New asymptotic properties for the robust ANMF

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Introduction and Motivations

In signal processing applications:

- complex multinormality is a poor approximation of the underlying physics
- non Gaussian data, missing data, outliers $\Longrightarrow$ robust estimators

A r. vect. $\mathbf{z}$ has a complex elliptically symmetric distribution $\mathbf{z} \sim \mathbb{C}ES_m(\boldsymbol{\mu}, \mathbf{M}, g_\mathbf{z})$, if its probability density function (PDF) can be written as

$$h_\mathbf{z}(\mathbf{z}) = |\mathbf{M}|^{-1} g_\mathbf{z}\big((\mathbf{z}-\boldsymbol{\mu})^H \mathbf{M}^{-1} (\mathbf{z}-\boldsymbol{\mu})\big) \quad (1)$$

where $h_\mathbf{z} : [0,\infty) \to [0,\infty)$ is any function such that (1) defines a PDF, $\boldsymbol{\mu}$ is the statistical mean and $\mathbf{M}$ is a scatter matrix (equal to the covariance matrix - CM - up to a scale factor).

- e.g., $g_\mathbf{x}(x) \propto e^{-x/2} \Longrightarrow$ Gaussian distribution $\mathbf{x} \sim \mathbb{C}N_m(\boldsymbol{\mu}, \mathbf{M})$

Let us consider an N-sample $(\mathbf{z}_1,...,\mathbf{z}_N)$ of i.i.d. vectors $\mathbf{z}_k \sim \mathbb{C}ES_m(\mathbf{0}, \mathbf{M}, g_\mathbf{z})$, $k = 1, ..., N$ with $\mathbf{z}_k = \sqrt{\tau_k}\mathbf{x}_k / \|\mathbf{x}_k\|$, where $\mathbf{x}_k \sim \mathbb{C}N_m(\mathbf{0}, \mathbf{M})$ and τ_k is an independent random variable. Let us consider the sample covariance matrix (SCM) $\widehat{\mathbf{M}}_{SCM}$ built with $(\mathbf{x}_1,...,\mathbf{x}_N)$

$$\widehat{\mathbf{M}}_{SCM} = \frac{1}{N}\sum_{k=1}^{N} \mathbf{x}_k \mathbf{x}_k^H \quad (2)$$

and the Tyler (TyE) estimator (also called fixed-point (FP) estimator) $\widehat{\mathbf{M}}_{FP}$ built with $(\mathbf{z}_1,...,\mathbf{z}_N)$

$$\widehat{\mathbf{M}}_{FP} = \frac{m}{N}\sum_{k=1}^{N} \frac{\mathbf{z}_k \mathbf{z}_k^H}{\mathbf{z}_k^H \widehat{\mathbf{M}}_{FP}^{-1} \mathbf{z}_k}. \quad (3)$$

Detection/estimation problem

For the detection/estimation problem, we focus on the Adaptive Normalized Matched Filter (ANMF):

$$H(\widehat{\mathbf{M}}) = \frac{|\mathbf{p}^H \widehat{\mathbf{M}}^{-1} \mathbf{y}|^2}{|\mathbf{p}^H \widehat{\mathbf{M}}^{-1} \mathbf{p}|\,|\mathbf{y}^H \widehat{\mathbf{M}}^{-1} \mathbf{y}|} \underset{H_1}{\overset{H_0}{\lessgtr}} \lambda, \quad (4)$$

where $\mathbf{p}$ is the steering vector, $\mathbf{y}$ is the observation under test and $\mathbf{M}$ is the CM estimator. The asymptotic distribution of (4) is given by

$$\sqrt{N}(H(\widehat{\mathbf{M}}) - H(\mathbf{M}))_\mathbf{z} \xrightarrow{d} N(0, \boldsymbol{\Sigma}_H) \quad (5)$$

where $\boldsymbol{\Sigma}_H$ is defined by

$$\boldsymbol{\Sigma}_H = 2\nu_1 H(\mathbf{M})(H(\mathbf{M}) - 1)^2. \quad (6)$$

where for the SCM $\nu_1 = 1$ and for TyE $\nu_1 = (m+1)/m$.
Assuming that the SCM is used, the theoretical relationship between the detection threshold and the Probability of False Alarm (PFA)is defined as

$$P_{fa} = P(H(\widehat{\mathbf{M}}) > \lambda | H_0) = (1-\lambda)^{a-1} {}_2F_1(a, a-1; b-1; \lambda) \quad (7)$$

where $K = \frac{(a-1)(m-1)}{N+1}$, $a = N - m + 2$, $b = N + 2$ and ${}_2F_1(\cdot)$ is the hypergeometric function.

GOAL: Derive the asymptotic distribution of the difference between the ANMF built with TyE and the one based on the SCM

Proposed result

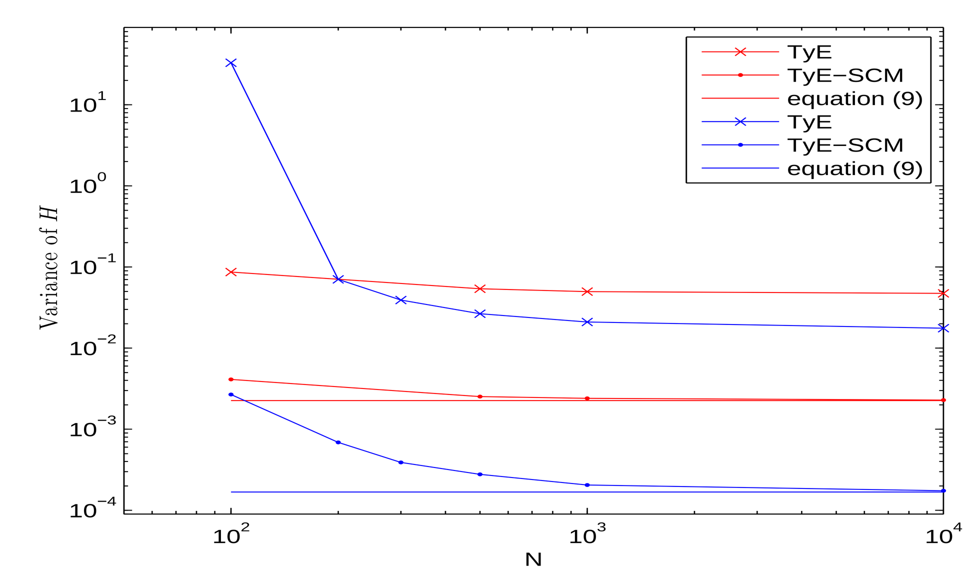
Let us consider the ANMF test defined by (4). Thus, conditionally to the distributions of $\mathbf{z}$, the asymptotic distribution of $H(\widehat{\mathbf{M}}_{FP}) - H(\widehat{\mathbf{M}}_{SCM})$ is

$$\sqrt{N}(H(\widehat{\mathbf{M}}_{FP}) - H(\widehat{\mathbf{M}}_{SCM}))_\mathbf{y} \xrightarrow{d} N(0, \boldsymbol{\Sigma}_T) \quad (8)$$

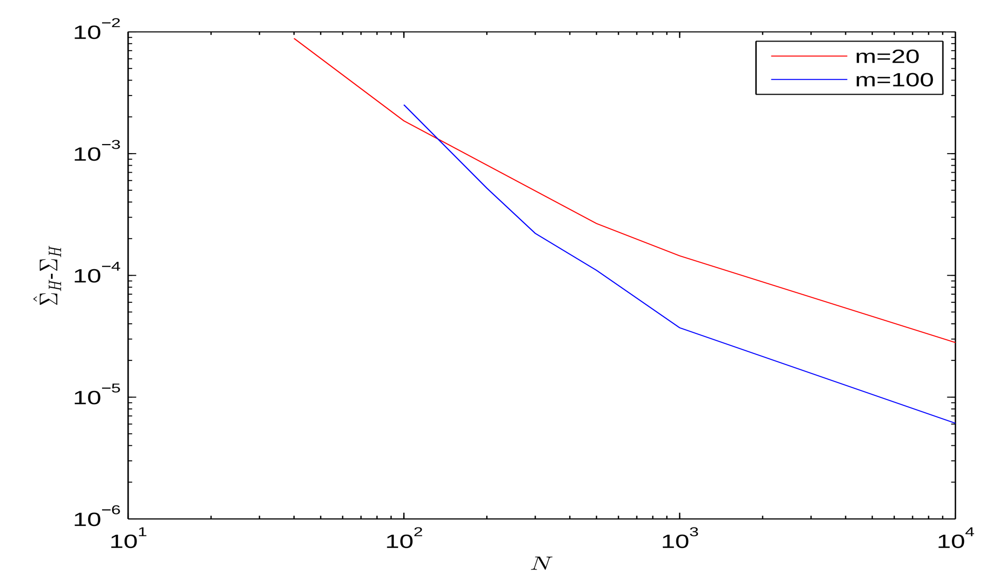
where

$$\boldsymbol{\Sigma}_T = \frac{2}{m} H(\mathbf{M})(H(\mathbf{M}) - 1)^2. \quad (9)$$

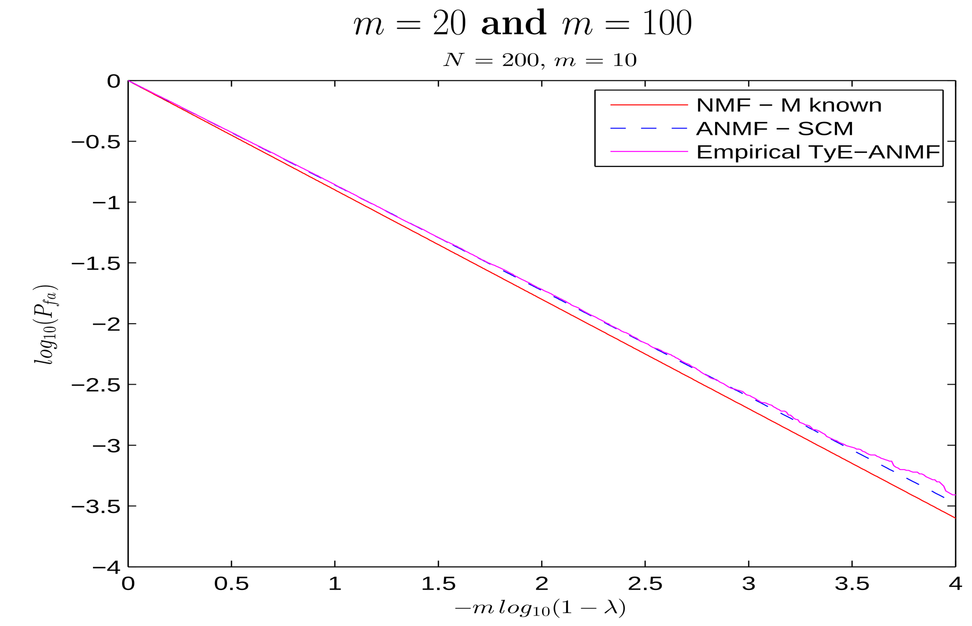
Simulation results



Empirical variance of the TyE-ANMF, of $var_N = \mathbf{var}(\sqrt{N}(H(\widehat{\mathbf{M}}_{FP}) - H(\widehat{\mathbf{M}}_{SCM})))$ and the theoretical result (Eq. (9)) for $m = 20$ (red) and $m = 100$ (blue)



Difference between $var_N = \mathbf{var}(\sqrt{N}(H(\widehat{\mathbf{M}}_{FP}) - H(\widehat{\mathbf{M}}_{SCM})))$ and the theoretical result (Eq. (9)) versus the number N of observations for $m = 20$ and $m = 100$



Comparison of PFA-threshold relationship for Tye-ANMF with the theoretical ones for NMF and SCM-ANMF

Remarks

The asymptotic variance in (9) is smaller than the ones of (6) when ν_1 is greater than one[a]. This result theoretically justifies that the behavior of $H(\widehat{\mathbf{M}}_{FP})$ is closer to $H(\widehat{\mathbf{M}}_{SCM})$ than to $H(\mathbf{M})$ ($\boldsymbol{\Sigma}_T < \boldsymbol{\Sigma}_H$). An important consequence is a better detection performance prediction when using $H(\widehat{\mathbf{M}}_{SCM})$ instead of $H(\mathbf{M})$.

[a] which is the case for all the considered CM estimators.

The asymptotic variance in (9) tends to 0 when the size m increases: for high dimensional observations, this approximation is more accurate since $\boldsymbol{\Sigma}_T \ll \boldsymbol{\Sigma}_H$. Interestingly, this is in agreement with recent results obtained using large random matrix theory in [4].

Conclusions and Perspectives

- The variance of the TyE-ANMF largely exceed the variance of the difference, which supports the idea of approximating the properties of TyE-ANMF with the theoretical ones of the SCM-ANMF (obtained in a Gaussian context)
- The variance of the difference decreases when the dimension m increases (in agreement with the theoretical result)
- The error decreases very fast as the number N of samples increases
- The PFA-threshold relationship for the TyE-ANMF is closer to the theoretical one for the SCM-ANMF than to the one for the NMF (test with the real covariance matrix)
- Conclusion: TyE behaves as the SCM in a Gaussian context, even when the sample are CES distributed $\Longrightarrow$ it allows one to **use the $H(\widehat{\mathbf{M}}_{FP})$ for detection purposes and to theoretically regulate parameters (e.g., the detection threshold) thanks to the $H(\widehat{\mathbf{M}}_{SCM})$ properties**

References

[1] Ollila, E. and Tyler, D.E. and Koivunen, V. and Poor, H.V., "Complex Elliptically Symmetric Distributions: Survey, New Results and Applications," *ICASSP-07IEEE Transactions on Signal Processing*, vol. 60, no. 11, pp. 5597-5625, November 2012.
[2] F. Pascal, and P. Forster, and J.-P. Ovarlez, and P. Larzabal, "Performance analysis of covariance matrix estimates in impulsive noise," *IEEE Transactions on Signal Processing*, vol. 56, no. 6, pp. 2206-2217, 2008.
[3] F. Pascal and J.-P. Ovarlez, "Asymptotic Properties of the Robust ANMF," *in Proc. ICASSP, Brisbane, Australia* vol. 3, pp. 1105-1108, April 2015.
[4] R. Couillet, and F. Pascal, and J. W. Silverstein, "The Random Matrix Regime of Maronna's M-estimator with elliptically distributed samples," *Journal of Multivariate Analysis* vol. 139, pp. 56-78, July 2015.