Low Rank Phase Retrieval

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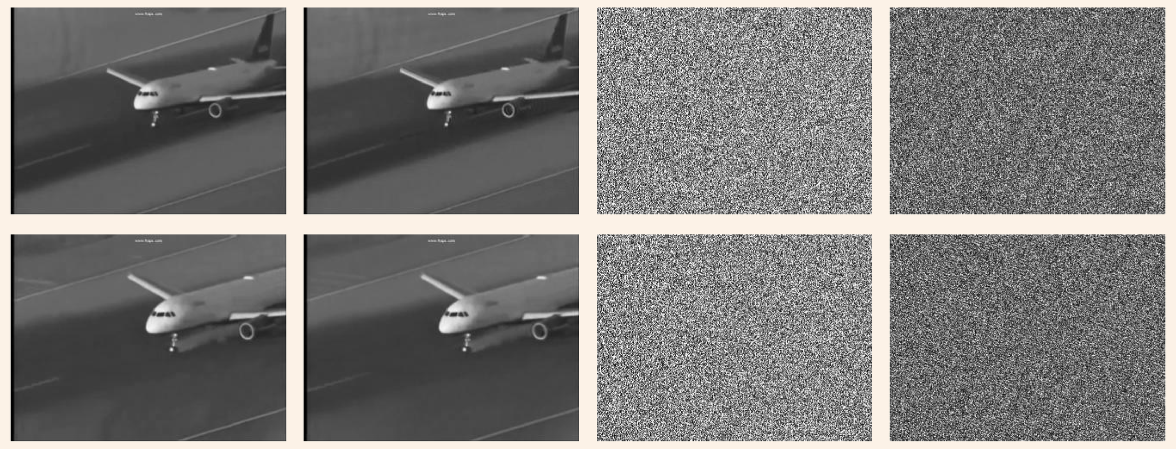
INTRODUCTION

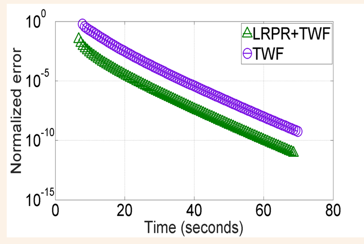
- **Goal:** recover a low-rank matrix, $\boldsymbol{X}$, from magnitude-only observations of random linear projections of its columns
- **Contributions:**
  1. AltMinTrunc that exploits low-rank structure of $\boldsymbol{X}$
  2. high probability sample complexity bounds for AltMinTrunc initialization
- **Problem Definition**
  - Instead of a single vector $\boldsymbol{x}$, we have a set of q vectors, $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_q$ which are such that the $n \times q$ matrix $\boldsymbol{X} := [\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_q]$ has rank $r \ll \min(n, q)$
  - For each $\boldsymbol{x}_k$, there are a set of m measurements of the form

$$\boldsymbol{y}_{i,k} := (\boldsymbol{a}_{i,k}{}'\boldsymbol{x}_k)^2, \ i = 1, 2, \dots m, \ k = 1, 2, \dots, q$$

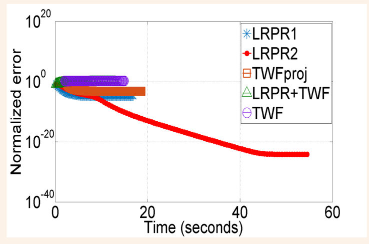
EXPERIMENT RESULTS



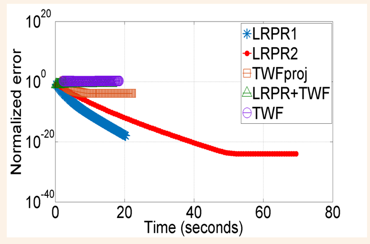
Original | AMT | TWFp | TWF



$m = 8n$



$m = 0.8n$



$m = 0.6n$

	$q = 100$			$q = 1000$		
m	AMT	TWF	TWFp	AMT	TWF	TWFp
10	1.32	1.62	1.00	0.46	1.62	0.73
50	0.53	1.48	0.77	0.11	1.48	0.57

EXPERIMENT DETAILS

- Plane Video Results
  - Shows frames 1 and 104 of original, and recovered videos of AltMinTrunc, TWF-proj and TWF
    - Settings: $m = 3n$ CDP measurements, $r = 25$, frame sizes: 240×320, total frames: $q = 105$
- Graphs
  - For each iteration, plot shows the error at iteration t against the time taken until iteration t
    - Settings: noise-free complex Gaussian, $n = 100$, $r = 2$, $q = 1000$
  - Table : Initialization error comparison
    - Settings are $n = 100$, $r = 2$, $\boldsymbol{b}_k \sim \mathcal{N}(0, \boldsymbol{I})$

COMPLETE ALGORITHM

- **Initialization**
  - Compute $\hat{\boldsymbol{U}}$ as top r eigenvectors of $\mathbf{Y_U} := \frac{1}{\mathbf{mq}} \sum_\mathbf{i} \sum_\mathbf{k} \mathbf{y_{i,k} a_{i,k} a_{i,k}{}' 1}_{\{\mathbf{y_{i,k}} \leq 9 \frac{\sum_\mathbf{i} \mathbf{y_{i,k}}}{\mathbf{m}}\}}$
  - For each $k = 1, 2, \dots, q$,
    - Compute $\hat{\boldsymbol{v}}_k$ as the top eigenvector of $\mathbf{Y_{b,k}} := \mathbf{\hat{U}' M_k \hat{U}}$ where $\mathbf{M_k} := \frac{1}{\mathbf{m}} \sum_\mathbf{i} \mathbf{y_{i,k} a_{i,k} a_{i,k}{}'}$
    - Compute $\hat{\eta}_k := \sqrt{\frac{1}{m} \sum_i \boldsymbol{y}_{i,k}}$; set $\hat{\boldsymbol{b}}_k = \hat{\boldsymbol{g}}_k = \hat{\boldsymbol{v}}_k \hat{\eta}_k$ and $\hat{\boldsymbol{x}}_k := \hat{\boldsymbol{U}} \hat{\boldsymbol{g}}_k$
- **Loop Iterations**
  - For $t = 1$ to T, repeat the following three steps:
    1. for all $k = 1, 2, \dots, q$, $\hat{\boldsymbol{C}}_k \leftarrow \text{diag}(\text{phase}(\boldsymbol{A}_k' \hat{\boldsymbol{U}} \hat{\boldsymbol{b}}_k))$
    2. $\hat{\boldsymbol{U}} \leftarrow \arg\min_{\tilde{\boldsymbol{U}}} \sum_k \|\hat{\boldsymbol{C}}_k \sqrt{\boldsymbol{y}_k} - \boldsymbol{A}_k{}' \tilde{\boldsymbol{U}} \hat{\boldsymbol{b}}_k\|^2$
    3. for all $k = 1, 2, \dots, q$, $\hat{\boldsymbol{b}}_k \leftarrow \arg\min_{\tilde{\boldsymbol{b}}_k} \|\hat{\boldsymbol{C}}_k \sqrt{\boldsymbol{y}_k} - \boldsymbol{A}_k{}' \hat{\boldsymbol{U}} \tilde{\boldsymbol{b}}_k\|^2$

Output $\hat{\boldsymbol{x}}_k = \hat{\boldsymbol{U}} \hat{\boldsymbol{b}}_k$'s for all $k = 1, 2, \dots, q$

Steps 2 and 3 involve solving a LS problem

ALGORITHM DERIVATION

- $\boldsymbol{X}$: rank $r \implies$ can be written as $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{B}$
  - $\boldsymbol{U}$ is an $n \times r$ matrix with mutually orthonormal columns
  - $\boldsymbol{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \dots \boldsymbol{b}_q]$ is an $r \times q$ matrix independent of $\boldsymbol{U}$
- Let $\frac{1}{q}\boldsymbol{X}\boldsymbol{X}' = \frac{1}{q}\sum_{k=1}^q \boldsymbol{x}_k \boldsymbol{x}_k{}' \overset{\text{EVD}}{=} \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}'$ denote the reduced eigenvalue decomposition (EVD) of $\boldsymbol{X}\boldsymbol{X}'/q$

Compute $\hat{\mathbf{U}}$:

- Define $\boldsymbol{Y}_{U,0} := \frac{1}{mq}\sum_{i=1}^m \sum_{k=1}^q \boldsymbol{y}_{i,k} \boldsymbol{a}_{i,k} \boldsymbol{a}_{i,k}{}'$
- It is not hard to see that $\mathbb{E}[\boldsymbol{Y}_{U,0} | \boldsymbol{U}] = 2\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}' + \text{trace}(\boldsymbol{\Lambda})\boldsymbol{I}$
- The subspace spanned by top r eigenvectors of above matrix is range$(\boldsymbol{U})$ and the gap between its r-th and $(r+1)$-th eigenvalue is $2\lambda_{\min}(\boldsymbol{\Lambda})$
- If m and q are large enough, L.L.N $\implies \boldsymbol{Y}_{U,0} \approx \mathbb{E}[\boldsymbol{Y}_{U,0}]$
  - So by $\sin\theta$ theorem (Davis-Kahan'71), the span of top r eigenvectors of $\boldsymbol{Y}_{U,0}$ is close to span of $\boldsymbol{U}$
- Let $\boldsymbol{w}_{i,k} = \sqrt{\boldsymbol{y}_{ik}} \boldsymbol{a}_{i,k}$; $\boldsymbol{w}_{i,k}$ is heavy-tailed
  - More samples are needed for law of large numbers to take effect
  - Solution : truncating $\boldsymbol{w}_{i,k}$'s
- Compute $\hat{\boldsymbol{U}}$ as the top r eigenvectors of $\boldsymbol{Y}_U$

Compute $\hat{\mathbf{b}}_\mathbf{k}'s$:

- If $\hat{\boldsymbol{U}}$ indep of $\boldsymbol{M}_k$, then $\mathbb{E}[\boldsymbol{Y}_{b,k} | \hat{\boldsymbol{U}}] = 2\boldsymbol{g}_k \boldsymbol{g}_k{}' + \|\boldsymbol{x}_k\|^2 \boldsymbol{I}$, and $\boldsymbol{g}_k = \hat{\boldsymbol{U}}'\boldsymbol{U}\boldsymbol{b}_k$
- Top eigenvector of this expectation is proportional to $\boldsymbol{g}_k$ and the gap between its first and second eigenvalues is $2\|\boldsymbol{g}_k\|^2 = 2\|\hat{\boldsymbol{U}}'\boldsymbol{U}\boldsymbol{b}_k\|^2$
- If $\hat{\boldsymbol{U}} \approx \boldsymbol{U}$, then $\|\boldsymbol{g}_k| \approx \|\boldsymbol{b}_k\|$ and $\hat{\boldsymbol{U}} g_k \approx \boldsymbol{U}\boldsymbol{b}_k$
- If $\hat{\boldsymbol{U}} \approx \boldsymbol{U}$ and m is large enough, then top eigenvector of $\boldsymbol{Y}_{b,k}$ is a good approximation of $\boldsymbol{b}_k$

Compute $\hat{\boldsymbol{b}}_k$ as the top eigenvector of $\boldsymbol{Y}_{b,k}$ and scale it

THEOREM

- Assume $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{B}$ and $\boldsymbol{B}$ independent of $\boldsymbol{U}$
- Let $\bar{\boldsymbol{\Lambda}} := \frac{1}{q}\sum_k \boldsymbol{b}_k \boldsymbol{b}_k{}'$, $\bar{\boldsymbol{\lambda}}_{\max}$ its maximum eigenvalue and κ condition number
- For each $\boldsymbol{x}_k$, $k = 1, 2, \dots, q$, we observe
  - m measurements $\boldsymbol{y}_{i,k} := (\boldsymbol{a}_{i,k}{}'\boldsymbol{x}_k)^2$ with $\boldsymbol{a}_{i,k} \overset{\text{iid}}{\sim} \mathcal{N}(0, \boldsymbol{I})$
  - $\tilde{m}$ measurements $\boldsymbol{y}_{i,k}^{new} := (\boldsymbol{a}_i^{new\prime}\boldsymbol{x}_k)^2$ with $\boldsymbol{a}_i^{new} \overset{\text{iid}}{\sim} \mathcal{N}(0, \boldsymbol{I})$, and with $\boldsymbol{a}_i^{new}$'s independent of $\boldsymbol{a}_{i,k}$'s
- Suppose that $r \leq cn^{1/5}$ and $q \leq cn^2$
- For an $\varepsilon < 1$, if

$$\tilde{m} \geq \frac{c\sqrt{n}}{\varepsilon^2}, \ m \geq \frac{c\kappa^2 r^4 (\log n)(\log \tilde{m})^2}{\varepsilon^2},$$
$$mq \geq \frac{c\kappa^2 n r^4 (\log \tilde{m})^2}{\varepsilon^2},$$

- With probability at least $1 - \frac{c}{n^2}$,
  1. $\text{SE}(\hat{\boldsymbol{U}}, \boldsymbol{U}) := \|(\boldsymbol{I} - \hat{\boldsymbol{U}}\hat{\boldsymbol{U}}')\boldsymbol{U}\| \leq \frac{c\varepsilon}{r \log \tilde{m}}$;
  2. $\text{NormErr}(\boldsymbol{X}, \hat{\boldsymbol{X}}) := \frac{\sum_{k=1}^q \text{dist}(\boldsymbol{x}_k, \hat{\boldsymbol{x}}_k)^2}{\sum_{k=1}^q \|\boldsymbol{x}_k\|^2} \leq c\varepsilon$

DISCUSSION

- If $r = c \log n$ and $q = cn$, **we only need $\mathbf{m + \tilde{m} \geq c\sqrt{n}}$,** in comparison with at least *cn* **which is needed by single vector PR methods**
- $c\sqrt{n}$ can be replaced by $cn^{1/d}$ for any $d \geq 2$ also
- To just recover $\boldsymbol{U}$ with $\text{SE}(\hat{\boldsymbol{U}}, \boldsymbol{U}) \leq c\varepsilon$, we need only $mq = cnr^2$ measurements
  - **when r is small, this is only a little more than the minimum required, nr**

Contact Information

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