RECOVERY OF SPARSE SIGNALS VIA BRANCH AND BOUND LEAST-SQUARES

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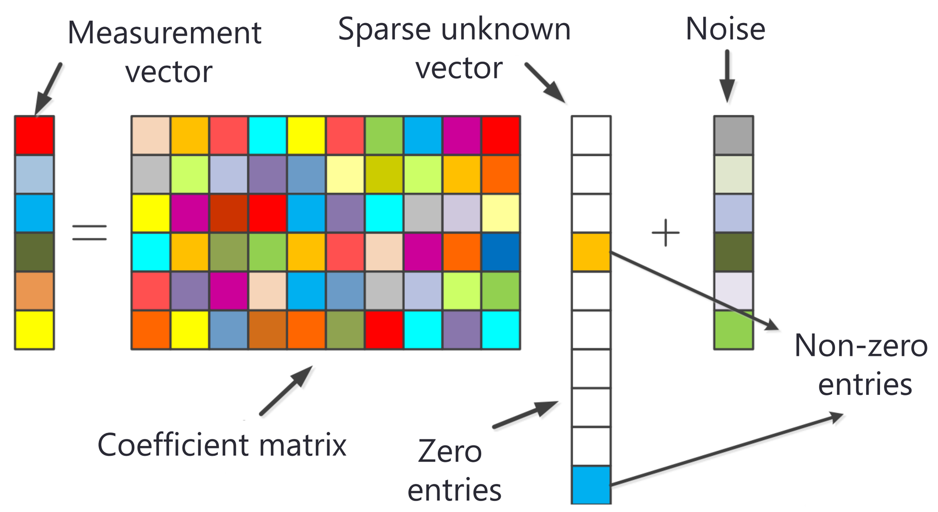
BACKGROUND

- Sparse Linear Regression:

$$\mathbf{y} = \mathbf{A}\boldsymbol{x} + \boldsymbol{\nu}$$

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \|\mathbf{y} - \mathbf{A}\boldsymbol{x}\|_2^2 \qquad \text{subject to} \quad \|\boldsymbol{x}\|_0 \le k$$



Existing Methods

- Convex Relaxation (e.g., LASSO)
- Tree Based (e.g., MMP)
- Greedy (e.g., OMP, OLS)

Tradeoff between Speed and accuracy

- OLS selection Criterion and update rule:

$$j_s = \arg\max_{j \in \mathcal{I}\backslash\mathcal{S}_{i-1}} \left| \mathbf{r}_{\mathcal{S}_{i-1}}^\top \frac{\mathbf{P}^\perp_{\mathcal{S}_{i-1}}\mathbf{a}_j}{\|\mathbf{P}^\perp_{\mathcal{S}_{i-1}}\mathbf{a}_j\|_2} \right| \qquad \mathbf{P}^\perp_{\mathcal{S}_i} = \mathbf{P}^\perp_{\mathcal{S}_{i-1}} - \frac{\mathbf{P}^\perp_{\mathcal{S}_{i-1}}\mathbf{a}_{j_s}\mathbf{a}_{j_s}^\top\mathbf{P}^\perp_{\mathcal{S}_{i-1}}}{\|\mathbf{P}^\perp_{\mathcal{S}_{i-1}}\mathbf{a}_{j_s}\|_2^2}$$

- Accelerated OLS (AOLS):

$$\mathbf{a}_j \leftarrow \mathbf{a}_j - \sum_{l=1}^{i} \frac{\mathbf{a}_j^\top\mathbf{u}_l}{\|\mathbf{u}_l\|_2^2}\mathbf{u}_l$$

$$\mathbf{q}_j = (\mathbf{a}_j^\top\mathbf{r}_i/\|\mathbf{a}_j\|_2^2)\,\mathbf{a}_j$$

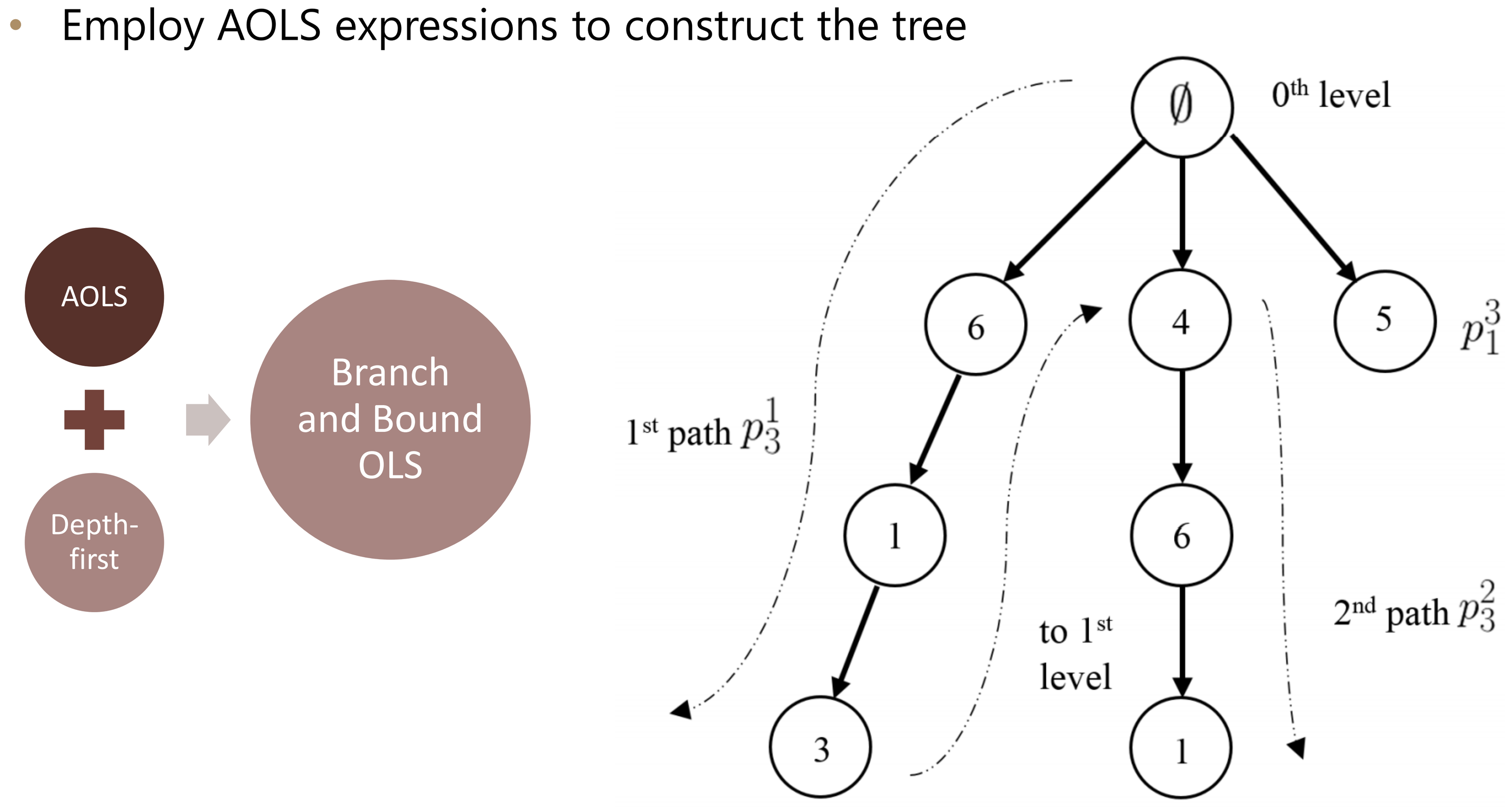
$$j_s = \text{argmax}_{j\in\mathcal{I}\backslash\mathcal{S}_i} \|\mathbf{q}_j\|_2$$

$$\mathbf{r}_{\mathcal{S}_i} = \mathbf{r}_{\mathcal{S}_{i-1}} - \mathbf{u}_{i+1}$$

$$\mathbf{u}_{i+1} = \mathbf{q}_{j_s}$$

BRANCH AND BOUND LEAST-SQUARES

- Construct a tree whose nodes represent columns of coefficient matrix
- A branch-and-bound search to traverse the tree in a depth-first manner
- Use a schedule $\mathbf{L} = [L_1, \ldots, L_k]$ to control the size of the search space
- Employ AOLS expressions to construct the tree



BRANCH AND BOUND LEAST-SQUARES

Algorithm 1 Branch and Bound Least-Squares (BBLS)

Input: $\mathbf{y}$, $\mathbf{A}$, sparsity level k, threshold ϵ, schedule $\mathbf{L}$, max number of paths N_p

Output: recovered support $\hat{\mathcal{S}}$, estimated signal $\hat{\boldsymbol{x}}$

1. (Initialize) $\mathcal{S} = \emptyset$, $\mathbf{r}_{p_0^\ell} = \mathbf{y}$, $r_{\ell_2} = \|\mathbf{y}\|_2$, $i = 1$, $\ell = 1$.
2. (Bounding) Let $\mathbf{S}_i = [\,]$ and $l_i = 0$,

for $j \in \mathcal{I}\backslash\mathcal{S}$ **do**

$$\mathbf{a}_j \leftarrow \mathbf{a}_j - \mathbb{I}(i>2)\frac{\mathbf{a}_j^\top\mathbf{u}_l}{\|\mathbf{u}_l\|_2^2}\mathbf{u}_l, \quad \mathbf{q}_j = \frac{\mathbf{a}_j^\top\mathbf{r}_{i-1}}{\|\mathbf{a}_j\|_2^2}\mathbf{a}_j$$

end for

Select $\mathbf{S}_i = [j_{s_1}, \ldots, j_{s_{L_i}}]$ corresponding to L_i largest terms $\|\mathbf{q}_j\|_2$

3. (Branching) $l_i = l_i + 1$. If $l_i > L_i$ go to 4, else $\mathcal{S} = \mathcal{S} \cup \{\mathbf{S}_i(l_i)\}$, $\mathbf{u}_i = \mathbf{q}_{j_{s_{l_i}}}$, $\mathbf{r}_{p_i^\ell} = \mathbf{r}_{p_{i-1}^\ell} - \mathbf{u}_i$, go to 5.
4. (Decrease i) If $i = 1$ go to 7, else $\mathcal{S} = \mathcal{S}\backslash\{\mathbf{S}_i(l_i)\}$, $i = i-1$, and go to 2.
5. (Increase i) If $i = k$ go to 6, else $i = i+1$ and go to 2.
6. (Solution found) Save the ℓ^{th} path $p_k^\ell = \mathcal{S}$ and its objective value $\left\|\mathbf{r}_{p_k^\ell}\right\|_2$. If $\left\|\mathbf{r}_{p_k^\ell}\right\|_2 < r_{\ell_2}$ update $r_{\ell_2} = \left\|\mathbf{r}_{p_k^\ell}\right\|_2$. $\ell = \ell+1$, if $\ell > N_p$ or $r_{\ell_2} < \epsilon$ go to 7, else go to 3.
7. Terminate the algorithm. Return the path $p_k^{\ell^*}$ with minimum residual norm as $\hat{\mathcal{S}}$, and the estimate $\hat{\boldsymbol{x}} = \mathbf{A}_{\hat{\mathcal{S}}}^\dagger\mathbf{y}$.

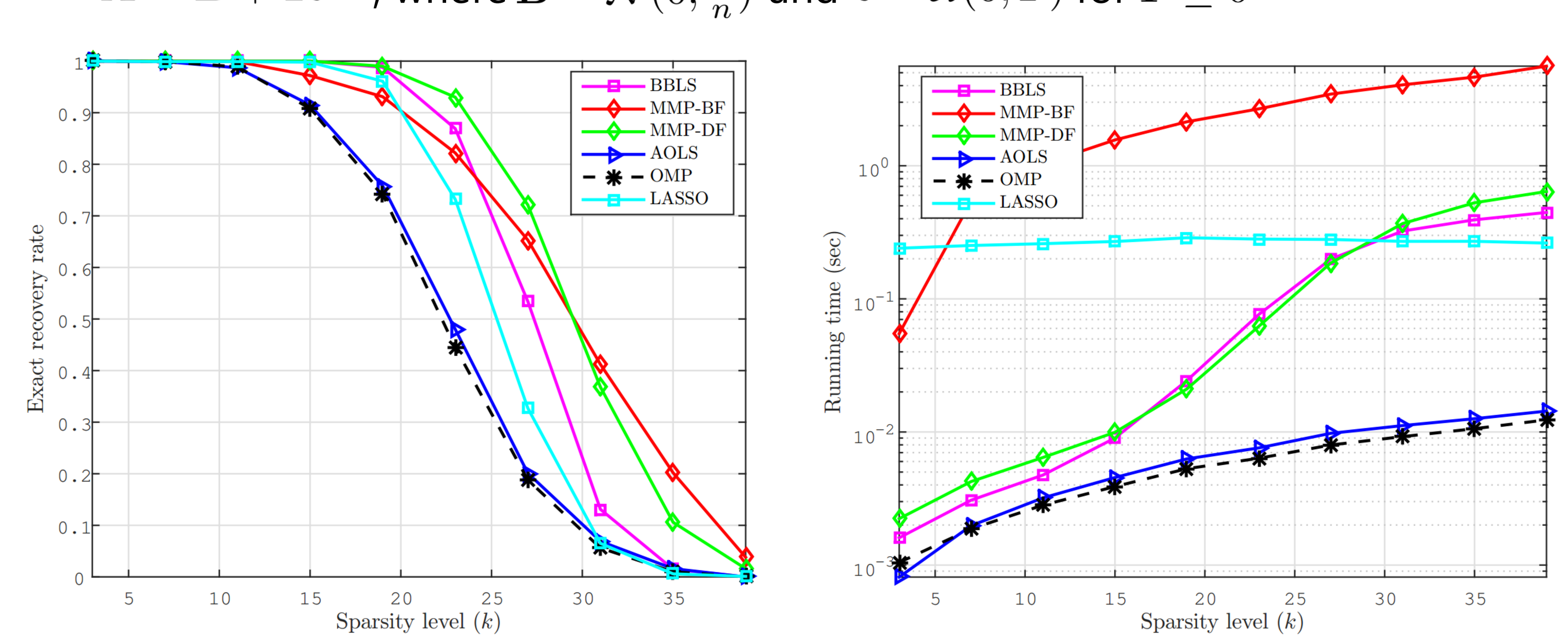
PERFORMANCE ANALYSIS

- Let $0 < \epsilon < 1$ and $0 < \delta < 1$ be universal constants, and $c_0(\epsilon) = \frac{\epsilon^2}{6}(1-\epsilon)$
- Assume $\mathbf{A} \sim \mathcal{N}(0, 1/n)$ or $\mathbf{A} \sim \mathcal{B}(\frac{1}{2}, \pm\frac{1}{\sqrt{n}})$, and noiseless measurements
- If $p_i^\ell = \{s_1^\ell, \ldots, s_i^\ell\} \subset \mathcal{S}_{true}$, then, at least one among L_{i+1} children of s_i^ℓ is in $\mathcal{S}_{true}$ with probability

$$p \ge \left(1 - 2e^{-(n-i)c_0(\epsilon)}\right)^2 \left(1 - 2(\frac{12}{\delta})^k e^{-nc_0(\frac{\delta}{2})}\right)\left(1 - 2e^{-\frac{n}{k-i}\frac{1-\epsilon}{1+\epsilon}(1-\delta)^2}\right)^{m-k-L_{i+1}+1}$$

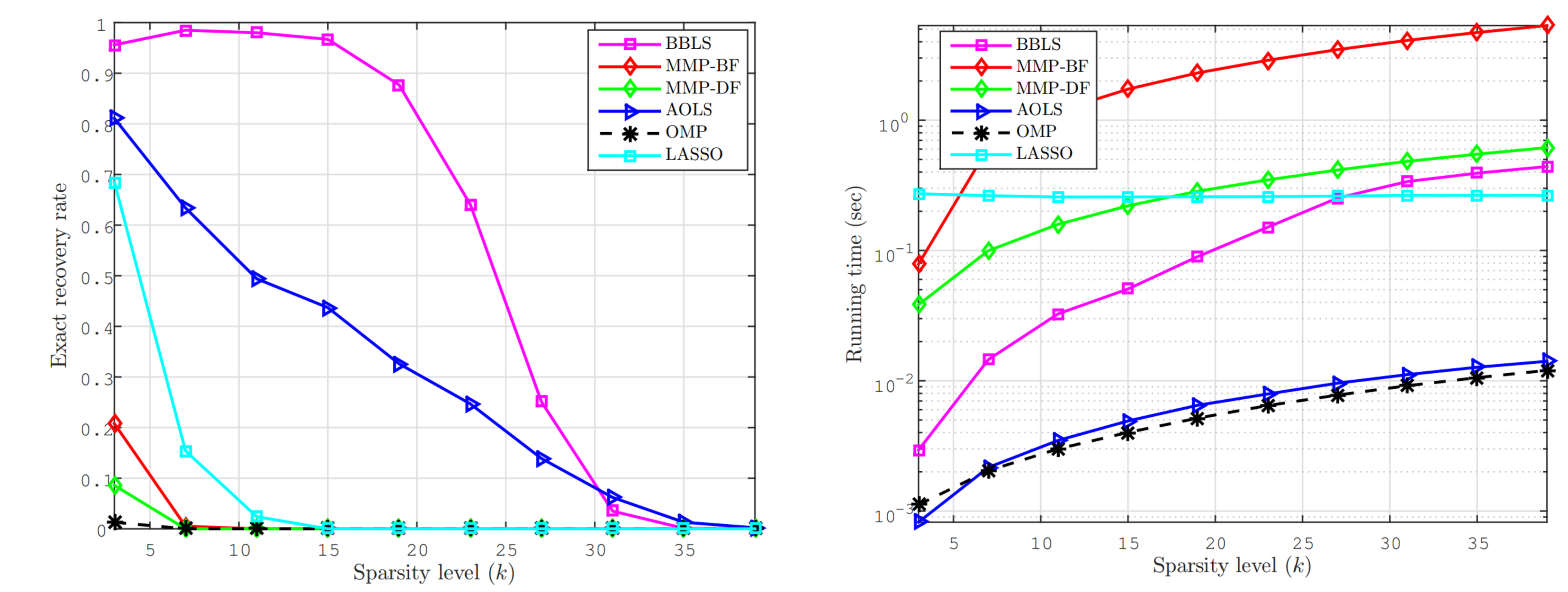
SIMULATION RESULTS

- n = 64, m = 128, $\boldsymbol{x} \sim \mathcal{N}(0,1)$, vary k for 1000 independent instances
- $\mathbf{A} = \mathbf{B} + \mathbf{1}\mathbf{t}^\top$, where $\mathbf{B} \sim \mathcal{N}(0, \frac{1}{n})$ and $\mathbf{t} \sim \mathcal{U}(0, T)$ for $T \ge 0$

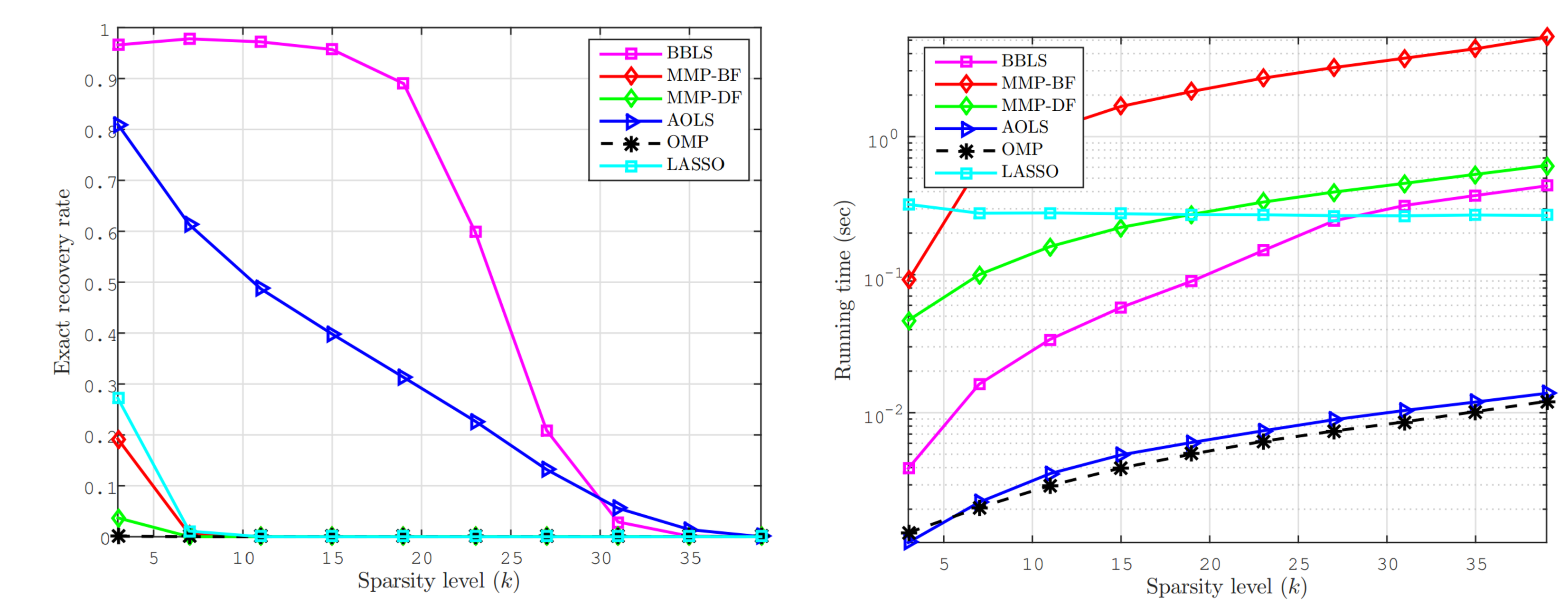


Comparison results for incoherent matrix (T=0)

SIMULATION RESULTS



Comparison results for coherent matrix (T=10)



Comparison results for highly coherent matrix (T=100)

CONCLUSIONS

The proposed algorithm, BBLS:

- is a depth-first search scheme for sparse reconstruction.
- selects different number of indices in each level according to a schedule.
- has guarantees for its achievable reconstruction probability.
- is capable of highly accurate recovery even for correlated dictionaries.

Future work: performance analysis under hybrid dictionaries.

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