SDR Approximation Bounds for the Robust Multicast Beamforming Problem with Interference Temperature Constraints

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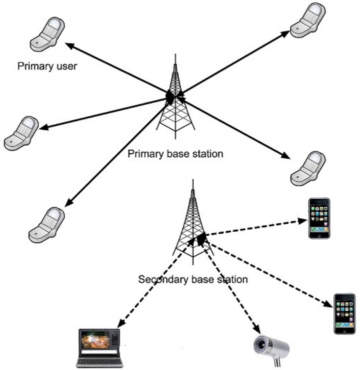
Multicast Transmission in a Cognitive Radio Network

Scenario Settings

- Primary group — with band license
- Secondary group — unlicensed, no exact CSIs of primary users
- Interference to primary users must not exceed certain thresholds
- Design a beamformer that maximizes multicast max-min-fair SNR.



New Challenges

- Robust design.
- Solution quality.

System Model

- A physical-layer multicasting cognitive radio system, SBS, equipped with N antennas, transmits a common signal to M single-antenna SUs.
- Our design problem is formulated as

$$
\begin{aligned}
\max_{\boldsymbol{W}} \quad & \gamma \\
\text{s.t.} \quad & \boldsymbol{h}_i^H \boldsymbol{W} \boldsymbol{h}_i \geq \gamma, && i = 1, \ldots, M, \\
& \max_{\|\boldsymbol{f}_j\| \leq \delta_j} (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \boldsymbol{W} (\boldsymbol{a}_j + \boldsymbol{f}_j) \leq \eta_j, && j = 1, \ldots, J, \\
& \mathrm{Tr}\,(\boldsymbol{W}) \leq P, \quad \boldsymbol{W} \succeq \mathbf{0}, \quad \mathrm{rank}(\boldsymbol{W}) \leq 1.
\end{aligned}
$$

  - $\boldsymbol{h}_i \in \mathbb{C}^N$ denotes the perfectly estimated channel between the SBS and SU i.
  - $\boldsymbol{a}_j \in \mathbb{C}^N$ is the estimated channel and $\boldsymbol{f}_j \in \mathbb{C}^N$ is the channel error.
- We are dealing with a class of NP-hard QCQP problems. Many researchers have done this before [SDL06, KSL08, GSS$^+$10, HLMZ12, WLMS14, LMS$^+$10].

The SDR and $\mathcal{S}$-lemma Techniques

- **S***tep 1:* Drop the rank constraint by using the SDR.
- **S***tep 2:* Denote $\boldsymbol{c}_j = \boldsymbol{W}\boldsymbol{a}_j,\ \zeta_j = \boldsymbol{a}_j^H \boldsymbol{W}\boldsymbol{a}_j$ and rewrite the robust constraints to

$$\forall \|\boldsymbol{f}_j\|^2 \leq \delta_j^2, \quad \left(\boldsymbol{f}_j^H \boldsymbol{W}\boldsymbol{f}_j + 2\mathrm{Re}\left\{\boldsymbol{c}_j^H \boldsymbol{f}_j\right\} + \zeta_j\right) \leq \eta_j,$$

- **S***tep 3:* Apply the $\mathcal{S}$-lemma and convert the relaxed problem into a system of linear matrix inequalities (LMIs):

$$\begin{aligned} \boldsymbol{W}^\star = \arg \max_{\boldsymbol{W},\gamma,\kappa_j} \quad & \gamma \\ \text{s.t.} \quad & \boldsymbol{h}_i^H \boldsymbol{W}\boldsymbol{h}_i \geq \gamma, \quad i = 1, \ldots, M, \\ & \begin{bmatrix} \kappa_j \boldsymbol{I}_N - \boldsymbol{W} & -\boldsymbol{c}_j \\ -\boldsymbol{c}_j^H & \eta_j - \zeta_j - \delta_j^2 \kappa_j \end{bmatrix} \succeq \boldsymbol{0}, j = 1, \ldots, J, \\ & \kappa_j \geq 0, \quad j = 1, \ldots, J, \\ & \mathrm{Tr}\left(\boldsymbol{W}\right) \leq P, \quad \boldsymbol{W} \succeq \boldsymbol{0}. \end{aligned}$$

This problem can be solved by a bisection method.

Non-rank-one Issue

NP-hardness: $\boldsymbol{W}^\star$ is generally not rank-one.

Algorithm 1 Gaussian Randomization Procedure

1: **input**: an optimal solution $\boldsymbol{W}^\star$, number of randomizations NR ≥ 1
2: **for** $\ell = 1, \ldots, \text{NR}$ **do**
3: generate $\boldsymbol{\xi}^\ell \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{W}^\star)$
4: set $\widetilde{\boldsymbol{\xi}}^\ell = \widehat{\boldsymbol{\xi}}^\ell \Big/ \sqrt{\max\left\{\pi^\ell, \max_{j=1,\ldots,J}\left\{\iota_j^\ell\right\}\right\}}$, where

$$\pi^\ell = \mathrm{Tr}(\widehat{\boldsymbol{W}}_j)/P, \quad \iota_j^\ell = \max_{\|\boldsymbol{f}_j\| \leq \delta_j} (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \widehat{\boldsymbol{\xi}}^\ell (\widehat{\boldsymbol{\xi}}^\ell)^H (\boldsymbol{a}_j + \boldsymbol{f}_j)/\eta_j$$

5: **end for**
6: let $\ell^\star = \arg\max_{\ell=1,\ldots,\text{NR}} |\boldsymbol{h}_i^H \widetilde{\boldsymbol{\xi}}^\ell|^2$
7: **output**: a feasible solution $\hat{\boldsymbol{w}} = \widetilde{\boldsymbol{\xi}}^{\ell^\star}$

A note: by using the triangular inequality, we can obtain ι_r^j in a closed form:

$$\iota_j^\ell = \max_{\|\boldsymbol{f}_j\| \leq \delta_j} \left|(\boldsymbol{a}_j + \boldsymbol{f}_j)^H \widehat{\boldsymbol{\xi}}^\ell\right|^2 = \left(\left|\boldsymbol{a}_j^H \widehat{\boldsymbol{\xi}}^\ell\right| + \delta_j \left\|\widehat{\boldsymbol{\xi}}^\ell\right\|\right)^2, \quad \boldsymbol{f}_j^\star = \delta_j \cdot \widehat{\boldsymbol{\xi}}^\ell / \left\|\widehat{\boldsymbol{\xi}}^\ell\right\|$$

Motivations

Key problem in this work: evaluate the quality of the SDR solution $\hat{\boldsymbol{w}}$.

- By using SDR to approximate the NP-hard QCQP, it is important to know the approximation quality.
- None of existing works study SDR approximation bounds for QCQPs applicable to imperfect CSIs,
  - Approximation bounds for standardized QCQPs under perfect CSIs [CLC08].
  - Approximation bounds for one-variable fractional QCQPs under perfect CSIs [JWSM13, WLSM16].
  - Approximation bounds for two-variable fractional QCQPs under perfect CSIs [WSPM16].
- It is essentially a fundamental problem in optimization theory.

Main Theorem

Theorem 1

Considering the design problem and Algorithm 1, we have

$$\Pr\left(\min_{i=1,\dots,M} \boldsymbol{h}_i^H \hat{\boldsymbol{w}}\hat{\boldsymbol{w}}^H \boldsymbol{h}_i = \Omega\left(\frac{1}{MN\log J}\right)\min_{i=1,\dots,M} \boldsymbol{h}_i^H \boldsymbol{W}^\star \boldsymbol{h}_i\right) \geq 1-(3/4)^{\mathrm{NR}},$$

where NR is the number of randomizations, M is the number of SU, J is the number of PU, and N is the number of antennas.

- Scaling with M is $1/M$.
- Scaling with N is $1/N$.
- Scaling with J is $1/\log J$.

Step 1: write an equivalent problem

- Equivalent problem: determining parameters $\beta \in (0,1)$ and $\gamma_1, \gamma_2 > 1$ such that

$$\Pr\left(\min_i \left| \boldsymbol{h}_i^H \widehat{\boldsymbol{\xi}}^\ell \right|^2 \geq \beta \min_i \boldsymbol{h}_i^H \boldsymbol{W}^\star \boldsymbol{h}_i \right.$$
$$\bigcap \ \left| (\widehat{\boldsymbol{\xi}}^\ell)^H \widehat{\boldsymbol{\xi}}^\ell \right|^2 \leq \gamma_1 \mathrm{Tr}(\boldsymbol{W}^\star) \bigcap \max_{\|\boldsymbol{f}_j\| \leq \delta_j} \left| (\widehat{\boldsymbol{\xi}}^\ell)^H (\boldsymbol{a}_j + \boldsymbol{f}_j) \right|^2$$
$$\left. \leq \gamma_2 \max_{\|\boldsymbol{f}_j\| \leq \delta_j} (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \boldsymbol{W}^\star (\boldsymbol{a}_j + \boldsymbol{f}_j), \forall j \right) \geq p, \quad (1)$$

  where $\widehat{\boldsymbol{\xi}}^\ell$ (cf. Step 4) is the randomized solution (may be infeasible) for rand. ℓ.

- Idea: If we set $\gamma_1 = \pi^\ell$, $\gamma_2 = \max_{j=1,\ldots,J} \{\iota_j^\ell\}$ and $\widetilde{\boldsymbol{\xi}}^\ell = \widehat{\boldsymbol{\xi}}^\ell \Big/ \sqrt{\max\{\gamma_1, \gamma_2\}}$, the resulting approximation ratio would be $\beta / \max\{\gamma_1, \gamma_2\}$, with a probability at least $1 - (1-p)^{\mathrm{NR}}$. We now determine β, γ_1 and γ_2 as follows.

Step 2: determine β and γ_1.

Lemma 1

Following our previous work in [WLSM16, WSPM16] and [SYZ08, Proposition 2.1],

$$\Pr\left(\mathrm{Tr}(\widehat{\boldsymbol{\xi}}^{\ell}(\widehat{\boldsymbol{\xi}}^{\ell})^H \mathbf{h}_i \mathbf{h}_i^H) \leq \beta \cdot \mathrm{Tr}(\mathbf{W}^\star \mathbf{h}_i \mathbf{h}_i^H)\right) \leq e^{1+\ln\beta},$$

$$\Pr\left(\mathrm{Tr}(\widehat{\boldsymbol{\xi}}^{\ell}(\widehat{\boldsymbol{\xi}}^{\ell})^H) \geq \alpha \cdot \mathrm{Tr}(\mathbf{W}^\star)\right) \leq e^{-\frac{1}{2}\left(\gamma_1 + 2\log\frac{1}{2}\right)}.$$

- Lemma 1 gives probability bounds parametrized by the scaling factors.
- By setting $\beta = (4eM)^{-1}$, $\gamma_1 = \log 64 \approx 4.16$ in (1) and then using the union bounds, we obtain

$$\Pr\left(\min_i \left|\boldsymbol{h}_i^H \widehat{\boldsymbol{\xi}}^{\ell}\right|^2 \leq \beta \min_i \boldsymbol{h}_i^H \boldsymbol{W}^\star \boldsymbol{h}_i\right) \leq M \cdot e^{1+\log\beta} = 1/4;$$

$$\Pr\left(\left|(\widehat{\boldsymbol{\xi}}^{\ell})^H \widehat{\boldsymbol{\xi}}^{\ell}\right|^2 \geq \gamma_1 \cdot \mathrm{Tr}(\mathbf{W}^\star)\right) \leq e^{-\frac{1}{2}\left(\gamma_1 + 2\log\frac{1}{2}\right)} = 1/4$$

for the first two events in (1).

The Difficulty in Determining γ_2

A naive attempt: we can deduce a lower bound

$$\Pr\left(\max_{\|\boldsymbol{f}_j\| \le \delta_j} (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \widehat{\boldsymbol{\xi}}^{\ell} (\widehat{\boldsymbol{\xi}}^{\ell})^H (\boldsymbol{a}_j + \boldsymbol{f}_j) \ge \kappa \max_{\|\boldsymbol{f}_j\| \le \delta_j} (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \boldsymbol{W}^{\star} (\boldsymbol{a}_j + \boldsymbol{f}_j) \right) \ge p_0. \quad (2)$$

We observe

$$\begin{aligned} & \max_{\|\boldsymbol{f}_j\| = \delta_j} (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \widehat{\boldsymbol{\xi}}^{\ell} (\widehat{\boldsymbol{\xi}}^{\ell})^H (\boldsymbol{a}_j + \boldsymbol{f}_j) \ge \kappa \max_{\|\boldsymbol{f}_j\| \le \delta_j} (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \boldsymbol{W}^{\star} (\boldsymbol{a}_j + \boldsymbol{f}_j) \\ = & \bigcup_{\|\boldsymbol{f}_j\| = \delta_j} (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \widehat{\boldsymbol{\xi}}^{\ell} (\widehat{\boldsymbol{\xi}}^{\ell})^H (\boldsymbol{a}_j + \boldsymbol{f}_j) \ge \kappa \max_{\|\boldsymbol{f}_j\| \le \delta_j} (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \boldsymbol{W}^{\star} (\boldsymbol{a}_j + \boldsymbol{f}_j), \end{aligned}$$

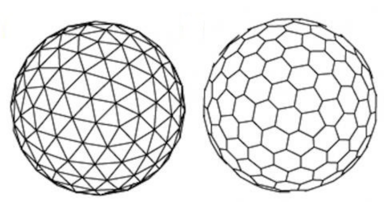
then a naive attempt may be to apply the union bound and use (2).

- No! Union bound does not work on an uncountable set.

Find a Proper Way to Represent the Uncountable Set

Definition[HW87, BG95, Ver12]

- Let $\mathcal{S}$ be a set. A subset $\mathcal{N} \subseteq \mathcal{S}$ is called an ϵ-net of $\mathcal{S}$ if for any point $x \in \mathcal{S}$, there exists a point $z \in \mathcal{N}$ such that $\|z - x\| \leq \epsilon$.



- Let $\mathcal{S}(\delta) \subset \mathbb{C}^n$ denote a sphere of radius δ. There exists an $(\delta/2)$-net $\mathcal{N}_\delta^{\delta/2}$ on $\mathcal{S}(\delta)$ with cardinality $|\mathcal{N}_\delta^{\delta/2}| \leq 5^{2n}$.

- Use the ϵ-net to approximate the uncountably infinite set $\|\boldsymbol{f}_j\| = \delta_j$ by a finite set.

Probability Bound Parametrized by ϵ and N

Lemma

Let $|\mathcal{N}_1^\epsilon|$ be the cardinality of an ϵ-net $\mathcal{N}_1^\epsilon$ of the unit sphere $S = S(1)$. Given $\boldsymbol{a} \in \mathcal{C}^n$ and $\boldsymbol{X}^\star \in \mathcal{H}_+^n$, let $\boldsymbol{\xi} \sim \mathcal{CN}(0, \boldsymbol{X}^\star)$. Then, for any $\kappa > 1$, $0 < \epsilon < 1$, we have

$$\Pr\left(\max_{\|\boldsymbol{f}\|\leq 1} |\boldsymbol{\xi}^H(\boldsymbol{a}+\boldsymbol{f})| \geq \kappa\left(\frac{1+\epsilon}{1-\epsilon}\right)^2 \max_{\|\boldsymbol{f}\|\leq 1}(\boldsymbol{a}+\boldsymbol{f})^H \boldsymbol{X}^\star(\boldsymbol{a}+\boldsymbol{f})\right)$$
$$\leq (|N_1^\epsilon|+1)\exp\left(-(\kappa-1)/6\right). \quad (3)$$

- The probability bound is parametrized by the approximation accuracy of the ϵ-net and the dimension of the ball, i.e., N

Step 3: determine γ_2

Key: combine ϵ-net approximation and union bounds.

- We choose $\epsilon = 1/2$, as well as $\gamma_2 = \big(6\log(4J(5^{2N}+1))+1\big)\cdot 3^2$ to obtain

$$\Pr\left(\max_{\|\boldsymbol{f}_j\|\le\delta_j}\left|(\widehat{\boldsymbol{\xi}}^{\ell})^H(\boldsymbol{a}_j+\boldsymbol{f}_j)\right|^2 \le \gamma_2 \max_{\|\boldsymbol{f}_j\|\le\delta_j}(\boldsymbol{a}_j+\boldsymbol{f}_j)^H\boldsymbol{W}^\star(\boldsymbol{a}_j+\boldsymbol{f}_j), \forall j,\right) \le 1/4. \quad (4)$$

- By further using the union bound, let $p = 1 - 3/4 = 1/4$ and $\beta/\max\{\gamma_1,\gamma_2\} = \beta/\gamma_2$. This immediately leads to Theorem 1, which completes the proof.

Numerical Simulations: approx. bounds scaling with M

- $\boldsymbol{h}_i, \boldsymbol{a}_j \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{I})$, $\delta_j = 0.1, \forall j$, $\sigma^2 = 1$, 1000 rand. and 100 channel realizations.

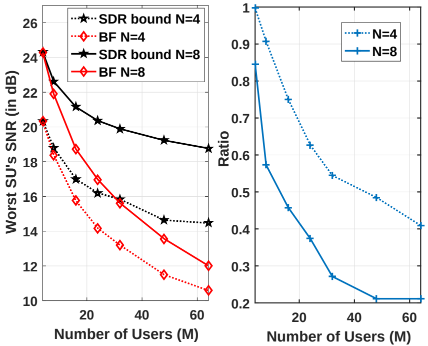


Figure: The worst SU's SNR and the approximation bound scale with M. The ratio is $\frac{\min_{i=1,...,M} \boldsymbol{h}_i^H \hat{\boldsymbol{w}} \hat{\boldsymbol{w}}^H \boldsymbol{h}_i}{\min_{i=1,...,M} \boldsymbol{h}_i^H \boldsymbol{W}^{\star} \boldsymbol{h}_i}$.

- As M increases, the SNR performance degrades and the gap between the SNRs associated with the SDR solution and the optimal solution is enlarged.
- Verify Theorem 1: the ratio is larger for $N = 8$ than that for $N = 4$

Numerical Simulations: approx. bounds scaling with J

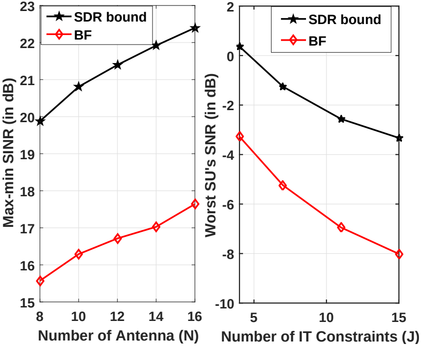


Figure: The worst SU's SNR scales with N and J. Left: $P = 20$dB and $J = 1$. Right: $P = 5$dB, $N = 4$ and $M = 32$.

- Left: N increases, SNR becomes better but the gap between the two lines becomes wider.
- Right: J increases, SNR becomes worse and the gap becomes winder.
- These observations are consistent with the analytical results in Theorem 1.

Conclusions

- We study the multicast beamforming design in a cognitive radio network.
- Our research object is the robust QCQPs: SDR and randomizations.
- Our main contribution is to provide the approximation bounds for robust QCQPs.
- Simulation results verify the theoretical analysis.

Appendix: Proof of the Lemma (1)

Since for any $\boldsymbol{X}^\star$, the maximum in (3) is attained at a point $\boldsymbol{f}^\star(\boldsymbol{X}^\star)$ with $\|\boldsymbol{f}^\star(\boldsymbol{X}^\star)\| = 1$, we focus on the set

$$\mathcal{U} = \{\boldsymbol{a} + \boldsymbol{f} : \|\boldsymbol{f}\| = 1\}.$$

Fixing $\boldsymbol{u} \in \mathcal{U}$, we have $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{f}(\boldsymbol{u})$ for some $\|\boldsymbol{f}(\boldsymbol{u})\| = 1$. By using the concept of the ϵ-net on the unit sphere $S = S(1)$, there exists an $\boldsymbol{f}_0(\boldsymbol{u}) \in \mathcal{N}_1^\epsilon$ such that $\|\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}_0(\boldsymbol{u})\| \le \epsilon$, which implies that

$$\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{f}_0(\boldsymbol{u}) + \epsilon_1(\boldsymbol{u})\tilde{\boldsymbol{f}}(\boldsymbol{u})$$

for some $\|\tilde{\boldsymbol{f}}(\boldsymbol{u})\| = 1$ and $0 \le \epsilon_1(\boldsymbol{u}) \le \epsilon$. In this way, we can express $\boldsymbol{u}$ as

$$\boldsymbol{u} = \boldsymbol{a} + \sum_{k \ge 0} \epsilon_k(\boldsymbol{u})\boldsymbol{f}_k(\boldsymbol{u}),$$

where $0 \le \epsilon_k(\boldsymbol{u}) \le \epsilon^k$ and $\boldsymbol{f}_k(\boldsymbol{u}) \in \mathcal{N}_1^\epsilon$ for all $k \ge 0$.

Appendix: Proof of the Lemma (2)

Continuing this fashion, by setting $D = \left(\sum_{k\geq 0} \epsilon_k(\boldsymbol{u})\right)^{-1}$, we can compute

$$\left|\boldsymbol{u}^H \boldsymbol{\xi}\right| \leq \sum_{k\geq 0} \epsilon_k(\boldsymbol{u}) \left|(D\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\right|$$

and

$$\left|(D\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\right| \leq \left|(\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\right| + |1 - D| \left|\boldsymbol{a}^H \boldsymbol{\xi}\right|.$$

It follows that

$$\begin{aligned}
\left|\boldsymbol{u}^H \boldsymbol{\xi}\right|^2 &\leq \left[\sum_{k\geq 0} \epsilon_k(\boldsymbol{u}) \left|(\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\right| + |(1 - D)/D| \left|\boldsymbol{a}^H \boldsymbol{\xi}\right|\right]^2 \\
&\leq \left[\frac{1}{D} \sup_{k\geq 0} \left|(\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\right| + \left|\frac{1 - D}{D}\right| \left|\boldsymbol{a}^H \boldsymbol{\xi}\right|\right]^2
\end{aligned}$$

Appendix: Proof of the Lemma (3)

Continuing this fashion, by setting $D = \left(\sum_{k\geq 0} \epsilon_k(\boldsymbol{u})\right)^{-1}$, we can compute

$$\left|\boldsymbol{u}^H \boldsymbol{\xi}\right| \leq \sum_{k\geq 0} \epsilon_k(\boldsymbol{u}) \left|(D\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\right|$$

and

$$\left|(D\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\right| \leq \left|(\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\right| + |1-D| \left|\boldsymbol{a}^H \boldsymbol{\xi}\right|.$$

It follows that

$$\left|\boldsymbol{u}^H \boldsymbol{\xi}\right|^2 \leq \left[\sum_{k\geq 0} \epsilon_k(\boldsymbol{u}) \left|(\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\right| + |(1-D)/D| \left|\boldsymbol{a}^H \boldsymbol{\xi}\right|\right]^2$$

$$\leq \left[\frac{1}{D} \sup_{k\geq 0} \left|(\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\right| + \left|\frac{1-D}{D}\right| \left|\boldsymbol{a}^H \boldsymbol{\xi}\right|\right]^2$$

Observe that for any $\boldsymbol{f} \in \mathcal{N}_1^{\epsilon}$, we have

$$\left\{\left|(\boldsymbol{a} + \boldsymbol{f})^H \boldsymbol{\xi}\right|^2\right\} \leq \kappa \cdot \left\{(\boldsymbol{a} + \boldsymbol{f})^H \boldsymbol{X}^\star (\boldsymbol{a} + \boldsymbol{f})\right\}$$

with probability at least $1 - \exp\left(-\frac{\kappa-1}{6}\right)$ [SYZ08], [WLSM16, Lemma 2].

Appendix: Proof of the Lemma (4)

Now, let $\boldsymbol{f}^\star = \arg\max_{\|\boldsymbol{f}\|\le 1} (\boldsymbol{a}+\boldsymbol{f})^H \boldsymbol{X}^\star (\boldsymbol{a}+\boldsymbol{f})$. Since $\boldsymbol{f}_k(\boldsymbol{u}) \in \mathcal{N}_1^\epsilon$ for all $u \in \mathcal{U}$ and $k \ge 0$, the inequalities

$$\sup_{\substack{\boldsymbol{u}\in\mathcal{U}\\ k\ge 0}} \left\{ \left| (\boldsymbol{a}+\boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi} \right|^2 \right\} \le \kappa \cdot \max_{\boldsymbol{f}\in\mathcal{N}_1^\epsilon} \left\{ (\boldsymbol{a}+\boldsymbol{f})^H \boldsymbol{X}^\star (\boldsymbol{a}+\boldsymbol{f}) \right\}$$

$$\le \kappa \cdot (\boldsymbol{a}+\boldsymbol{f}^\star)^H \boldsymbol{X}^\star (\boldsymbol{a}+\boldsymbol{f}^\star)$$

hold with probability at least $1 - |\mathcal{N}_1^\epsilon| \exp\left(-\frac{\kappa-1}{6}\right)$ for $\kappa > 1$, where the second inequality is due to the optimality of $\boldsymbol{f}^\star$.
Similarly, the inequalities

$$\left| \boldsymbol{f}^H \boldsymbol{\xi} \right|^2 \le \kappa \cdot \boldsymbol{f}^H \boldsymbol{X}^\star \boldsymbol{f} \le \kappa \cdot (\boldsymbol{a}+\boldsymbol{f}^\star)^H \boldsymbol{X}^\star (\boldsymbol{a}+\boldsymbol{f}^\star)$$

hold with probability at least $1 - \exp\left(-\frac{\kappa-1}{6}\right)$ for $\kappa > 1$. Observing that $(1+|1-D|)/D \le (1+\epsilon)/(1-\epsilon)$ and combining all the pieces together, we have

$$\max_{\|\boldsymbol{f}\|=1} \left| (\boldsymbol{a}+\boldsymbol{f})^H \boldsymbol{\xi} \right|^2 \le \kappa \left(\frac{1+\epsilon}{1-\epsilon} \right)^2 \max_{\|\boldsymbol{f}\|=1} (\boldsymbol{a}+\boldsymbol{f})^H \boldsymbol{X}^\star (\boldsymbol{a}+\boldsymbol{f}).$$

This completes the proof of Lemma 1.

References

[BG95] Hervé Brönnimann and Michael T Goodrich.
Almost optimal set covers in finite VC-dimension.
Discrete & Computational Geometry, 14(4):463–479, 1995.

[CLC08] Tsung-Hui Chang, Zhi-Quan Luo, and Chong-Yung Chi.
Approximation bounds for semidefinite relaxation of max–min–fair multicast transmit beamforming problem.
56(8):3932–3943, Aug. 2008.

[GSS+10] Alex B. Gershman, Nicholas D. Sidiropoulos, Shahram Shahbazpanahi, Mats Bengtsson, and Björn Ottersten.
Convex optimization–based beamforming: From receive to transmit and network designs.
27(3):62–75, May 2010.

[HLMZ12] Yongwei Huang, Qiang Li, Wing-Kin Ma, and Shuzhong Zhang.
Robust multicast beamforming for spectrum sharing–based cognitive radios.
60(1):527–533, Jan. 2012.

[HW87] David Haussler and Emo Welzl.
ϵ-nets and simplex range queries.
Discrete & Computational Geometry, 2(2):127–151, 1987.

[JWSM13] Senshan Ji, Sissi Xiaoxiao Wu, Anthony Man-Cho So, and Wing-Kin Ma.
Multi–group multicast beamforming in cognitive radio networks via rank–two transmit beamformed Alamouti space–time coding.
In *Proc. 2013 IEEE Int. Conf. Acoust., Speech, and Signal Process. (ICASSP)*, pages 4409–4413, May 2013.

[KSL08] Eleftherios Karipidis, Nicholas D. Sidiropoulos, and Zhi-Quan Luo.
Semidefinite relaxation of quadratic optimization problems.
27(3):20–34, May 2010.

[SDL06] Nicholas D. Sidiropoulos, Timothy N. Davidson, and Zhi-Quan Luo.
Transmit beamforming for physical–layer multicasting.
54(6):2239–2251, Jun. 2006.

[SYZ08] Anthony Man-Cho So, Yinyu Ye, and Jiawei Zhang.
A unified theorem on SDP rank reduction.
Math. of Oper. Res., 33(4):910–920, 2008.

[Ver12] Roman Vershynin.
Introduction to the non-asymptotic analysis of random matrices.
In Yonina C. Eldar and Gitta Kutyniok, editors, *Compressed Sensing*, pages 210–268. Cambridge University Press, 2012.
Cambridge Books Online.

[WLMS14] S. X. Wu, Q. Li, W. K. Ma, and A. M. C. So.
Robust transmit designs for an energy harvesting multicast system.
In *2014 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 4748–4752, May 2014.

[WLSM16] S. X. Wu, Q. Li, A. M.-C. So, and W.-K. Ma.
A stochastic beamformed amplify-and-forward scheme in a multigroup multicast MIMO relay network with per-antenna power constraints.
IEEE Trans. Wireless Commun., 15(7):4973–4986, July 2016.

[WSPM16] S. X. Wu, A. M.-C. So, Jiaxian Pan, and W.-K. Ma.

Thank You

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Question Welcomed!