NON-CONVEX SPARSE OPTIMIZATION FOR PHOTON-LIMITED IMAGING

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Introduction

Background: In many real world applications, the observation noise does not follow the Gaussian statistics. In particular, when the number of observed photon counts is relatively low at the camera detector, the measurement noise follows a Poisson distribution.

Applications: Astronomy, night vision, and medical imaging such as Positron Emission Tomography (PET) and Single Photon Emission Computed Tomography (SPECT).

Goal: Recover sparse high-resolution signals from low-dimensional measurements corrupted by Poisson noise.

Approach: Explicitly model noise using Poisson statistics and further enforce sparsity and structure in the solution using the ℓ_p-norm $(0 \leq p < 1)$. Solve time-dependent sparse recovery problems in several steps: Recover the support of the signal using the time-averaged data and reconstruct the signal intensity using the time-dependent data.

Observation Model

The Poisson noise model [1] is used to accurately model the received photon counts at the detector:

$$\mathbf{y} \sim \text{Poisson}(\mathbf{A}\mathbf{f}^*),$$

where

$\mathbf{y} \in \mathbb{Z}_+^m$ = a vector of observed photon counts,
$\mathbf{f}^* \in \mathbb{R}_+^n$ = the true signal or image of interest,
$\mathbf{A} \in \mathbb{R}_+^{m\times n}$ = the linear projection matrix.

Method: The true signal $\mathbf{f}^*$ is estimated using the *maximum likelihood principle*.

Optimization Problem

The Poisson intensity reconstruction problem has the following constrained minimization form:

$$\begin{aligned} &\underset{\mathbf{f}\in\mathbb{R}^n}{\text{minimize}} \quad \Phi(\mathbf{f}) \equiv F(\mathbf{f}) + \tau\ \text{pen}(\mathbf{f}) \\ &\text{subject to} \quad \mathbf{f} \succeq 0, \end{aligned} \tag{1}$$

where $\tau > 0$ is a regularization parameter, $F(\mathbf{f})$ is the negative Poisson log-likelihood function $F(\mathbf{f}) = \mathbf{1}^T\mathbf{A}\mathbf{f} - \sum_{i=1}^m y_i \log(\mathbf{e}_i^T\mathbf{A}\mathbf{f} + \beta)$, where $\mathbf{1}$ is an m-vector of ones, $\mathbf{e}_i$ is the ith canonical basis unit vector, $\beta > 0$ (typically $\beta \ll 1$) and pen : $\mathbb{R}^n \longrightarrow \mathbb{R}$ is a sparsity-promoting penalty functional.

Proposed approach: Use SPIRAL framework [2] with $\text{pen}(\mathbf{f}) = \|\mathbf{f}\|_p^p$ $(0 \leq p < 1)$ to enhance the sparsity of the reconstruction.

Subproblem Formulation

To solve the minimization problem in (1), $F(\mathbf{f})$ is approximated by second-order Taylor series expansion, where the Hessian in the Taylor series is replaced by a scaled identity matrix $\alpha_k\mathbf{I}$, where $\alpha_k > 0$ [3]. A simple manipulation to this quadratic approximation will lead into a sequence of subproblems of the form

$$\begin{aligned} \mathbf{f}^{k+1} = &\underset{\mathbf{f}\in\mathbb{R}^n}{\arg\min} \quad \frac{1}{2}\parallel \mathbf{f} - \mathbf{s}^k \parallel_2^2 + \frac{\tau}{\alpha_k}\|\mathbf{f}\|_p^p \\ &\text{subject to} \quad \mathbf{f} \succeq 0, \end{aligned} \tag{2}$$

where $\mathbf{s}^k = \mathbf{f}^k - \frac{1}{\alpha_k}\nabla F(\mathbf{f}^k)$. The subproblem (2) can be uncoupled into scalar minimization problems and solved using the *generalized soft-thersholding function* – a zero finding method such as Newton's method or fixed-point iteration method is used along with a threshold value to find the global minimum [4, 5].

Significance

Our approach is significant for the following reasons:

1) We incorporate a nonconvex ℓ_p-norm to further promote sparsity in the solution.

2) The p-value can be tuned to highlight different structural properties of the signal.

3) We solve time-dependent sparse recovery problems using a multistep process.

Application: Fluorescence Lifetime Imaging

Goal: Reconstruct the fluorescence lifetime along with the support and fluorophore concentration from **time-dependent CCD camera measurements corrupted by Poisson noise** (see Fig. 1) [6].

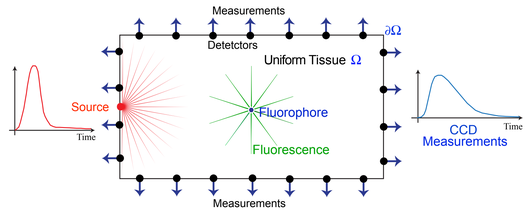


Figure 1: Schematic diagram of time-dependent fluorescence lifetime tomography.

Forward problem: Transportation of excitation light I^e is governed by an initial-boundary value problem for the diffusion approximation. A portion of I^e is absorbed by the fluorophores and re-emitted. The transportation of emitted light I^f is then modeled by

$$\begin{aligned} \frac{1}{c}\frac{\partial I^f}{\partial t} - \nabla\cdot(\kappa^f\nabla I^f) + \mu_a^f I^f &= Q(\mathbf{r},t) \quad \text{in } \Omega\times(0,T], \\ I^f + \alpha^f\kappa^f\frac{\partial I^f}{\partial n} &= 0 \quad \text{on } \partial\Omega, \\ I^f(\mathbf{r},0) &= 0 \quad \text{in } \Omega, \end{aligned} \tag{3}$$

I^f : the intensity of emission light,
κ^f : the diffusion coefficient,
μ_a^f : the absorption coefficient,
$Q(\mathbf{r},t) = \chi(\mathbf{r})h(\mathbf{r})\int_0^t e^{-(t-t')/\tau(\mathbf{r})}I^e dt'$,
$\chi(\mathbf{r})$: indicator function,
$h(\mathbf{r})$: fluorphore concentration,
$\tau(\mathbf{r})$: fluorescence-lifetime,
α^f, c : some physical constants.

Then, we model measurements of scattered light leaving the boundary of the medium, $u(\mathbf{r},t)$, through evaluation of

$$u(\mathbf{r},t) = -\kappa^f\frac{\partial I^f}{\partial n} = \frac{1}{\alpha^f}I^f \quad \text{on } \partial\Omega\times(0,T].$$

Both initial-boundary value problems are solved numerically using the **Crank-Nicolson method**. In the discrete setting, the time-dependent measurements are obtained by restricting the numerical solution of emission light, say $\mathbf{V}$, to the boundary:

$$\mathbf{y} = \frac{1}{\alpha^f}\mathbf{R}\mathbf{V} = \frac{1}{\alpha^f}\mathbf{R}\mathbf{L}^{-1}\tilde{\mathbf{Q}}, \tag{4}$$

where $\mathbf{R}$ is a boundary restriction operator, $\mathbf{L}$ is the finite difference operator and $\tilde{\mathbf{Q}}$ is $Q(\mathbf{r},t)$ averaged between consecutive time steps.

Forward operator: We define $\frac{1}{\alpha^f}\mathbf{R}\mathbf{L}^{-1}$ in (4) as the system matrix $\mathbf{A}$. Instead of generating the system matrix $\mathbf{A}$ explicitly, we compute the action $\mathbf{A}(\cdot)$ and $\mathbf{A}^T(\cdot)$ on-the-fly using the forward and backward substitution techniques.

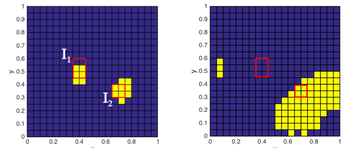
Inverse problem: The inverse problem seeks to reconstruct the sparse spatial distribution of fluorescence lifetime $\tau(\mathbf{r})$ from the set of Poisson noisy measurements in $\mathbf{u}$.

Step 1: We apply our nonconvex Poisson noise-based sparsity promoting method, to determine the spatial support, $\chi(\mathbf{r})$ from the time-averaged data (see Fig. 2(a)).

Step 2: Using the determined support $\chi(\mathbf{r})$ of the sources from Step 1, we apply the same Poisson recovery method to determine $Q(\mathbf{r},t)$ from the time-dependent measurements with a negligible regularization penalty parameter.

Step 3: Using $\chi(\mathbf{r})$ and $Q(\mathbf{r},t)$ from Steps 1 and 2, we apply Matlab's nonlinear least squares solver (`lsqnonlin`) to recover the fluorophore concentration $h(\mathbf{r})$ and the lifetime $\tau(\mathbf{r})$.

Numerical experiments: We used a unit square domain $\Omega = (0,1)\times(0,1)$ with non-dimensionalized optical properties: the absorption coefficient $\mu_a = 0.05$ and the diffusion coefficient $\kappa = 0.0476$. A sampling rate of 0.05 and 5 exterior near-infrared sources were used.



(a) Proposed approach (b) $\ell_2 - \ell_1$ minimization

	Ground Truth	Range of Estimate
$h(I_1)$	2.00×10^3	1.29×10^3 to 2.72×10^3
$h(I_2)$	2.00×10^3	0.58×10^3 to 0.95×10^3
τ	5.70	5.76

Figure 2: (a) Reconstructed support of the fluorophore islands (true support is shown by red boxes) using the Step 1. (b) Support from $\ell_2 - \ell_1$ minimization using the GPSR method [7].

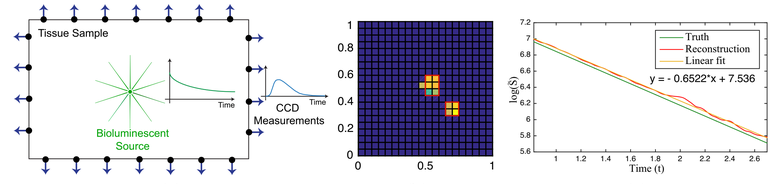
Application: Time-Dependent Bioluminescence Tomography

Goal: We seek to reconstruct sources of light contained within a tissue sample from noisy boundary measurements of scattered light. Compared to the fluorescence lifetime imaging problem, here we do not have an excitation source (see Fig. 3(a)). Therefore, we model only the emission light using the diffusion approximation.

Inverse problem: We propose two-stage based inverse approach [8]:

Step 1: Use our nonconvex Poisson noise-based sparsity promoting method to recover the support using the time-averaged data (see Fig. 3(b)).

Step 2: With the given support determined from Step 1, we recover the characteristic time decay using the time-dependent data (see Fig. 3(c)).



(a) (b) (c)

Figure 3: (a) Schematic diagram of time-dependent bioluminescence tomography. (b) Reconstructed support $(p = 0.3)$ (true support is shown by red boxes). (c) Approximated decay rate through a linear fit to the reconstruction is 1.53, while the true decay rate is 1.50.

Concluding Remarks: We developed and implemented a nonconvex sparsity promoting method that is able to solve time-dependent tomography problems such as fluorescence lifetime imaging and bioluminescence tomography, using a multistep process.

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