Sparse Eigenvectors of Graphs

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42nd International Conference on
Acoustics, Speech and Signal Processing

Caltech

Outline

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1. Graph Signal Processing
2. Motivation
3. Sparse Eigenvectors of Graphs
   - Disconnected Graphs
   - Connected Graphs
   - Generalizations
4. Real-World Examples
5. Conclusions

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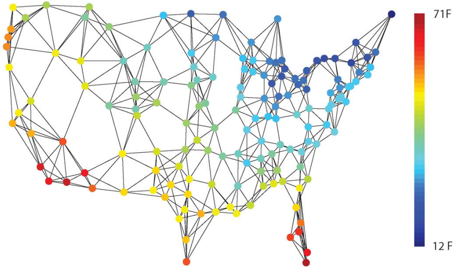
Preliminaries



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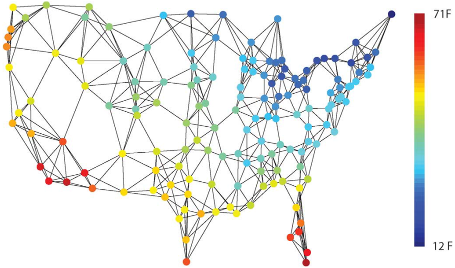

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Other selections[3]		

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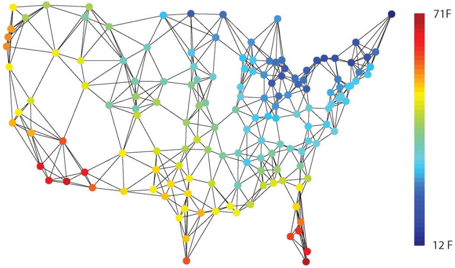

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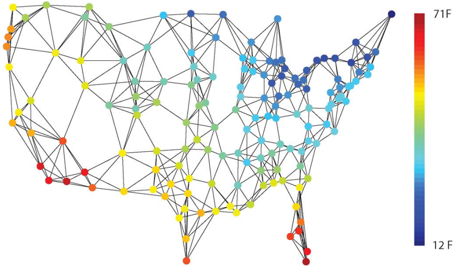

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Graph Fourier Basis : $\boldsymbol{V}$
Graph Fourier Transform : $\boldsymbol{F} = \boldsymbol{V}^{-1}$

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Sparse eigenvectors in *connected* graphs.

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Connected graph $\Longrightarrow$ No 1-sparse eigenvector!

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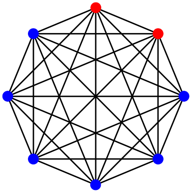

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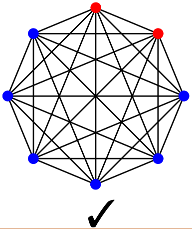

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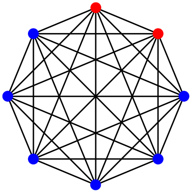

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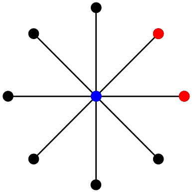

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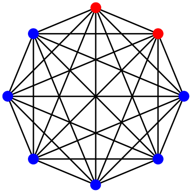

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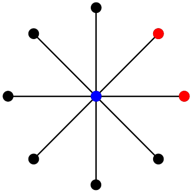

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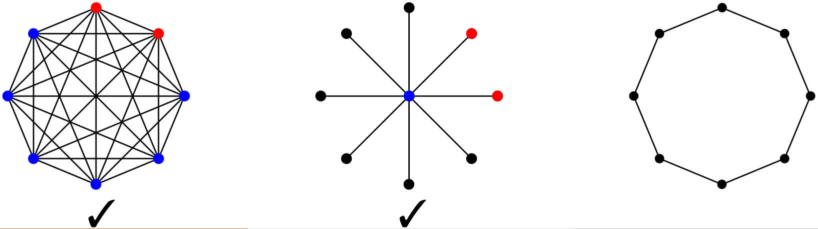

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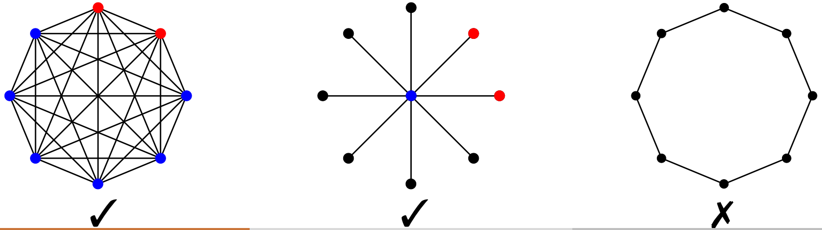

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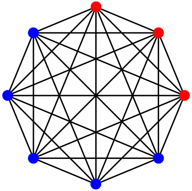

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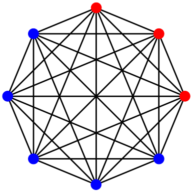

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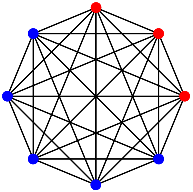

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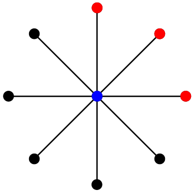

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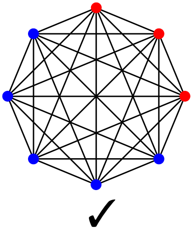

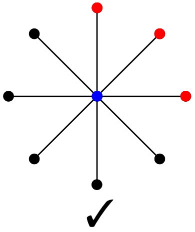

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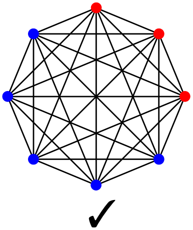

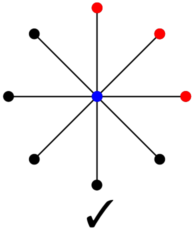

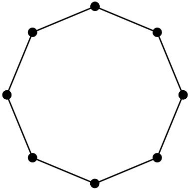

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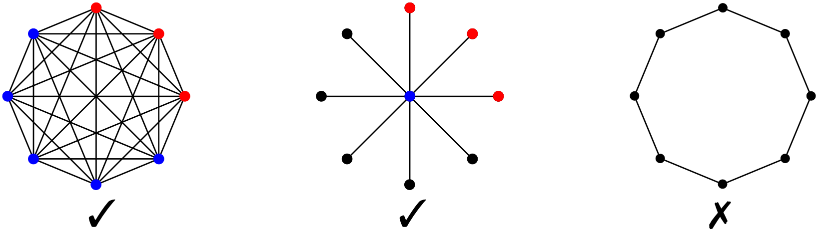

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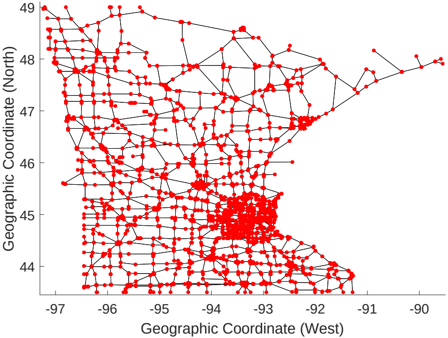
This is only sufficient, but not necessary.

(Counter-examples to follow)

Outline

Minnesota Road Graph



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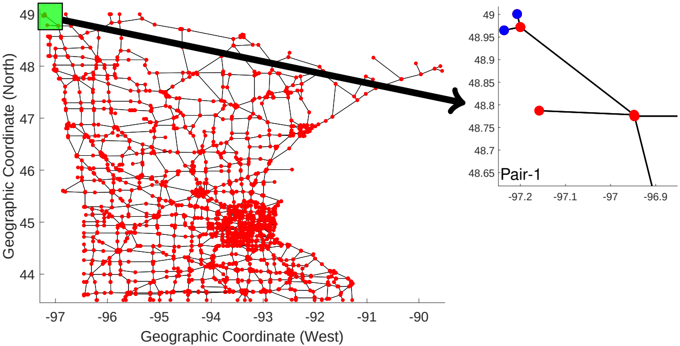
2-Sparse Case: $\mathcal{N}(i)\backslash\{j\} = \mathcal{N}(j)\backslash\{i\}$



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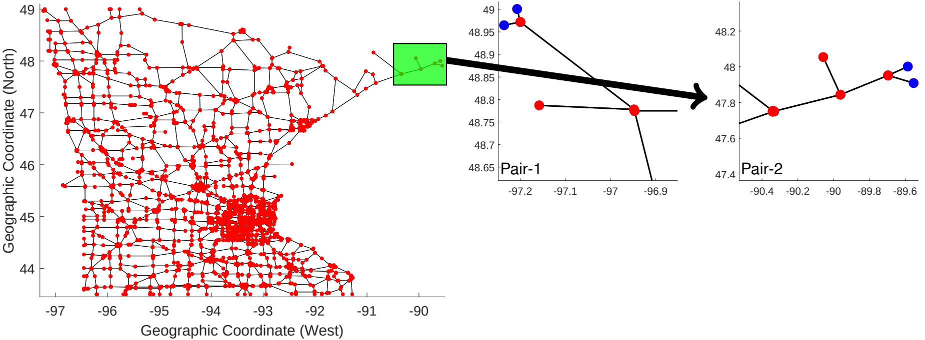

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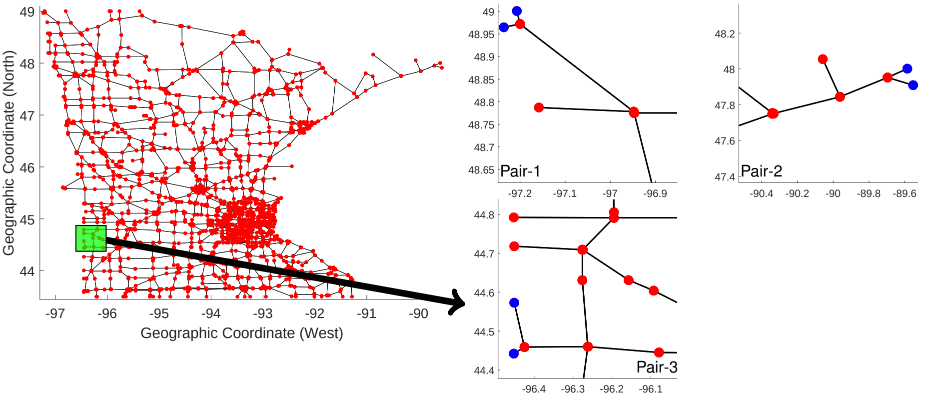

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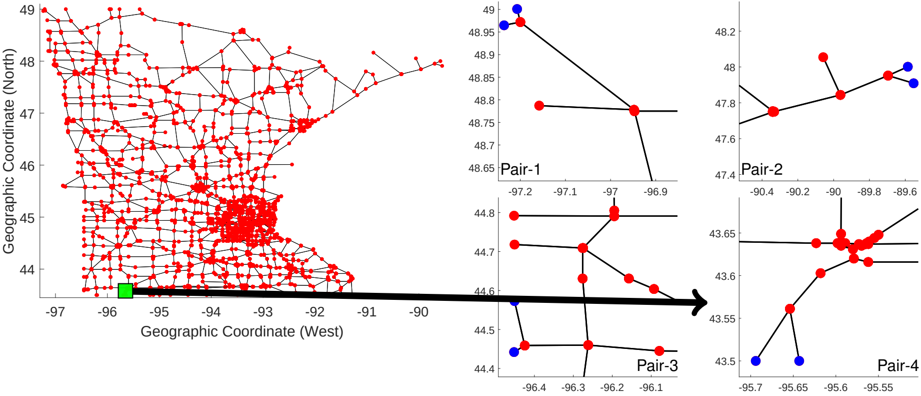

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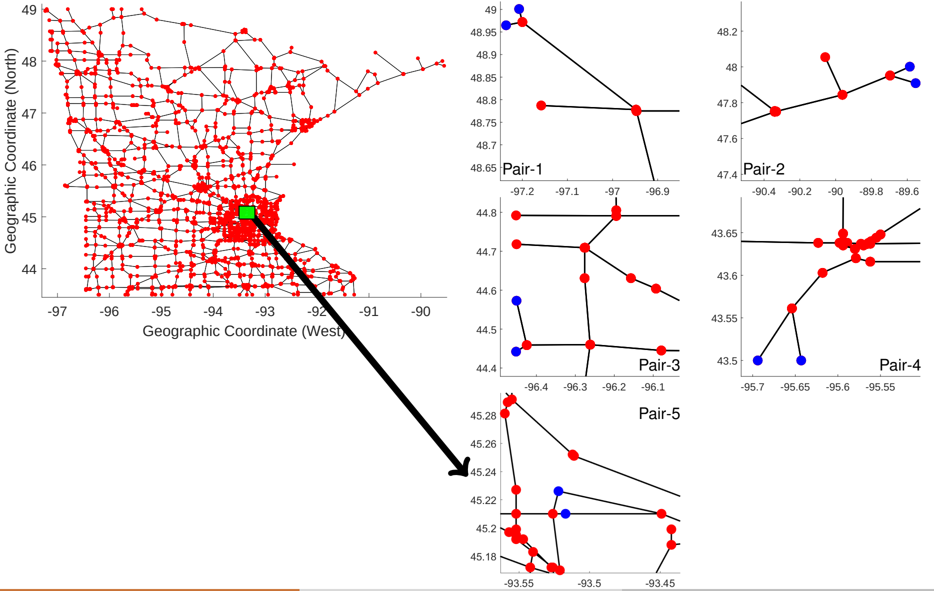

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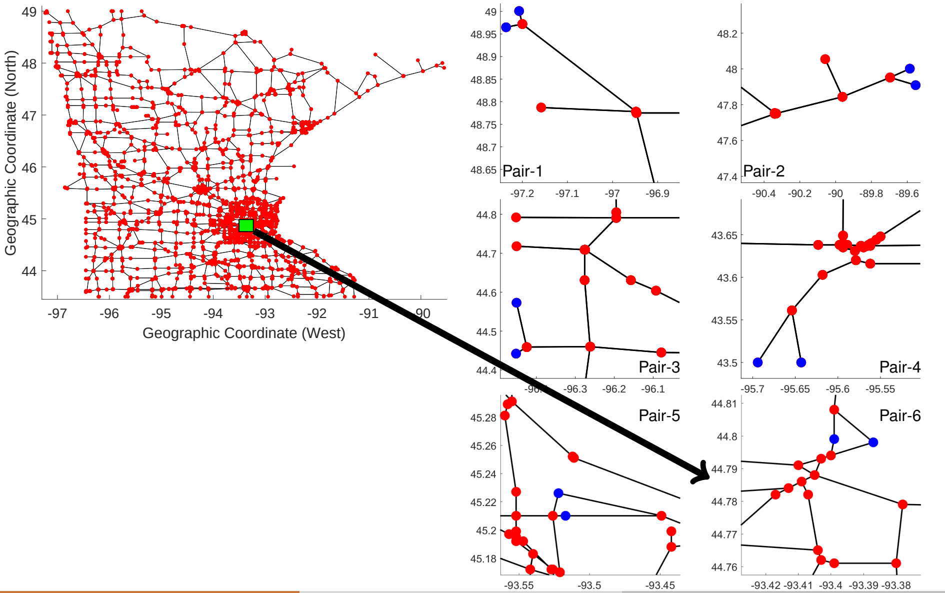

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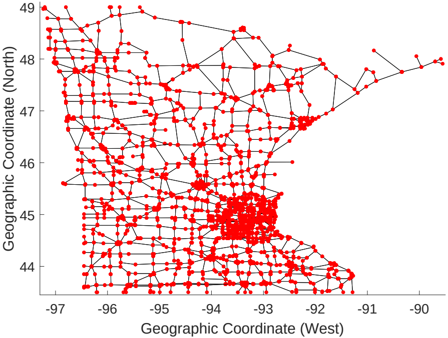

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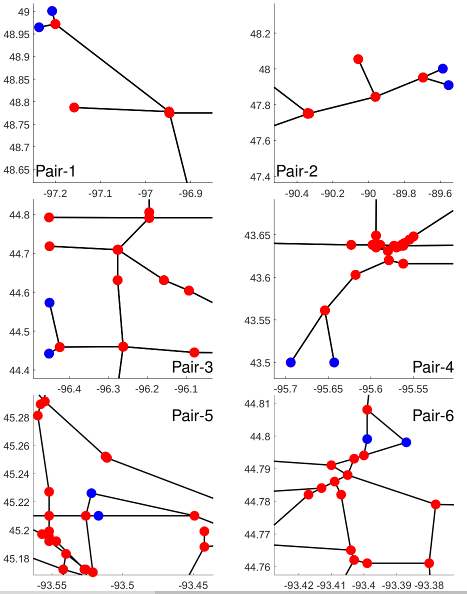

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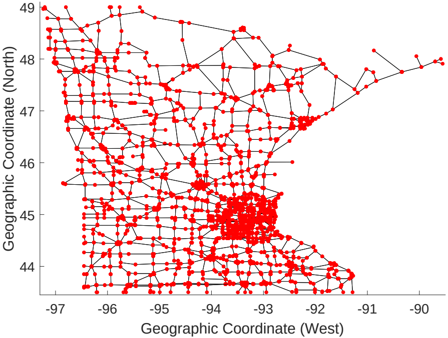

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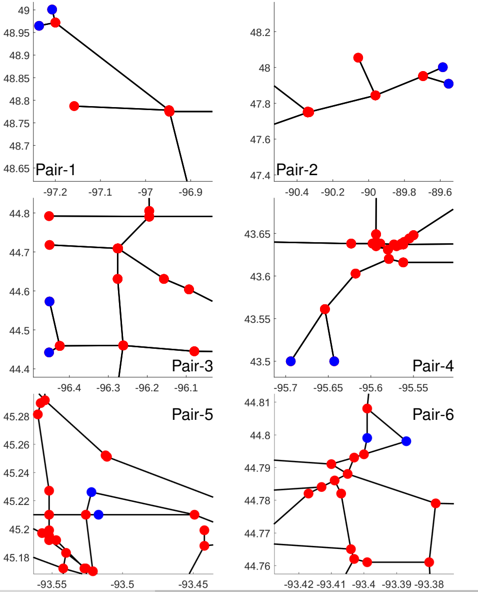

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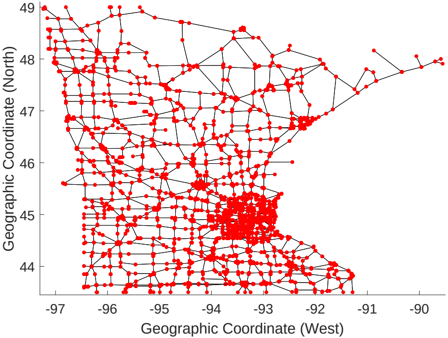

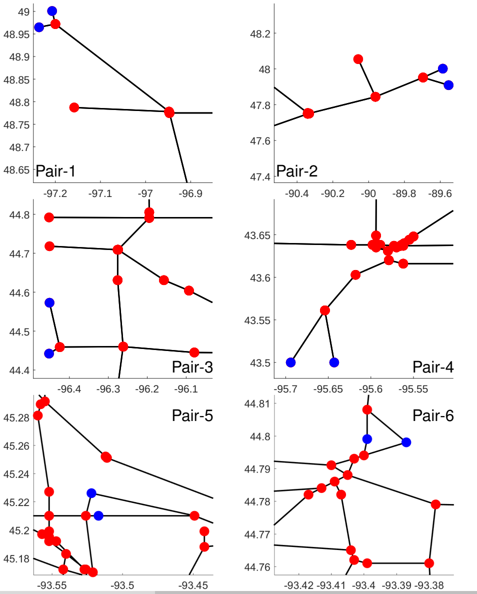

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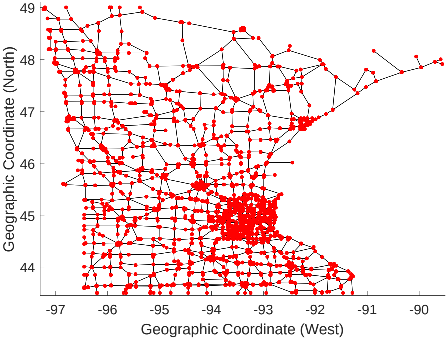

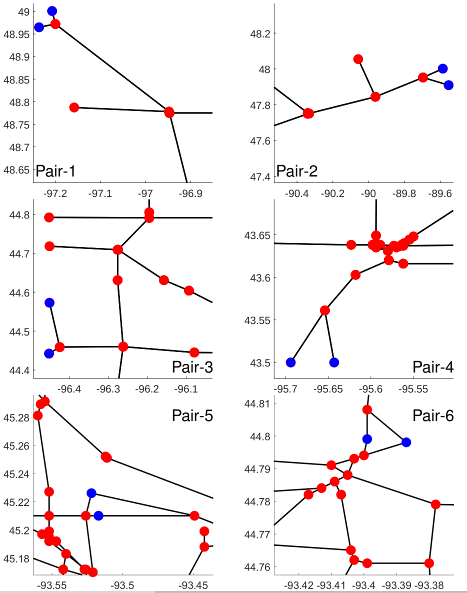

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Nodes = Characters (77)

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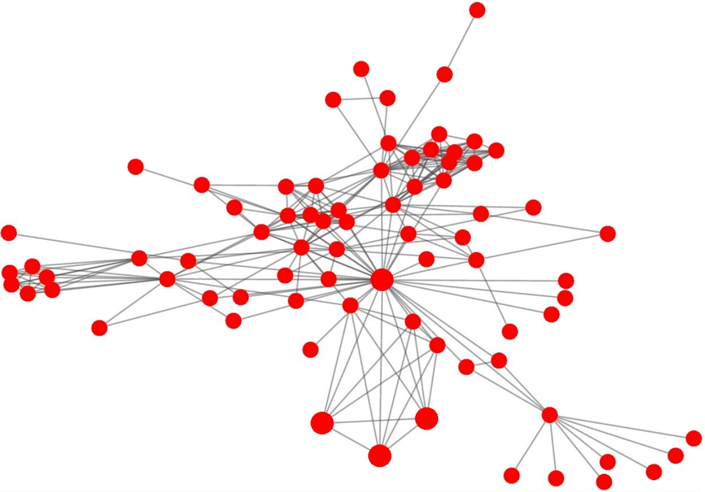

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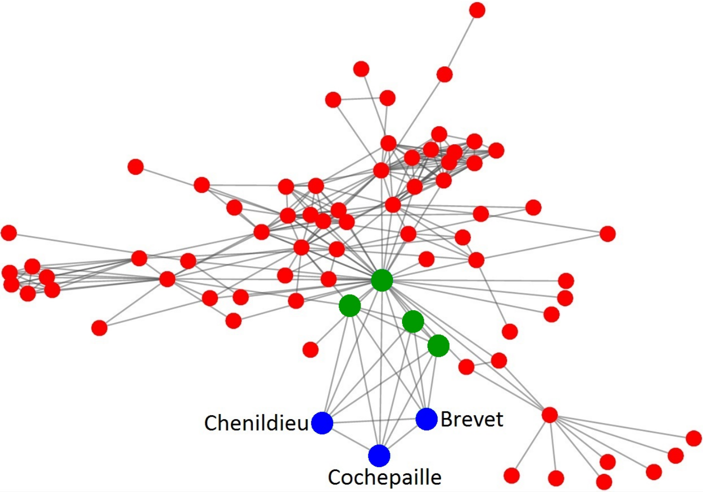

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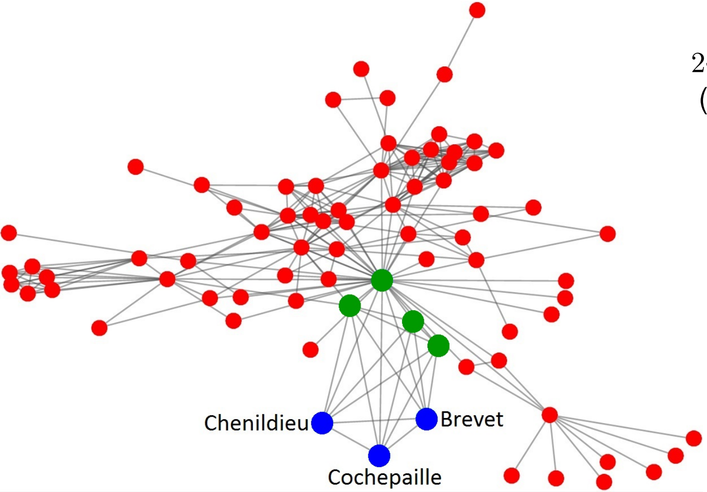

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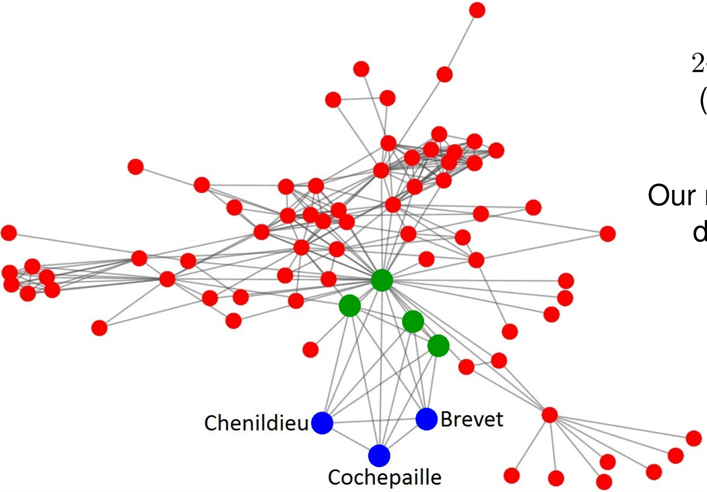

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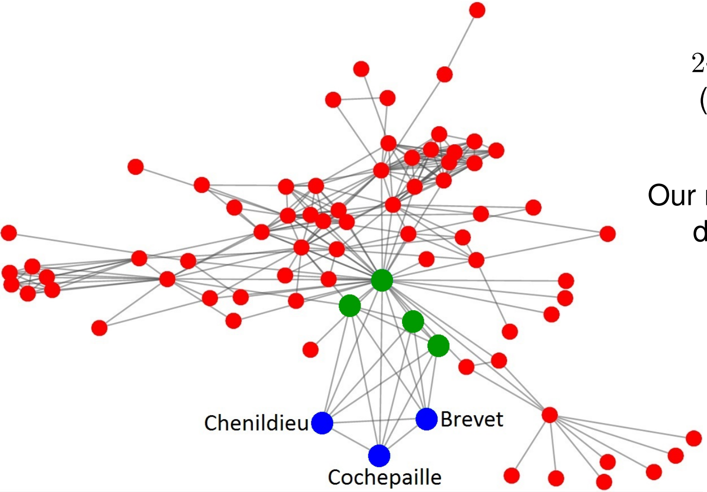

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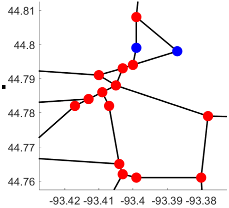

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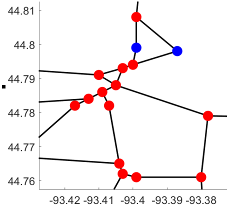

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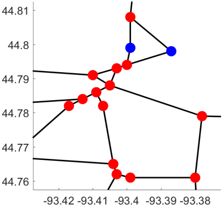

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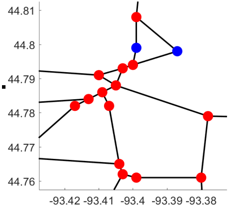

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Theorem

Assume the graph is simple and connected.
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Thank you!