An M-Channel Critically Sampled Graph Filter Bank

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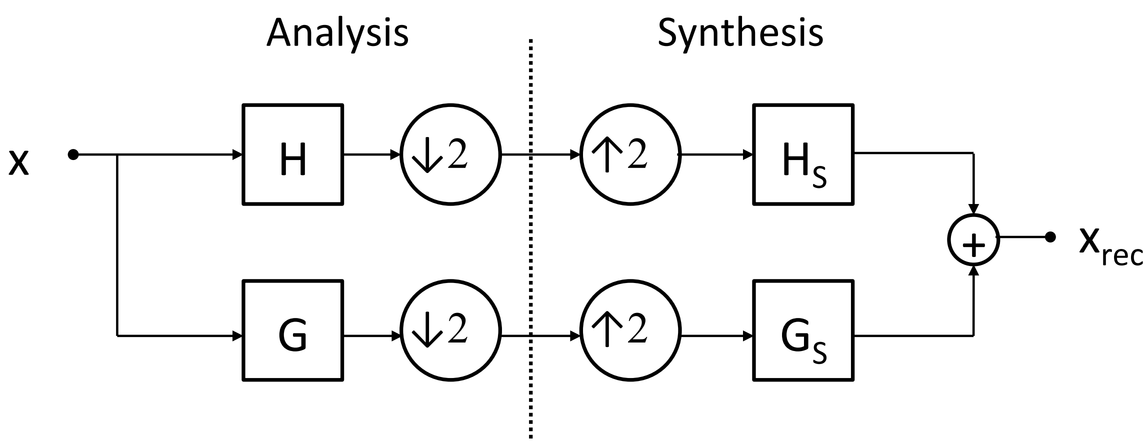
Designing Graph Spectral Filter Banks

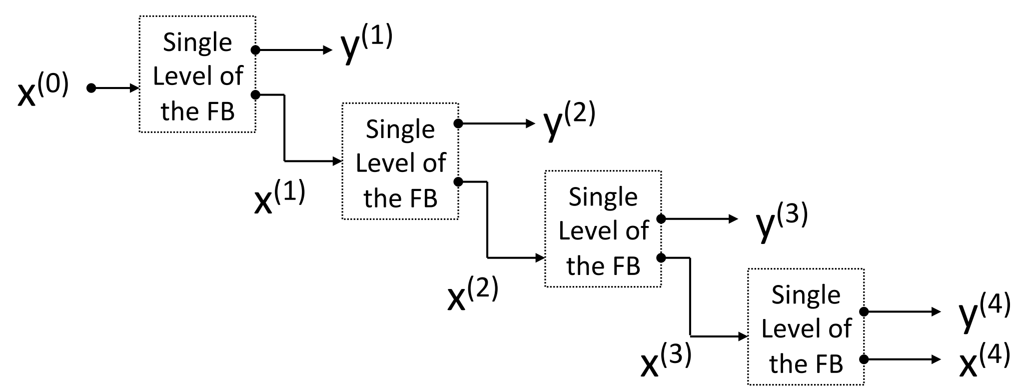
Classical 2-Channel Critically Sampled Filter Bank



For irregular graphs, it is difficult to generalize conditions on filters ensuring properties such as perfect reconstruction, orthogonality

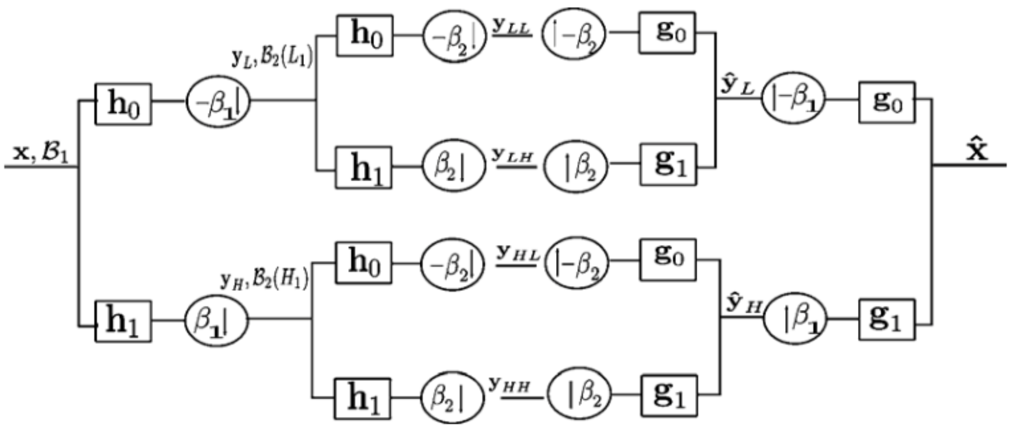
Iterating Low Pass Branch Yields Wavelets

Need appropriate notions of downsampling, upsampling, filtering, graph reduction that preserve a meaningful correspondence between filtering at different resolution levels



Approach 1: Decompose into Structured Subgraphs

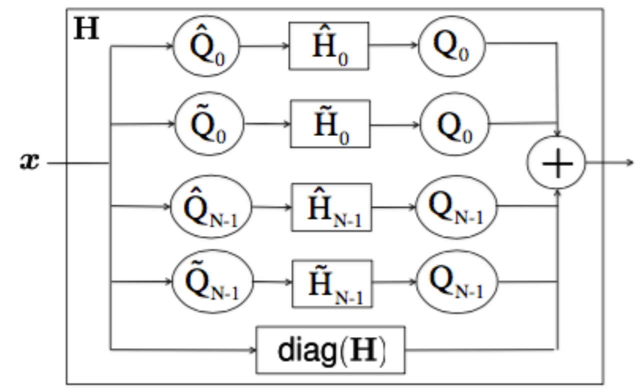
Bipartite Subgraph Decomposition



Source: Narang and Ortega, TSP, 2012

- Narang and Ortega, Perfect reconstruction two-channel wavelet filter banks for graph structured data, TSP, 2012
- Narang and Ortega, Compact support biorthogonal wavelet filterbanks for arbitrary undirected graphs, TSP, 2013

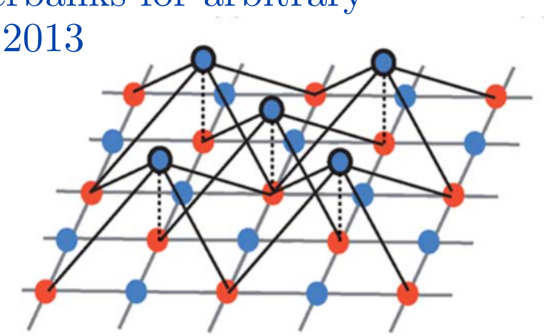
Circulant Subgraph Decomposition



Source: Ekambaram, Ph.D. Thesis, 2013

- Enkambaram et al., Critically-sampled perfect reconstruction spline-wavelet filterbanks for graph signals, GlobalSIP, 2013
- Kotzagiannidis and Dragotti, The graph FRI framework - spline wavelet theory and sampling on circulant graphs, ICASSP, 2016

Oversampled Filter Bank

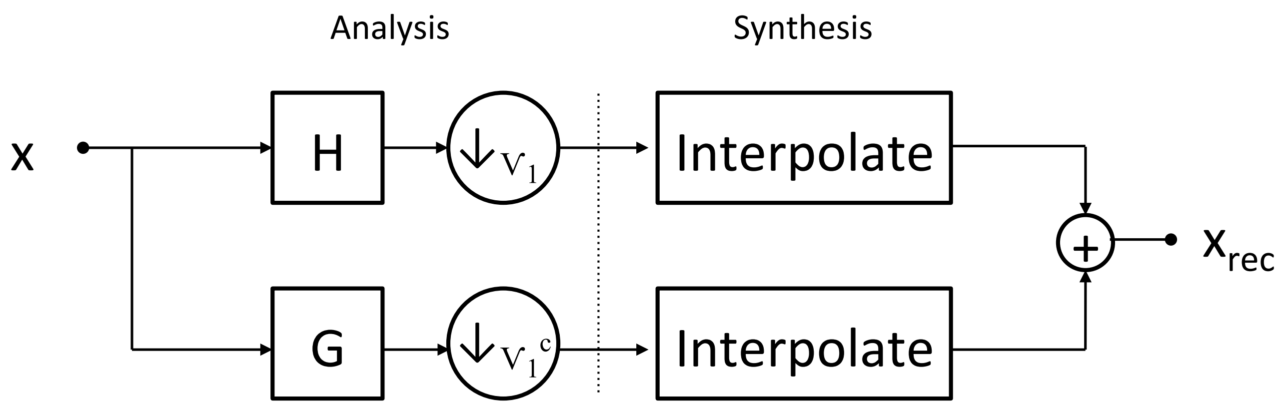


Source: Sakiyama and Tanaka, 2014

- Sakiyama and Tanaka, Oversampled graph Laplacian matrix for graph filter banks, TSP, 2014

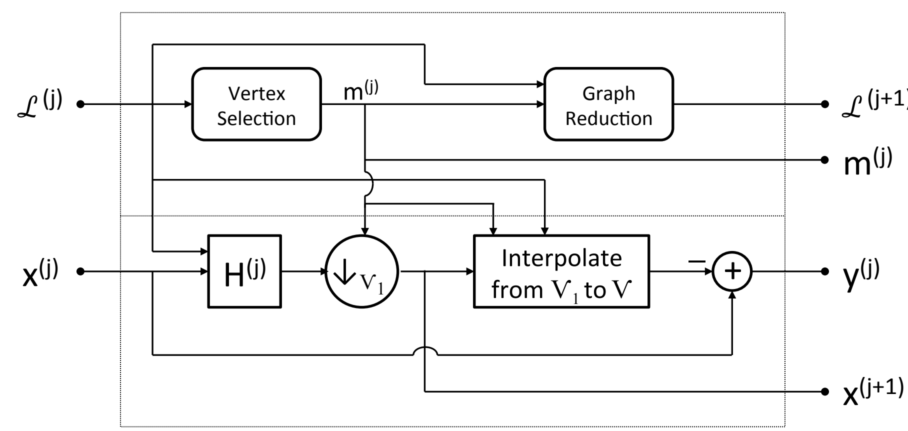
Approach 2: Replace Upsampling and Synthesis Filters with Interpolation Operators

Synthesis Via Interpolation



Chen et al., Discrete signal processing on graphs: sampling theory, TSP, 2015

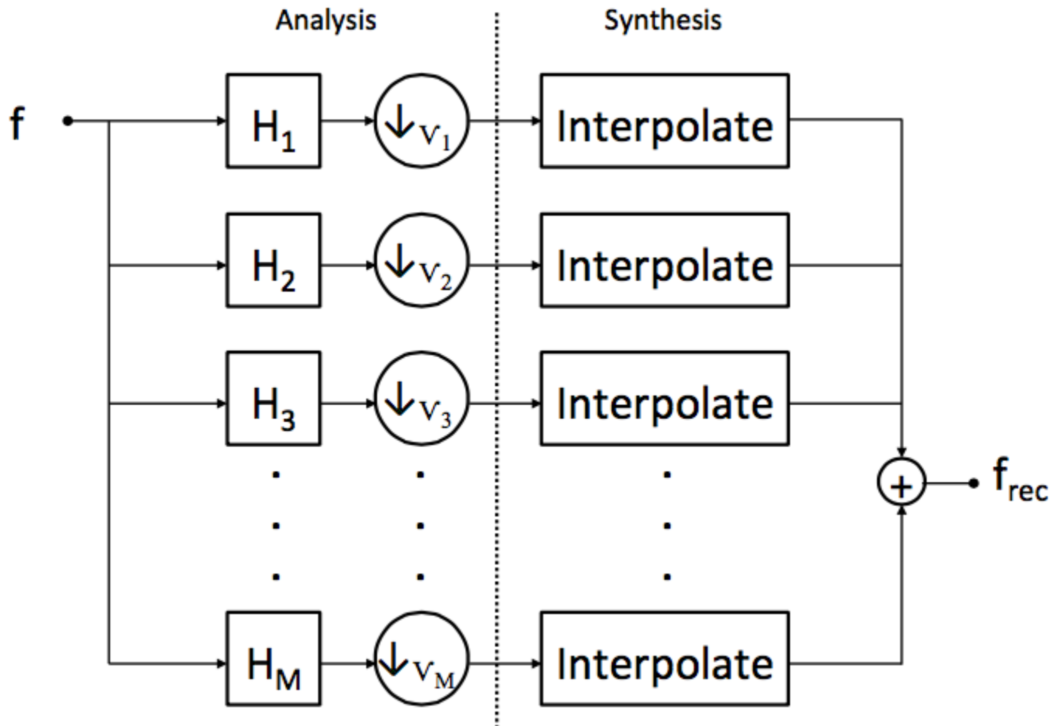
Pyramid



Shuman et al., A multiscale pyramid transform for graph signals, TSP, 2016

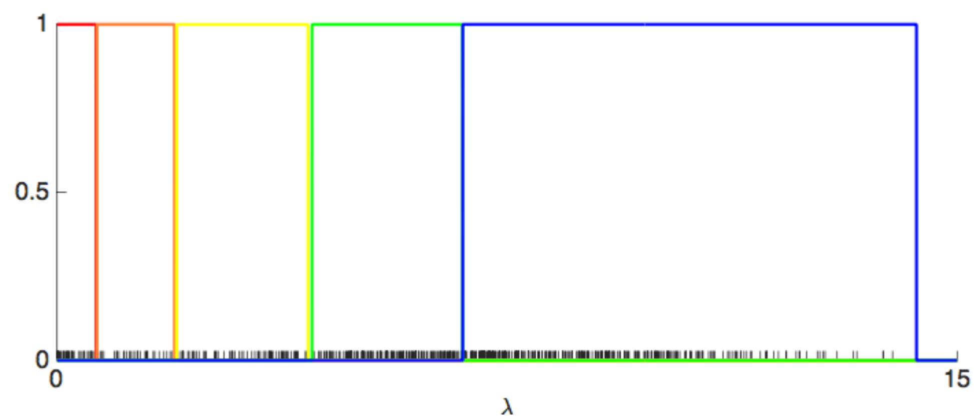
M-Channel Critically Sampled Graph Filter Bank

Architecture



- Number of vertices in V_i is equal to the number of eigenvalues in the support of the corresponding filter

Ideal Filter Bank



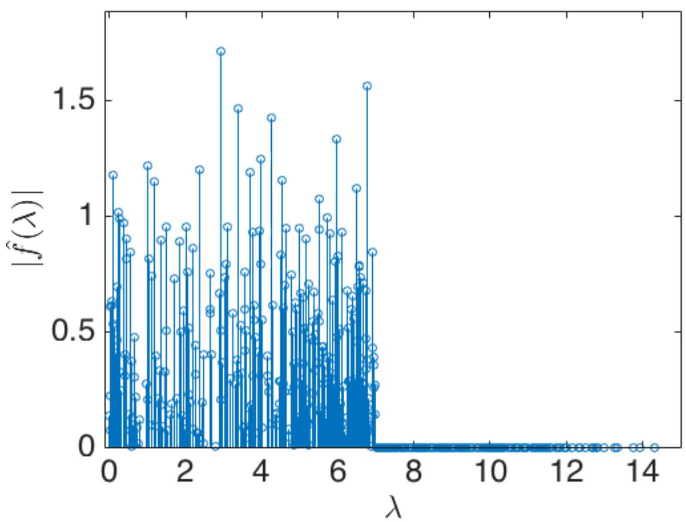
Sampling and Interpolation

- How to sample a graph signal and interpolate from the samples?
- Subset V_s of vertices is a uniqueness set for a subspace P iff:

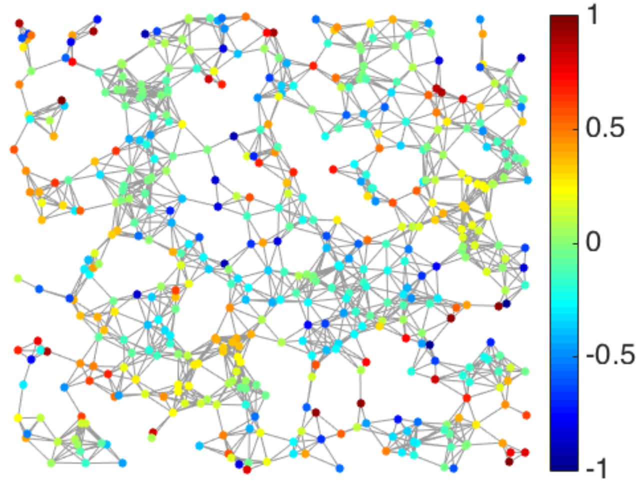
  If two signals in the subspace P have the same values on the vertices in the uniqueness set, then they are the same signal

I. Pesenson, "Sampling in Paley-Wiener spaces on combinatorial graphs," Trans. Amer. Math. Soc., 2008

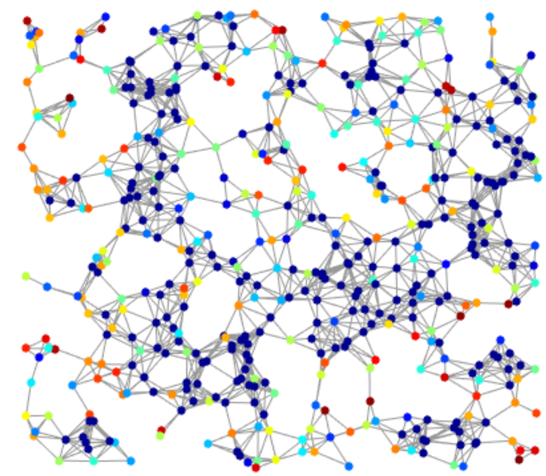
Graph Spectral Domain



Vertex Domain



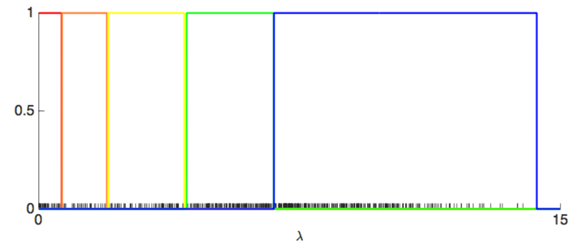
Signal Sampled on a Uniqueness Set for col($U_{1:266}$)

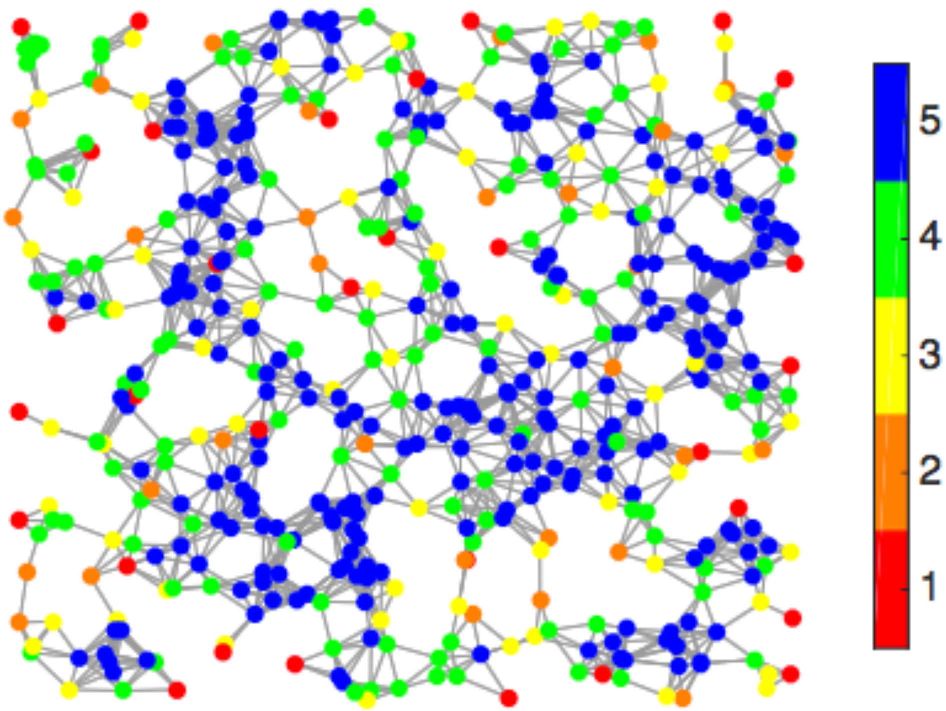


- Interpolation (noiseless case):

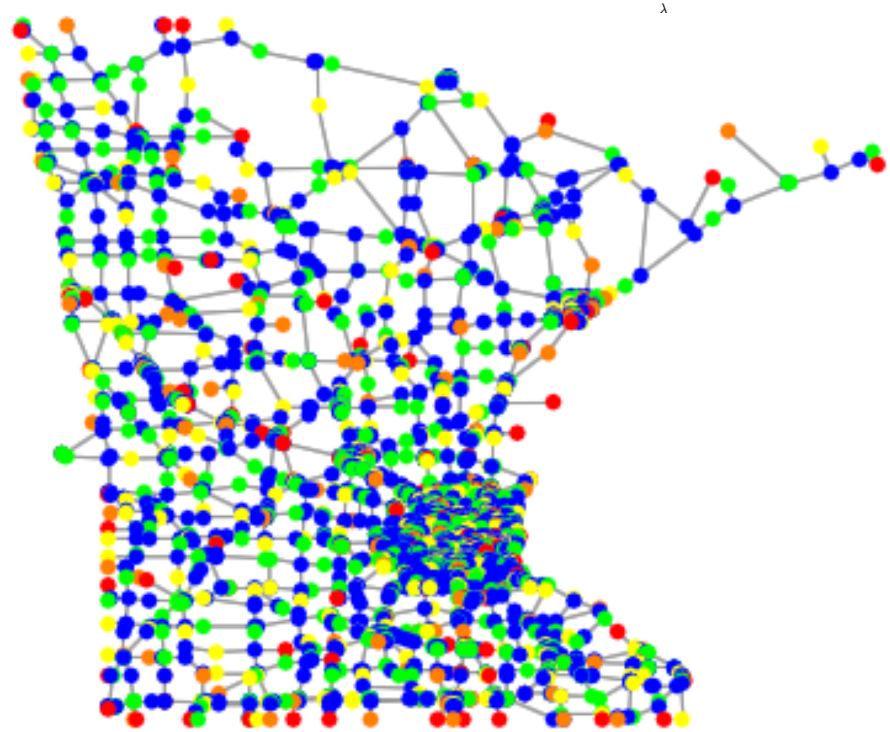
$$f_{\text{rec}} = U_{1:266}\alpha, \text{ where } f_S = U_{S,1:266}\alpha$$

Objective: Partition into M Uniqueness Sets for Ideal Filter Bank Subspaces





500-node Random Sensor Network



Minnesota Road Network

Algorithm to Create Uniqueness Set Partitions *Case 1: M=2*

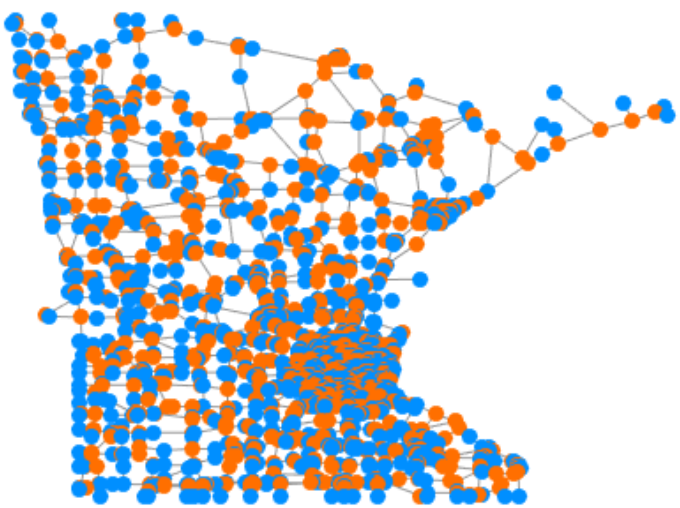
- Goal: Find a permutation matrix P such that the submatrices along the diagonal of PU are full rank:

$$PU = \tilde{U} = \left[\begin{array}{c|c} \tilde{U}_1 & \\ \hline & \tilde{U}_2 \end{array}\right]$$

- Proposition: If M=2 and the space spanned by first k columns of U is orthogonal to the space spanned by last N-k columns, then S is a uniqueness set for $U_{1:k}$ if and only if S^c is a uniqueness set for $U_{k+1:N}$

- Steinitz exchange lemma guarantees that we can find such a permutation

- Equivalently, we can find two complementary uniqueness sets for the corresponding spectral subspaces



Finding a Single Uniqueness Set

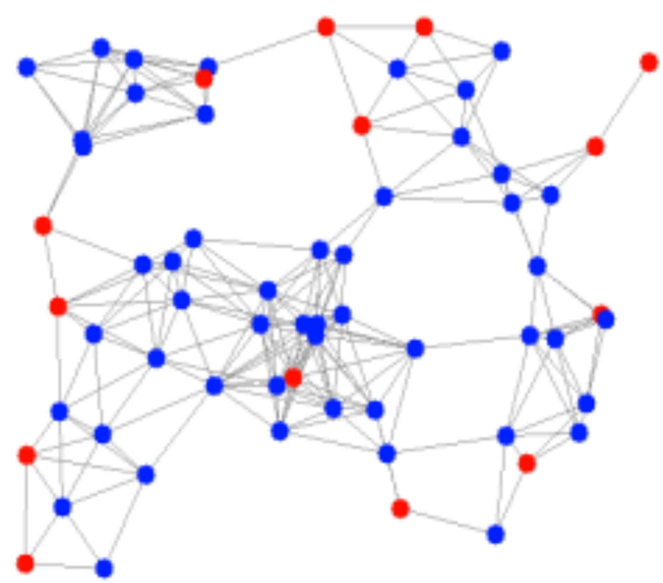
Numerous algorithms have been proposed recently

- Shomorony and Avestimehr, Sampling large data on graphs, GlobalSIP, 2014
- Chen et al., Discrete signal processing on graphs: Sampling theory, TSP, 2015
- Anis et al., Efficient sampling set selection for bandlimited graph signals using graph spectral proxies, TSP, 2016
- Puy et al., Random sampling of bandlimited signals on graphs, ACHA, 2016

Different objectives: minimal set size, speed, recovery robustness to noise

Algorithm 1 (Shomorony and Avestimehr (2014)) Compute the smallest $\mathcal{S}$ with designated bandwidth k

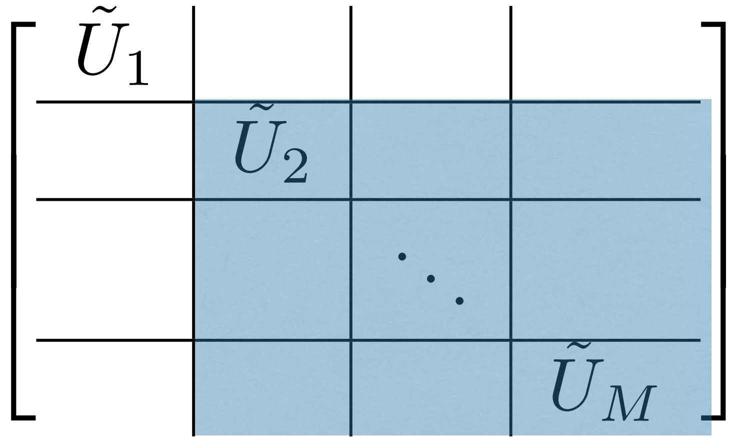
Input $\{u_1, u_2, \cdots, u_m\}$
$\mathcal{S} \leftarrow \emptyset$
$b_1, \cdots, b_n \leftarrow \delta_1, \cdots, \delta_n$
for $u = u_1, \cdots, u_m$ **do**
  Write u as $u = \sum_{i=1}^{n} \alpha_i b_i$
  $l \leftarrow \arg\max_{i \notin \mathcal{S}} |\alpha_i|$
  $\mathcal{S} \leftarrow \mathcal{S} \cup \{l\}$
  $b_l \leftarrow u$
end for
Output $\mathcal{S}$



Algorithm to Create Uniqueness Set Partitions
Case 2: M>2

- Goal: Find permutation P s.t.

$$PU = \tilde{U} =$$



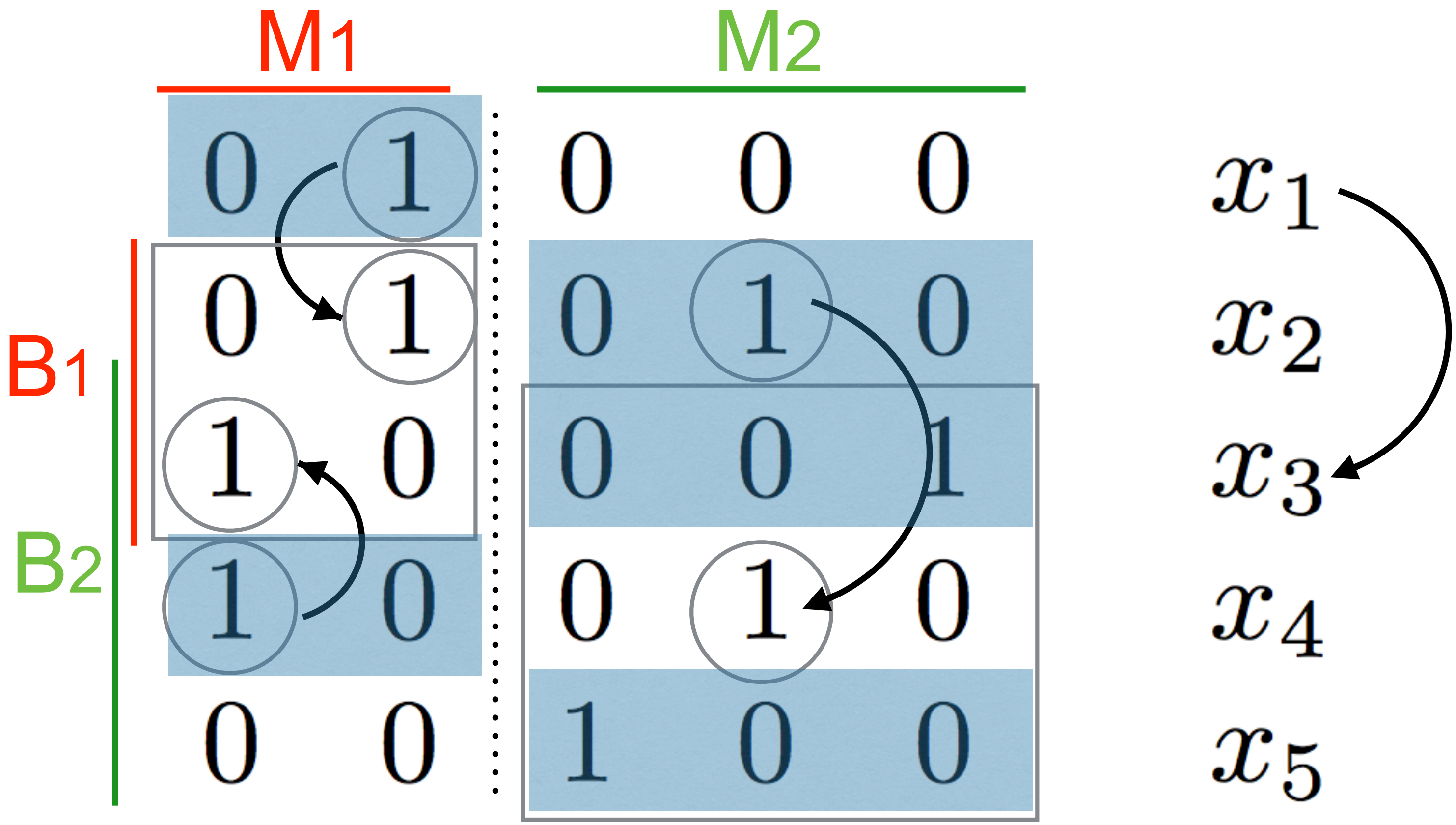
- Challenge: After first set of vertices is identified, the shaded submatrix no longer features orthogonal columns, so you cannot simply greedily iterate the M=2 method block by block
- May need to do extra row exchanges at each step
- Techniques initially discovered in the context of matroid theory tell us how to perform these exchanges

Greene, A multiple exchange property for bases, Proc. AMS, 1973

Greene and Magnanti, Some abstract pivot algorithms, SIAM J. Appl. Math., 1975

Algorithm to Create Uniqueness Set Partitions
Case 2: M>2

$$B_1 = \{x_2, x_3\} \quad B_2 = \{x_3, x_4, x_5\}$$



$$x_1 \xrightarrow[B_1]{} x_2 \xrightarrow[B_2]{} x_4 \xrightarrow[B_1]{} x_3$$

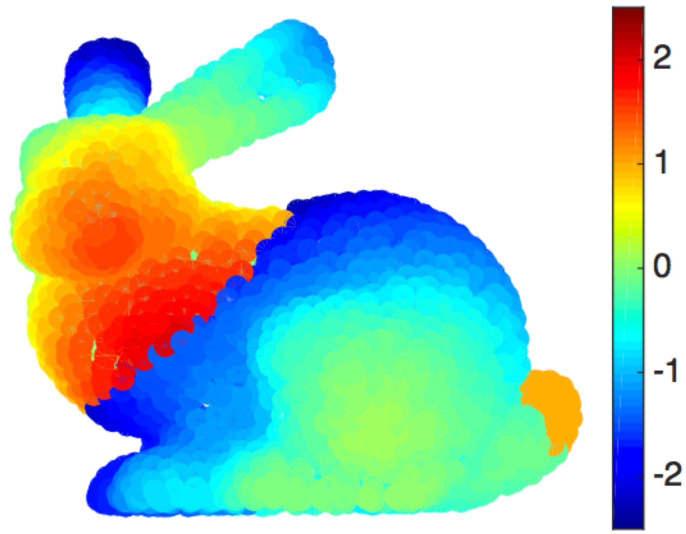
$$B_1' = (B_1 - x_2 - x_3) \cup (x_1, x_4) = \{x_1, x_4\}$$

$$B_2' = (B_2 - x_4) \cup (x_2) = \{x_2, x_3, x_5\}$$

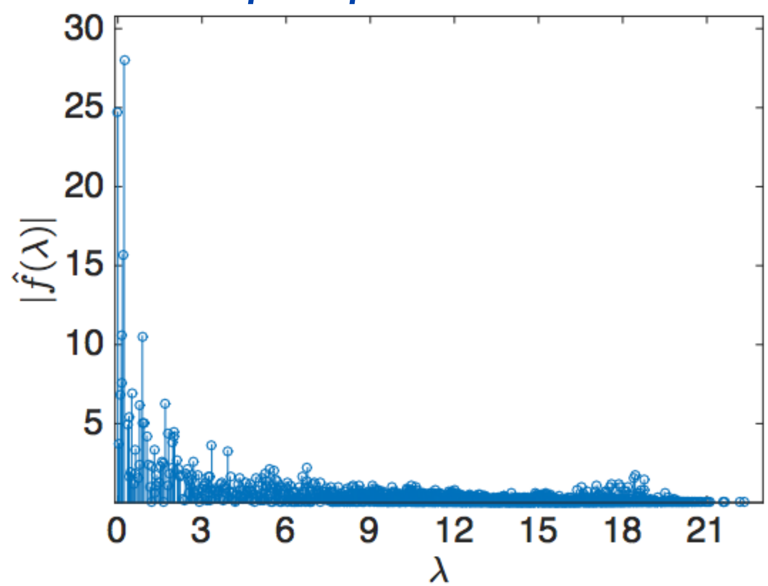
Greene and Magnanti, Some abstract pivot algorithms, SIAM J. Appl. Math., 1975

Example: Piecewise Smooth Signal
Partition and Analysis Coefficients

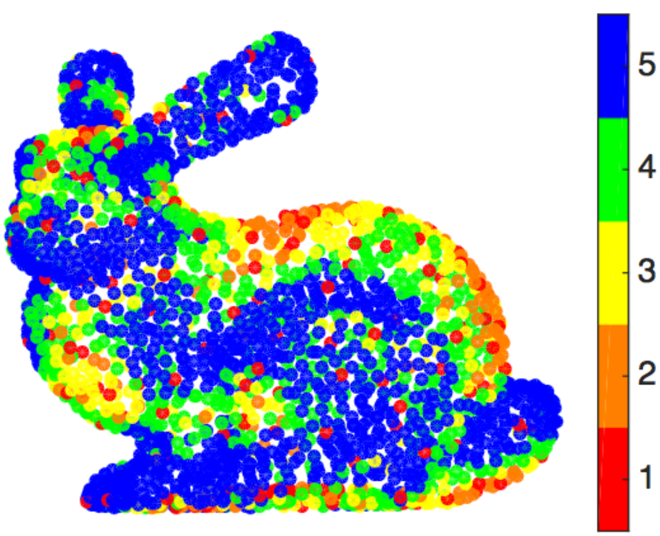
Vertex Domain



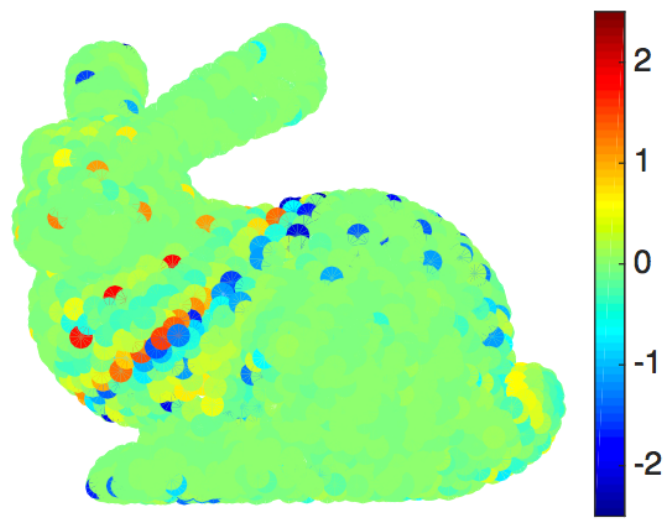
Graph Spectral Domain



Partition into Uniqueness Sets

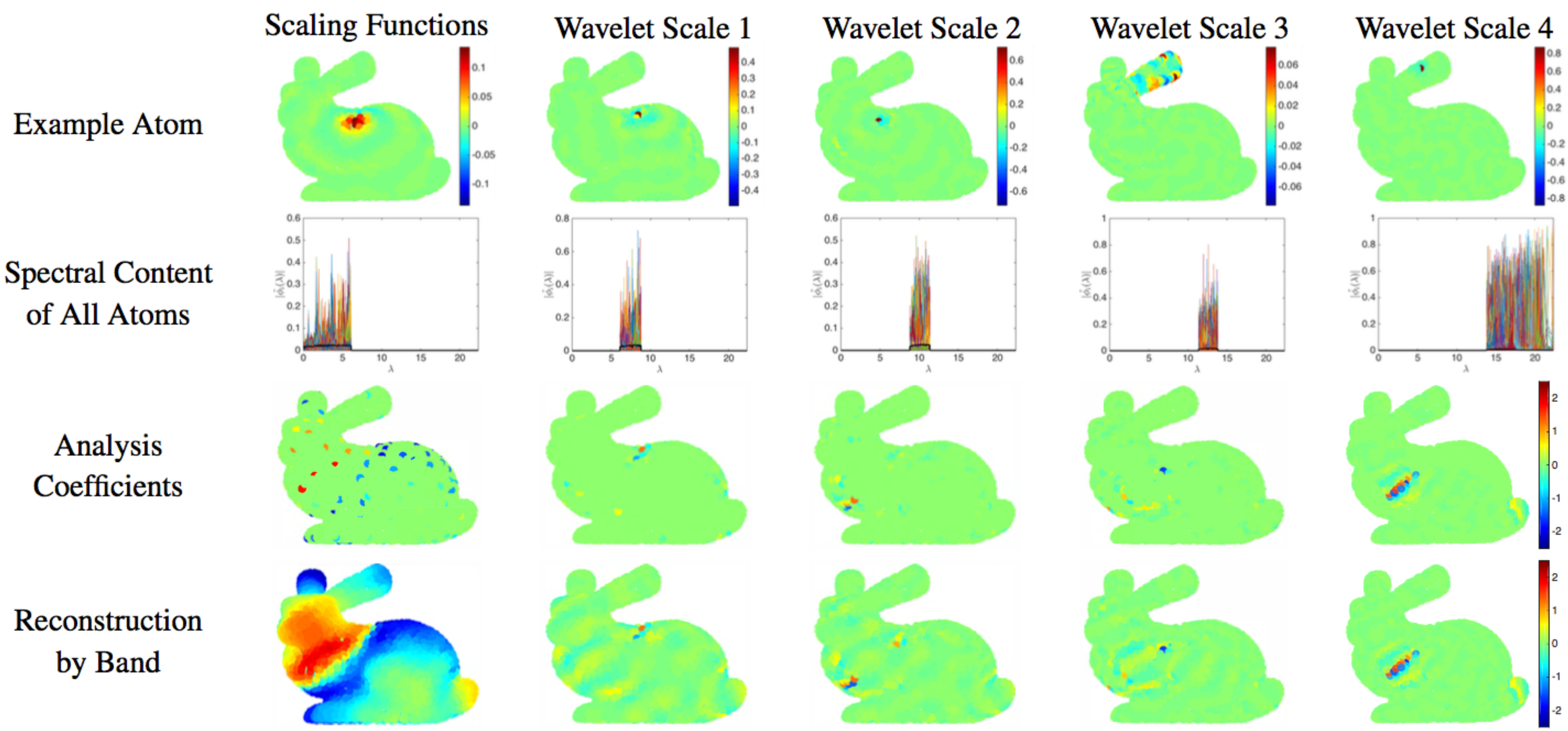


Analysis Coefficients



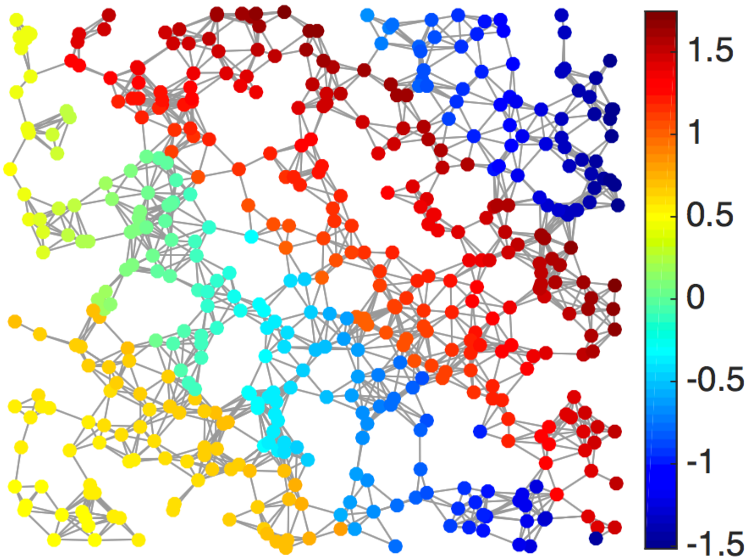
Example: Piecewise Smooth Signal *Atoms*

- Atoms jointly localized in vertex and graph spectral domains
- Non-zero wavelet coefficients clustered around discontinuities

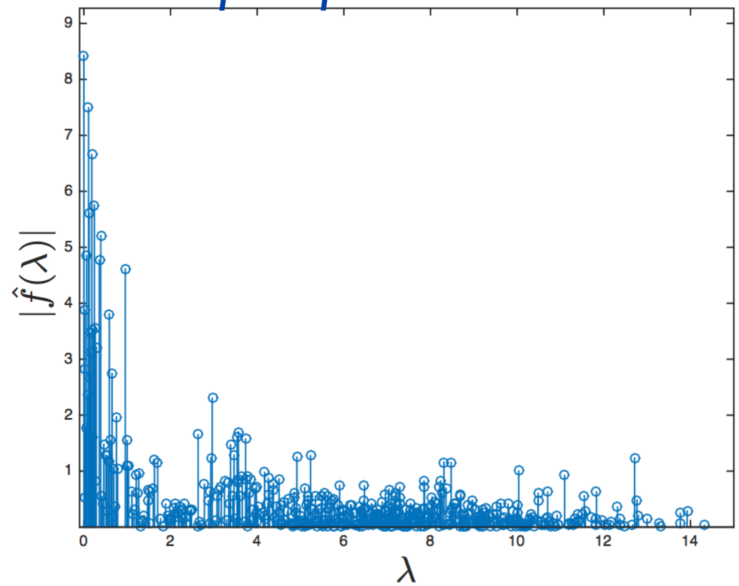


Compression Example

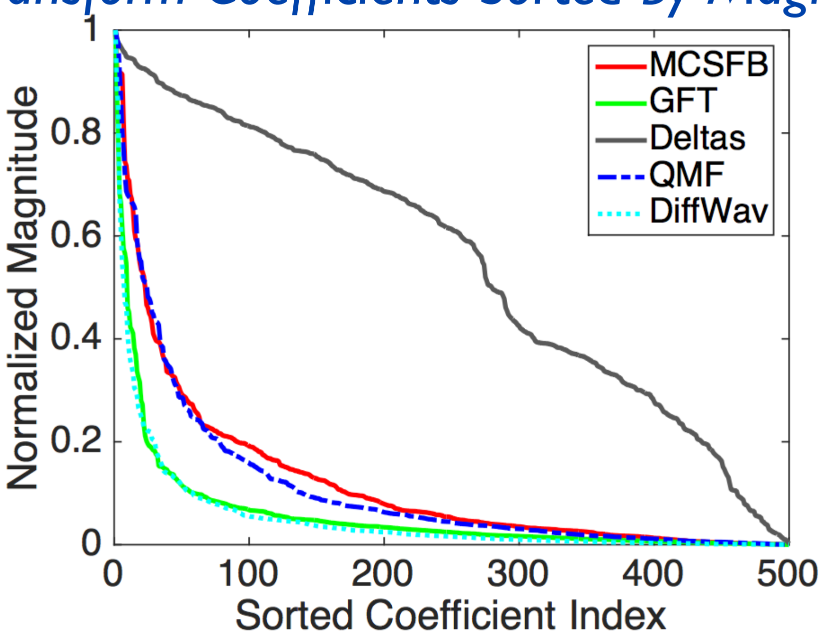
Vertex Domain



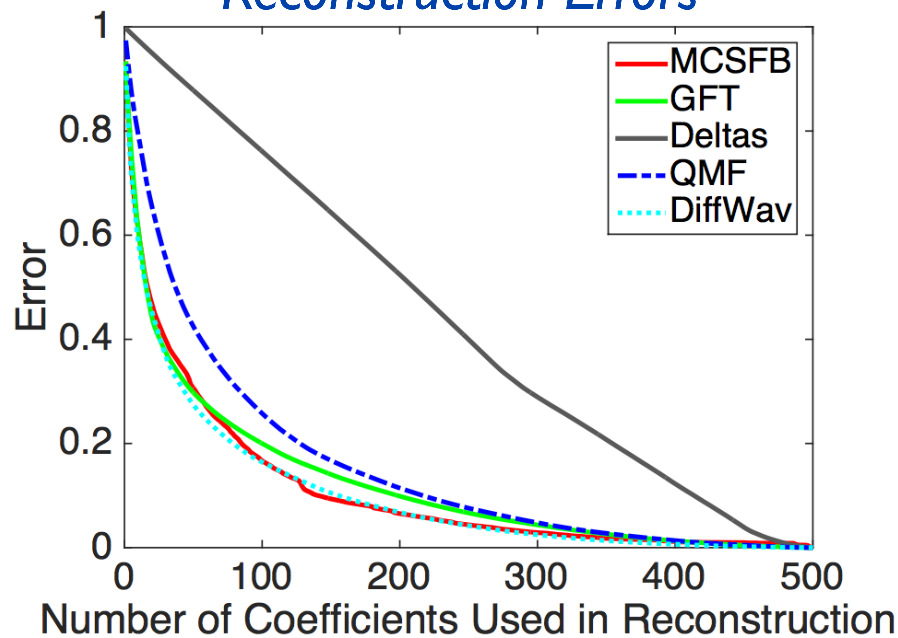
Graph Spectral Domain



Transform Coefficients Sorted by Magnitude



Reconstruction Errors



Ongoing Work

- Computational approximations to improve scaling
  - Non-uniform random sampling (c.f., Puy et al., 2015)
  - Stably reconstruct signals supported on a specific spectral band without requiring a full eigendecomposition
- Reconstruction robustness
  - Many different partitions into uniqueness sets; which ones makes reconstruction more stable when transform coefficients are noisy or missing?
- Iterated filter bank: how does iterating with fewer channels compares to a single level with more channels?
- Formally characterize the relationships between the decay of the analysis coefficients, properties of the graph signals, and the underlying graph structure