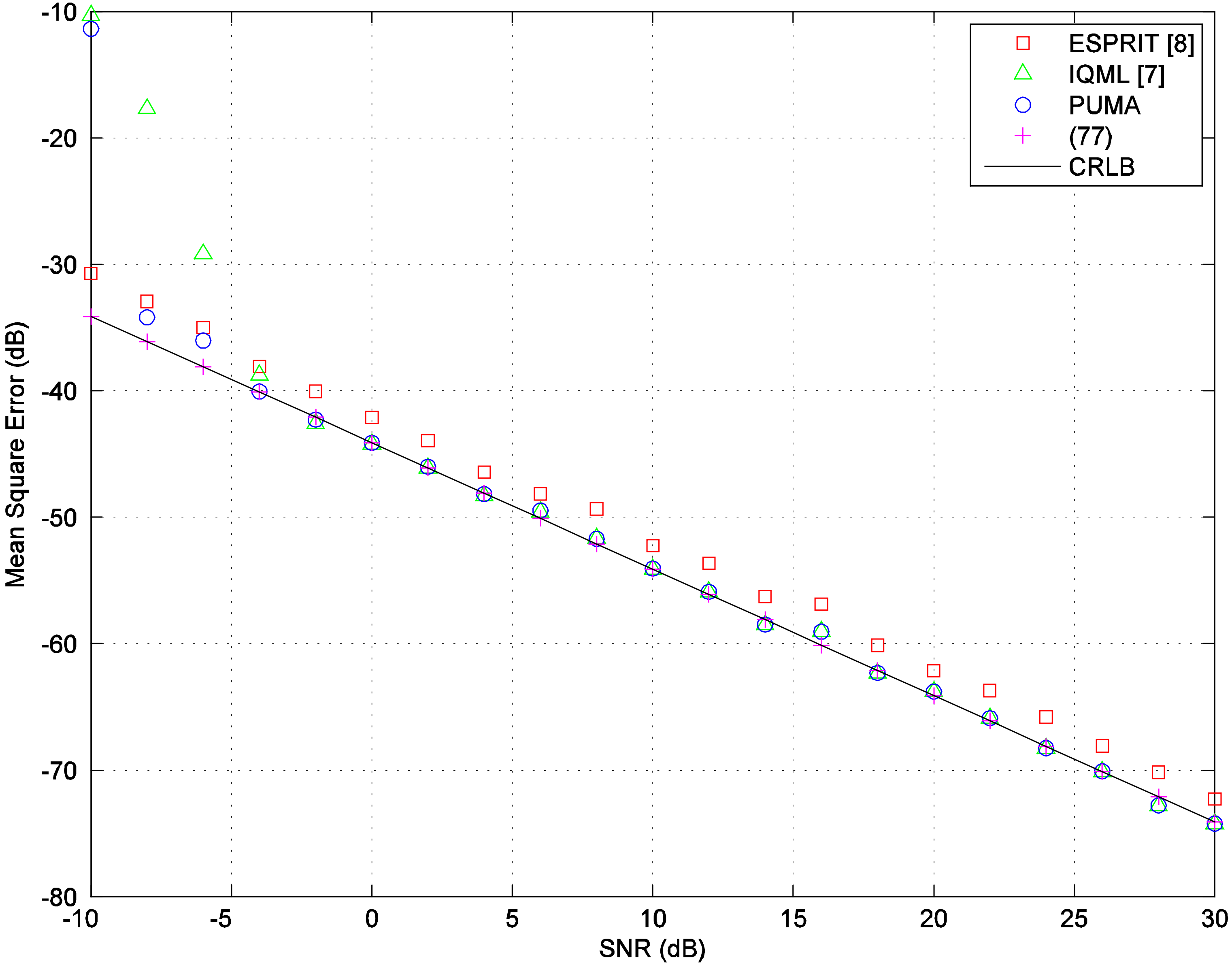
$$\hat{\mu} = \angle(\hat{a})$$



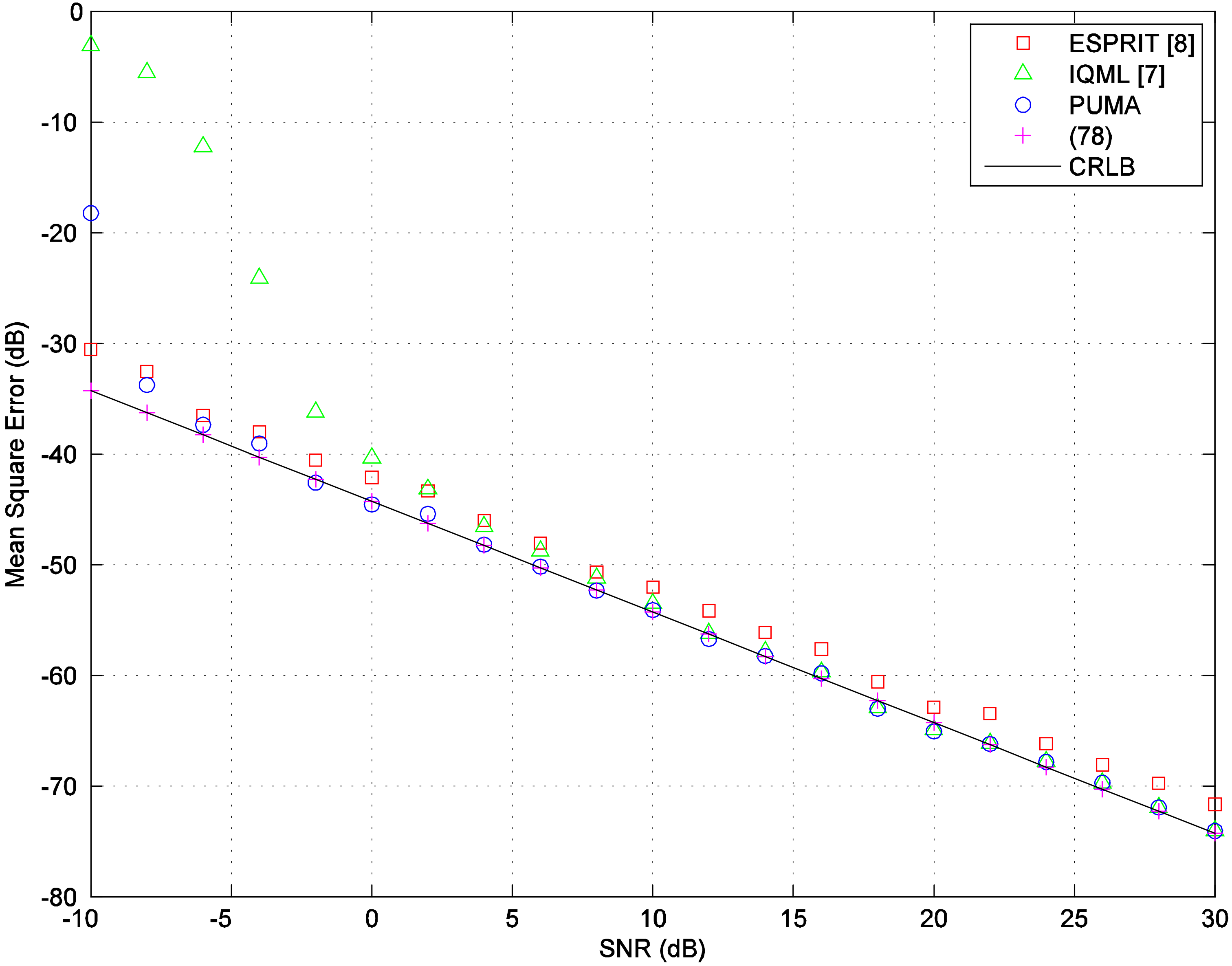
Mean square error of μ versus SNR



Mean square error of α versus SNR



Mean square error of ν versus SNR



Mean square error of β versus SNR

- Real 2D damped single-tone model [9]:

$$r_{m,n} = \gamma \alpha^m \beta^n \cos(\mu m + \phi) \cos(\nu n + \theta) + q_{m,n}$$

Now $\mathbf{S}$ can be factorized as:

$$\mathbf{S} = \gamma \mathbf{g} \mathbf{h}^T$$

where

$$\mathbf{g} = \left[\alpha \cos(\mu + \phi) \;\; \alpha^2 \cos(2\mu + \phi) \;\; \cdots \;\; \alpha^M \cos(M\mu + \phi) \right]^T$$

$$\mathbf{h} = \left[\beta \cos(\nu + \theta) \;\; \beta^2 \cos(2\nu + \theta) \;\; \cdots \;\; \beta^N \cos(N\nu + \theta) \right]^T$$

The LP property in $\mathbf{g}$ and $\mathbf{h}$ can be observed as:

$$[\mathbf{g}]_m = a_1[\mathbf{g}]_{m-1} + a_2[\mathbf{g}]_{m-2}, \quad m = 3, 4, \cdots, M$$

$$[\mathbf{h}]_n = b_1[\mathbf{h}]_{n-1} + b_2[\mathbf{h}]_{n-2}, \quad n = 3, 4, \cdots, N$$

where

$$a_1 = 2\alpha\cos(\mu)$$

$$a_2 = -\alpha^2$$

$$b_1 = 2\beta\cos(\nu)$$

$$b_2 = -\beta^2$$

Utilizing the LP property, we have:

$$\mathbf{X}\mathbf{a} \approx \mathbf{x}$$

where

$$\mathbf{X} = \text{Toeplitz}\left(\left[[\mathbf{u}_1]_2 \ [\mathbf{u}_1]_3 \ \cdots \ [\mathbf{u}_1]_{M-1}\right]^T, \left[[\mathbf{u}_1]_2 \ [\mathbf{u}_1]_1\right]\right)$$

$$\mathbf{x} = \left[[\mathbf{u}_1]_3 \ [\mathbf{u}_1]_4 \ \cdots \ [\mathbf{u}_1]_M\right]^T$$

$$\mathbf{a} = \left[a_1 \ a_2\right]^T$$

Following the GWLP development, $\mathbf{a}$ is estimated as:

$$\hat{\mathbf{a}} = (\mathbf{X}^T \mathbf{W}_M(\mathbf{a})\mathbf{X})^{-1}\mathbf{X}^T \mathbf{W}_M(\mathbf{a})\mathbf{x}$$

where

$$\mathbf{W}_M(\mathbf{a}) = (\mathbf{A}\mathbf{A}^H)^{-1}$$

$$\mathbf{A} = \text{Toeplitz}\left(\left[-a_2 \ \mathbf{0}_{1\times(M-3)}\right]^T, \left[-a_2 \ -a_1 \ 1 \ \mathbf{0}_{1\times(M-3)}\right]\right)$$

After algorithm convergence, the damping factor and frequency are estimated as

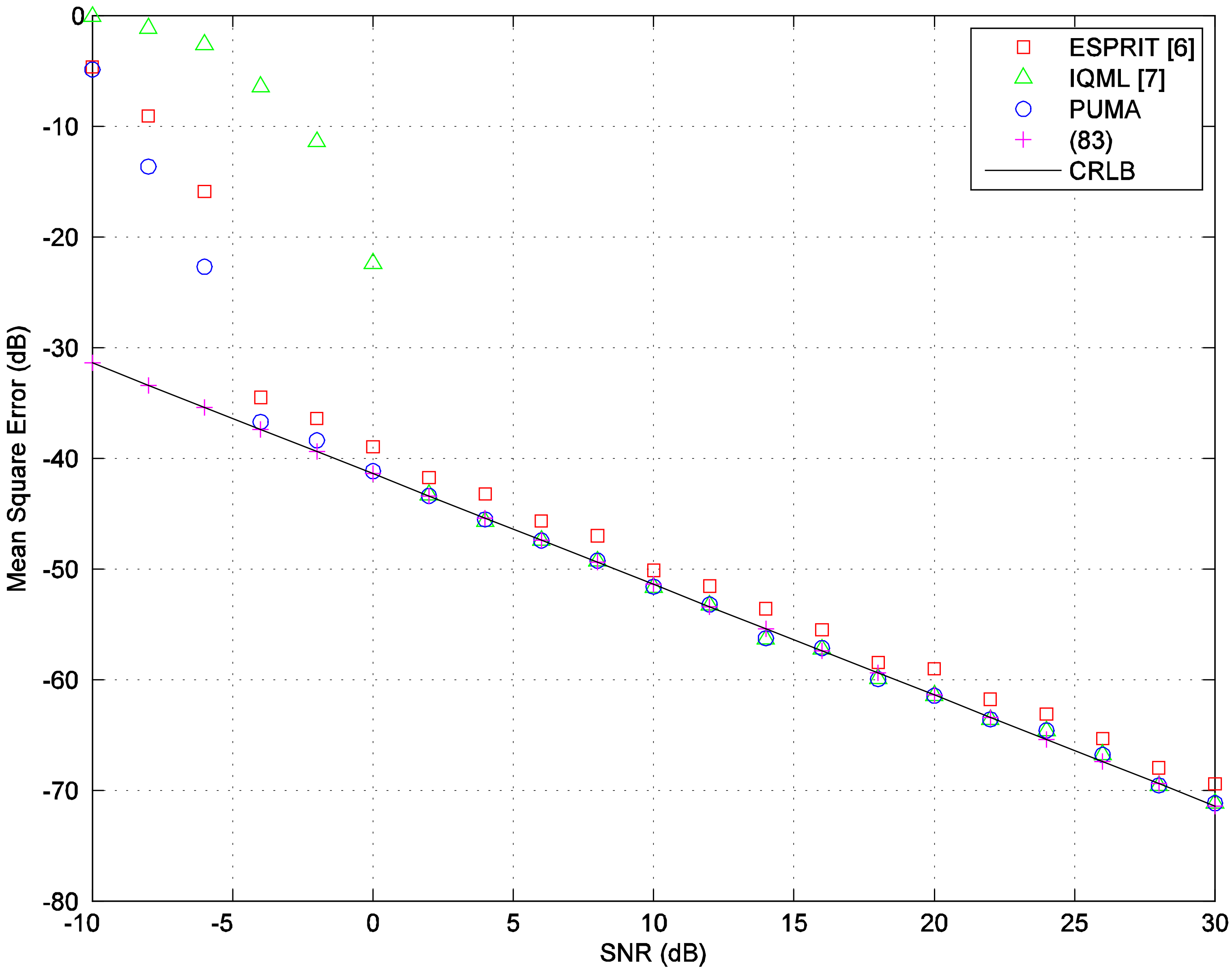
$$\hat{\alpha} = \sqrt{-\hat{a}_2}$$

$$\hat{\mu} = \cos^{-1}\left(\frac{\hat{a}_1}{2\hat{\alpha}}\right)$$

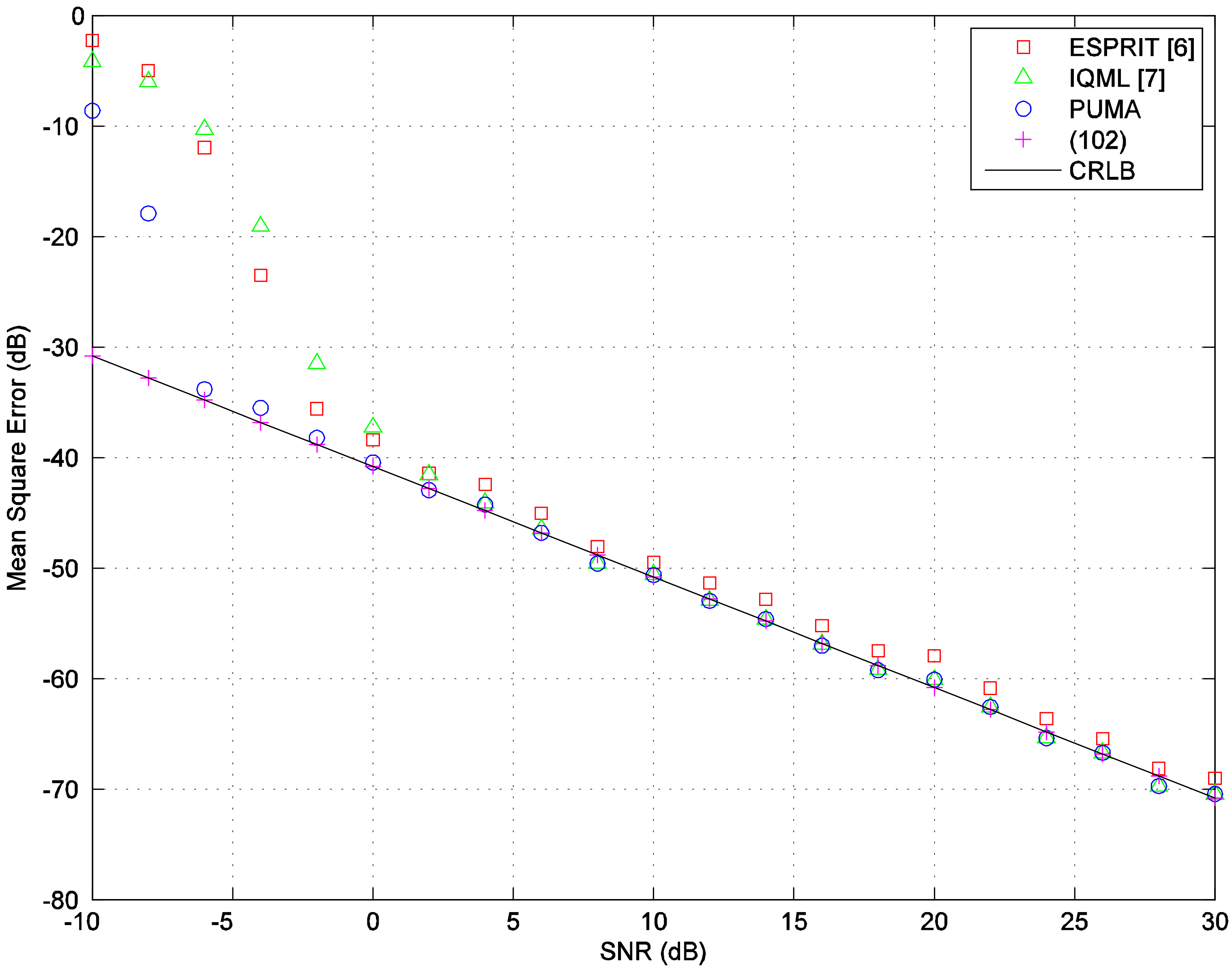
Similarly, β and ν are estimated independently from the right principal singular vector:

$$\hat{\beta} = \sqrt{-\hat{b}_2}$$

$$\hat{\nu} = \cos^{-1}\left(\frac{\hat{b}_1}{2\hat{\beta}}\right)$$



Mean square error of μ versus SNR



Mean square error of ν versus SNR

- Complex 2D damped multiple-tone model [11]:

$$r_{m,n} = \sum_{k=1}^{K} \gamma_k \alpha_k^m \beta_k^n \exp\{j(\mu_k m + \nu_k n)\} + q_{m,n}$$

Now $\mathbf{S}$ can be factorized as:

$$\mathbf{S} = \mathbf{G}\mathbf{\Gamma}\mathbf{H}^T$$

where

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_1 & \mathbf{g}_2 & \cdots & \mathbf{g}_K \end{bmatrix}$$

$$\mathbf{\Gamma} = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_K)$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \cdots & \mathbf{h}_K \end{bmatrix}$$

$$\mathbf{g}_k = \begin{bmatrix} a_k & a_k^2 & \cdots a_k^M \end{bmatrix}^T, \quad a_k = \alpha_k \exp\{j\mu_k\}$$

$$\mathbf{h}_k = \begin{bmatrix} b_k & b_k^2 & \cdots b_k^N \end{bmatrix}^T, \quad b_k = \beta_k \exp\{j\nu_k\}$$

On the other hand, the SVD of $\mathbf{S}$ is:

$$\mathbf{S} = \bar{\mathbf{U}}_s \bar{\mathbf{\Lambda}}_s \bar{\mathbf{V}}_s^H$$

where

$$\bar{\mathbf{U}}_s = \left[\bar{\mathbf{u}}_1 \; \bar{\mathbf{u}}_2 \; \cdots \; \bar{\mathbf{u}}_K\right]$$

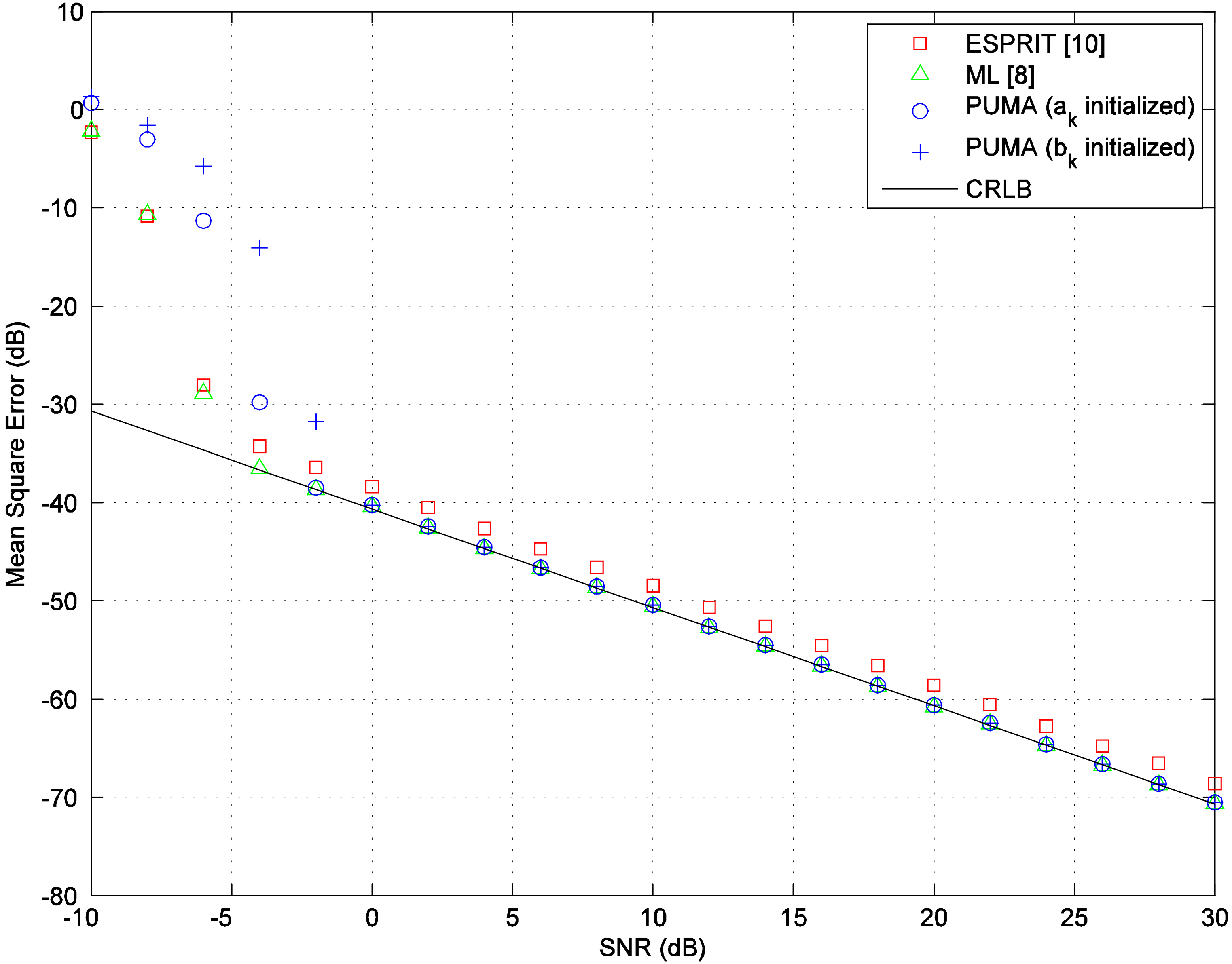
$$\bar{\mathbf{\Lambda}}_s = \mathrm{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_K)$$

$$\bar{\mathbf{V}}_s = \left[\bar{\mathbf{v}}_1 \; \bar{\mathbf{v}}_2 \; \cdots \; \bar{\mathbf{v}}_K\right]$$

Comparing $\mathbf{G}\mathbf{\Gamma}\mathbf{H}^T$ and $\bar{\mathbf{U}}_s \bar{\mathbf{\Lambda}}_s \bar{\mathbf{V}}_s^H$ yields

$$\bar{\mathbf{U}}_s = \mathbf{G}\mathbf{\Omega}_G$$

where $\mathbf{\Omega}_G$ is unknown. That is, each column of $\bar{\mathbf{U}}_s$ is a sum of K multiple tones with damping factors $\{\alpha_k\}$ and frequencies $\{\mu_k\}$



Average mean square frequency error versus SNR

- Real 2D damped multiple-tone model:

$$r_{m,n} = \sum_{k=1}^{K} \gamma_k \alpha_k^m \beta_k^n \cos(\mu_k m + \phi_k) \cos(\nu_k n + \theta_k) + q_{m,n}$$

Now $\mathbf{S}$ can be factorized as:

$$\mathbf{S} = \mathbf{G}\mathbf{\Gamma}\mathbf{H}^T$$

where

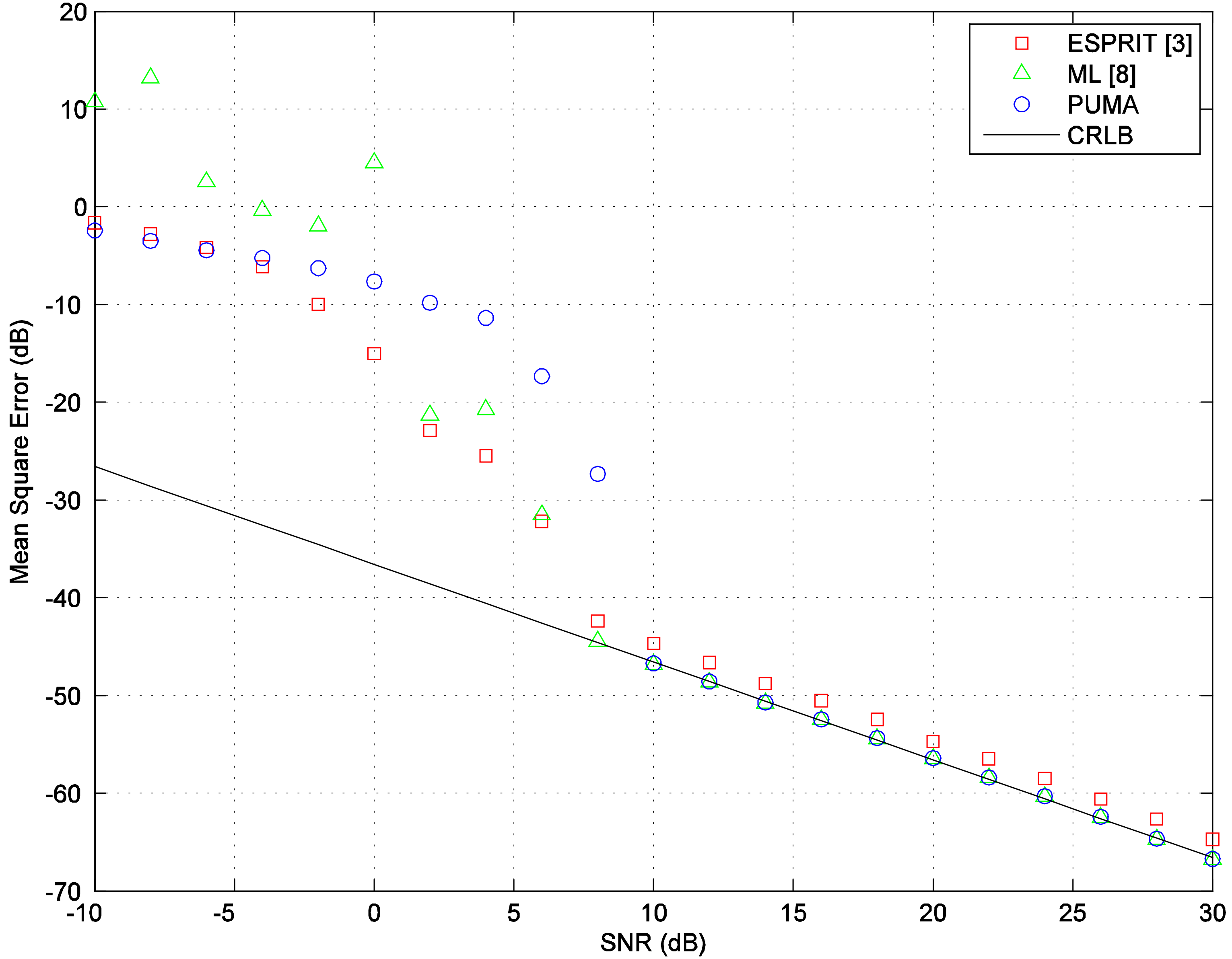
$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_1 & \mathbf{g}_2 & \cdots & \mathbf{g}_K \end{bmatrix}$$

$$\mathbf{\Gamma} = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_K)$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \cdots & \mathbf{h}_K \end{bmatrix}$$

$$\mathbf{g}_k = \begin{bmatrix} \alpha_k \cos(\mu_k + \phi_k) & \alpha_k^2 \cos(2\mu_k + \phi_k) & \cdots & \alpha_k^M \cos(M\mu_k + \phi_k) \end{bmatrix}^T$$

$$\mathbf{h}_k = \begin{bmatrix} \beta_k \cos(\nu_k + \theta_k) & \beta_k^2 \cos(2\nu_k + \theta_k) & \cdots & \beta_k^N \cos(N\nu_k + \theta_k) \end{bmatrix}^T$$



Average mean square frequency error versus SNR

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