Information Processing & Algorithms Laboratory
Everything can be optimized

ADAPTIVE MATCHING PURSUIT FOR SPARSE SIGNAL RECOVERY

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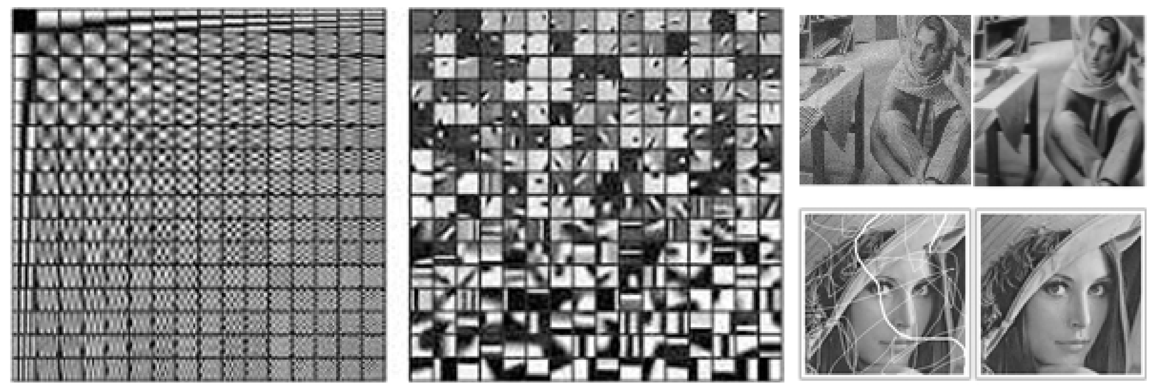
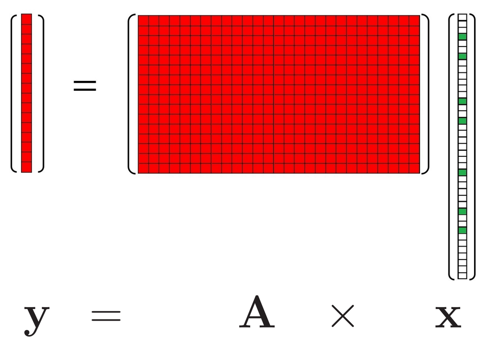
1. Sparsity in Signal/Image Processing

- Parsimony in signal representation has value in many applications.
- A large class of signals can be represented in a compact manner with respect to some basis (dictionary).
- Recent advances using spare representation for a variety of applications:



16x16 DCT bases — Adaptive bases — Denoising/Inpainting

- A measurement (dictionary) matrix $\mathbf{A} \in \mathbb{R}^{q \times p}$
- Compressive Sensing framework ($\mathbf{x} \in \mathbb{R}^p$):
  $\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_0$ subject to $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \le \epsilon$
- The ℓ_1 relaxation of regularized problem:
  $\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 + \lambda\|\mathbf{x}\|_1$



2. Motivation and Idea

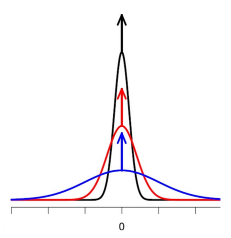
- Model-based Compressive Sensing:

$$\max_{\mathbf{x}} f(\mathbf{x}) \text{ subject to } \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 < \epsilon$$

Laplacian MAP estimation

$$\arg\max_{\mathbf{x}} \exp\left\{-\lambda \sum_i |x_i|\right\} \Leftrightarrow \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1$$

- Introduce a prior $f_{\mathbf{x}}$: which can capture the sparse structure
- Spike-and-Slab Priors: The gold standard to induce sparsity [1]
- Goal: Find the sparse signal $\mathbf{x} \in \mathbb{R}^p$ from a set of fewer measurements $\mathbf{y} \in \mathbb{R}^q$. $(q \ll p)$



3. Bayesian Interpretation

Bayesian formulation

$$\mathbf{y}|\mathbf{A}, \mathbf{x}, \gamma, \sigma^2 \sim \mathcal{N}(\mathbf{A}\mathbf{x}, \sigma^2\mathbf{I}),$$
$$\mathbf{x}|\gamma, \lambda, \sigma^2 \sim \prod_{i=1}^{p} \gamma_i \mathcal{N}(0, \sigma^2\lambda^{-1}) + (1-\gamma_i)\mathbb{I}(x_i = 0),$$
$$\gamma \sim \prod_{i=1}^{p} \text{Bernoulli}(\kappa_i),$$

- The optimal $\mathbf{x}^*, \gamma^*$ are obtained by MAP estimation on joint posterior density:

$$(\mathbf{x}^*, \gamma^*) = \arg\max_{\mathbf{x},\gamma} \left\{p(\mathbf{x}, \gamma | \mathbf{A}, \mathbf{y}, \boldsymbol{\kappa}, \lambda, \sigma^2)\right\}. \quad (1)$$

Optimization problem [2]

$$(1) \Leftrightarrow (\mathbf{x}^*, \gamma^*) = \arg\min_{\mathbf{x},\gamma} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_2^2 + \sum_{i=1}^{p} \rho_i\gamma_i \quad (2)$$

where $\rho_i \triangleq \sigma^2 \log\left(\frac{2\pi\sigma^2(1-\kappa_i)^2}{\lambda\kappa_i^2}\right)$.

- This is a hard nonconvex mixed-integer programming to find both the support and the values of the solution.

5. Existing Solutions/Special Cases

1. Markov Chain Monte Carlo (MCMC algorithm) [3]
2. Majorization minimization (MM algorithm) [4] by assuming only one κ

$$\arg\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_2^2 + \rho\|\mathbf{x}\|_0$$

3. Adaptive Elastic Net [5] by relaxing ℓ_0 norm to ℓ_1 norm

$$\arg\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_2^2 + \rho\|\mathbf{x}\|_1$$

4. Sparse Reconstruction by Separable Approximation(SpaRSA) [6]

6. AMP: Adaptive Matching Pursuit - First observations

- First, let $\mathbf{D} = \begin{bmatrix}\mathbf{A} \\ \sqrt{\lambda}\mathbf{I}\end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix}\mathbf{y} \\ \mathbf{0}\end{bmatrix}$ with $\mathbf{I} \in \mathbb{R}^{p \times p}$ and $\mathbf{0} \in \mathbb{R}^{p \times 1}$ being the identity matrix and zero vector, we can rewrite (2) as:

$$(\mathbf{x}^*, \gamma^*) = \arg\min_{\mathbf{x},\gamma} \|\mathbf{z} - \mathbf{D}\mathbf{x}\|_2^2 + \sum_{i=1}^{p} \rho_i\gamma_i. \quad (3)$$

where $\|\mathbf{d}_i\|_2^2 = \|\mathbf{a}_i\|_2^2 + \lambda = 1 + \lambda,\ \forall i = 1, \dots, p$

- **If we know the true support of the signal**, i.e. $\mathcal{S} = \{i : \gamma_i \neq 0\}$, **we can infer the solution** of (3) by:

$$\mathbf{x}_{\mathcal{S}} = \arg\min_{\mathbf{x}_{\mathcal{S}}} \|\mathbf{z} - \mathbf{D}_{\mathcal{S}}\mathbf{x}_{\mathcal{S}}\|_2^2 \Rightarrow \boxed{\mathbf{D}_{\mathcal{S}}^T\mathbf{D}_{\mathcal{S}}\mathbf{x}_{\mathcal{S}} = \mathbf{D}_{\mathcal{S}}^T\mathbf{z} \triangleq \mathbf{w}_{\mathcal{S}}} \text{ and } \mathbf{r}_{\mathcal{S}} = \mathbf{z} - \mathbf{D}_{\mathcal{S}}\mathbf{x}_{\mathcal{S}}$$

7. AMP: The central idea

Define: $g(\mathcal{S}) = \min_{\mathbf{x}^{\mathcal{S}}} \|\mathbf{z} - \mathbf{D}_{\mathcal{S}}\mathbf{x}^{\mathcal{S}}\|_2^2 + \sum_{i \in \mathcal{S}} \rho_i.$

best improvement if we **insert**: $U_{\mathcal{S}} = \min_{i \notin \mathcal{S}} g(\mathcal{S} \cup \{i\}) - g(\mathcal{S}),$

best improvement if we **remove**: $V_{\mathcal{S}} = \min_{j \in \mathcal{S}} g(\mathcal{S} \backslash \{j\}) - g(\mathcal{S}).$

- If $U_{\mathcal{S}}, V_{\mathcal{S}} > 0 \Rightarrow$ stop. If $U_{\mathcal{S}} < V_{\mathcal{S}} \Rightarrow$ insert, else remove.
- **Challenge:** calculating $U_{\mathcal{S}}, V_{\mathcal{S}}$ is expensive.
- **Solution:** find their *tight upper bounds.*

8. AMP: Supporting lemmas and Cholesky decomposition

Lemma 1: Initialization of the support set $\mathcal{S}$

If $\rho_i < 0$ ($\kappa_i >$ a threshold), then $i \in \hat{\mathcal{S}}$ – the optimal active set $\Rightarrow$ initialize $\mathcal{S}_0 = \{i : \rho_i < 0\}$

Lemma 2: Approximation of $U_{\mathcal{S}}$

$$U_{\mathcal{S}} \le \min_{i \notin \mathcal{S}} \left\{\rho_i - \frac{(\mathbf{r}_{\mathcal{S}}^T\mathbf{d}_i)^2}{1+\lambda}\right\} \triangleq \bar{U}_{\mathcal{S}}$$

Lemma 3: Approximation of $V_{\mathcal{S}}$

$$V_{\mathcal{S}} \le \min_{j \in \mathcal{S}} \left\{(1+\lambda)(x_j^{\mathcal{S}})^2 + 2\mathbf{d}_j^T\mathbf{r}_{\mathcal{S}}x_j^{\mathcal{S}} - \rho_j\right\} \triangleq \bar{V}_{\mathcal{S}}.$$

- Challenge: Once $\mathcal{S}$ is found, we need to solve: $\mathbf{D}_{\mathcal{S}}^T\mathbf{D}_{\mathcal{S}}\mathbf{x}^{\mathcal{S}} = \mathbf{w}^{\mathcal{S}}$ which may be expensive.
- Solution: Use the Cholesky decomposition: $\mathbf{D}_{\mathcal{S}}^T\mathbf{D}_{\mathcal{S}} = \mathbf{L}_{\mathcal{S}}^T\mathbf{L}_{\mathcal{S}}$ where $\mathbf{L}_{\mathcal{S}}$ is a low triangular matrix.

Forward Cholesky decomposition

If $\mathbf{v}$ is the solution of $\mathbf{L}_{\mathcal{S}}\mathbf{v} = \mathbf{D}_{\mathcal{S}}^T\mathbf{d}_i$ via forward substitution, then:

$$\mathbf{L}_{\mathcal{S}\cup\{i\}} = \begin{bmatrix}\mathbf{L}_{\mathcal{S}} & \mathbf{0} \\ \mathbf{v}^T & \sqrt{1+\lambda-\mathbf{v}^T\mathbf{v}}\end{bmatrix}.$$

Backward Cholesky decomposition

$$\text{If } \mathbf{L}_{\mathcal{S}} = \begin{bmatrix}\mathbf{L}_{11} & 0 & 0 \\ \mathbf{l}_{21}^T & l_{22} & 0 \\ \mathbf{L}_{31} & \mathbf{l}_{32} & \mathbf{L}_{33}\end{bmatrix} \leftarrow j^{\text{th}} \text{ row}$$

$$\Rightarrow \mathbf{L}_{\mathcal{S}\backslash\{j\}} = \begin{bmatrix}\mathbf{L}_{11} & 0 \\ \mathbf{L}_{31} & \bar{\mathbf{L}}_{33}\end{bmatrix} \text{ where } \bar{\mathbf{L}}_{33}\bar{\mathbf{L}}_{33}^T = \mathbf{L}_{33}\mathbf{L}_{33}^T + \mathbf{l}_{32}\mathbf{l}_{32}^T$$

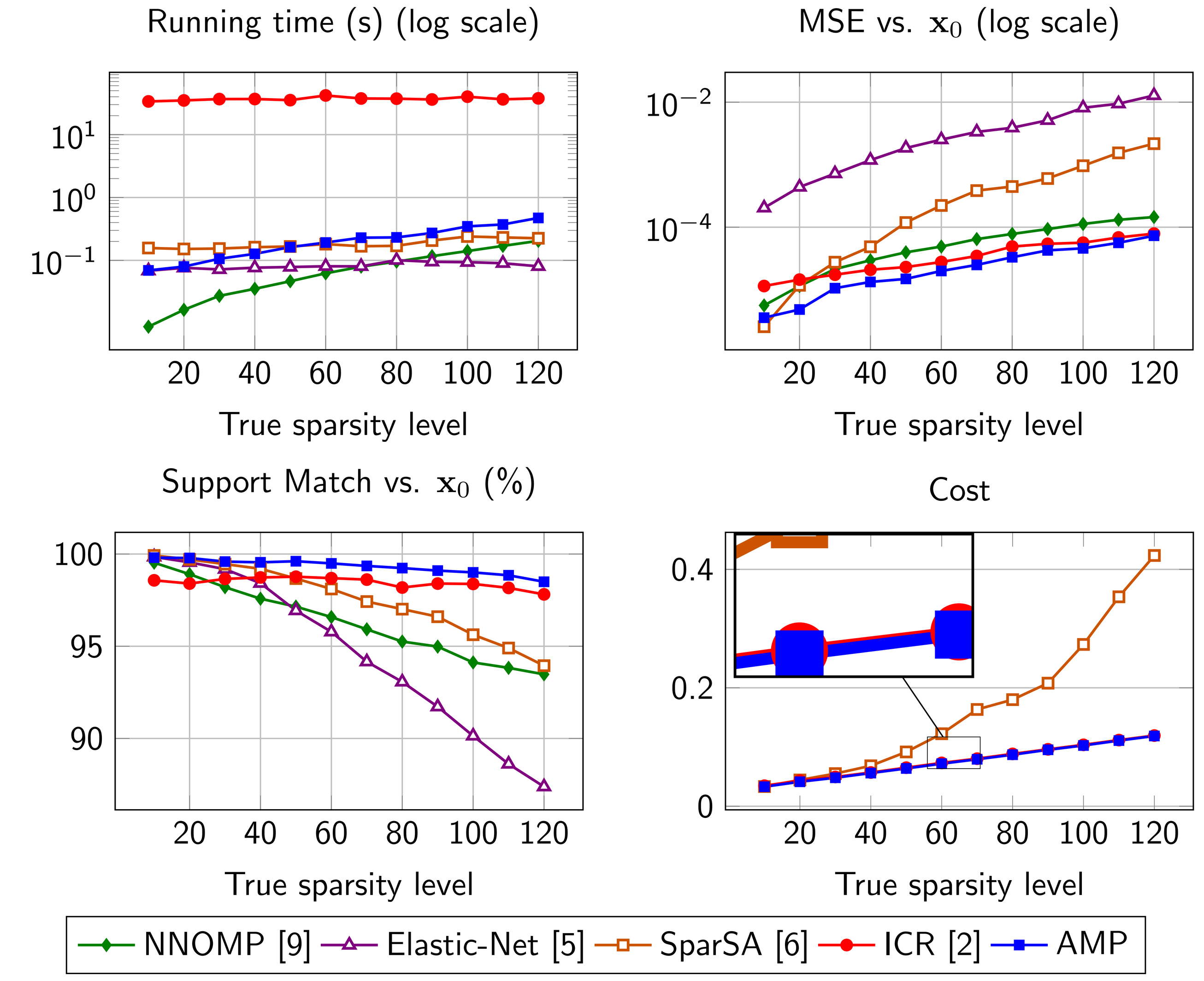
9. Experiments with simulated data

- 1000 realizations of $\mathbf{A}$, sparse $\mathbf{x}_0$ and Gaussian noise ε: $\mathbf{y} = \mathbf{A}\mathbf{x}_0 + \varepsilon$
- Coefficient vector $\mathbf{x}_0 \in \mathbb{R}^{512}$. Test vector $\mathbf{y} \in \mathbb{R}^{256}$. Sparsity level $= 100$.

Table: Comparison of methods for $p = 512$, $q = 256$.

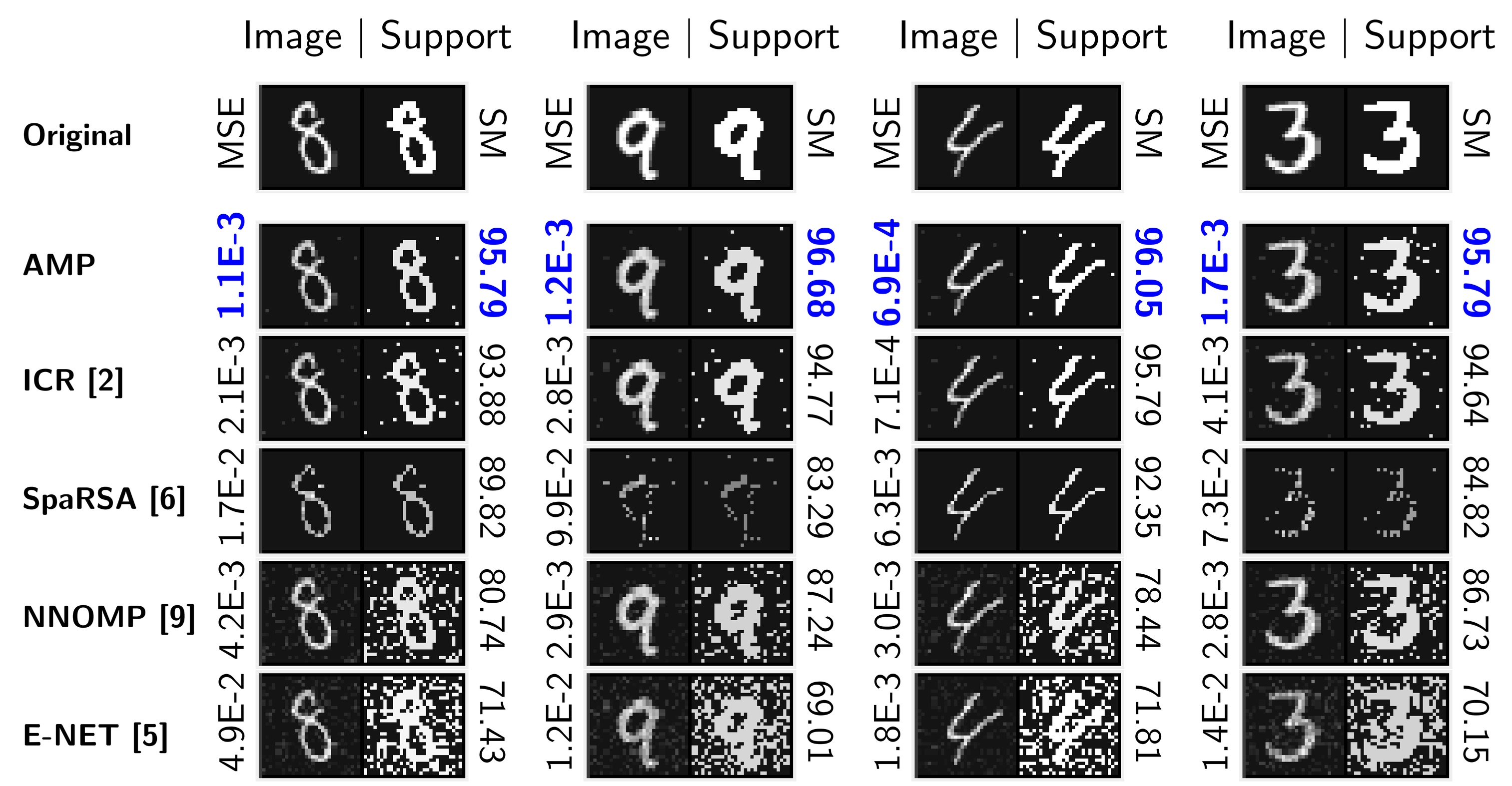
Average	CoSaMP [7]	E-NET [5]	RW-L1 [8]	SpaRSA [6]	ICR [2]	**AMP**
Time (s)	1.1E0	**7.9E-2**	4.7E-1	2.2E-1	2.9E1	***1.6E-1***
MSE	3.0E-4	4.5E-3	2E-4	5.4E-4	3.1E-3	**6.1E-5**
Cost	-	-	-	1.6E-1	2.1E-1	**9.5E-2**
SM (%)	93.52	85.55	92.71	93.87	83.79	**97.32**

- With Nonnegativity constraints:



10. Experiments with real data

- Reconstruction of handwritten digit images from the MNIST dataset.
- MNIST contains 60000 digit images (0 to 9) of size 28×28 pixels.
- Sparse signal $\mathbf{x}_0$ (vectorized image) is to be reconstructed from a smaller set of random measurements $\mathbf{y}$: $\mathbf{y} = \mathbf{A}\mathbf{x}_0 + \varepsilon$
- Coefficient vector $\mathbf{x}_0 \in \mathbb{R}^{784}$. measured vector $\mathbf{y} \in \mathbb{R}^{350}$.



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[9] A. Bruckstein et. al, TIT 2008

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