Atomic Norm Minimization for Modal Analysis with Random Spatial Compression

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Introduction

- Physical structures:

Mississippi River Bridge (2007) Sampoong Department Store (1995)


- How to detect the damage that can be caused over time by continuous use?
  - Natural frequencies
  - Mode shapes
  - Damping ratios
  - Uniform sampling
  - Synchronous random sampling
  - Asynchronous random sampling
  - Random temporal compression
  - **Random spatial compression**

Modal Expansion Theorem

- Second-order equations of motion for an N degree of freedom linear system:

$$[\boldsymbol{M}]\{\ddot{\boldsymbol{x}}(t)\} + [\boldsymbol{C}]\{\dot{\boldsymbol{x}}(t)\} + [\boldsymbol{K}]\{\boldsymbol{x}(t)\} = \{\boldsymbol{f}(t)\}$$

- Modal expansion with K active modes:

$$\{\boldsymbol{x}(t)\} = [\boldsymbol{\Psi}]\{\boldsymbol{q}(t)\} = \sum_{k=1}^{K}\{\boldsymbol{\psi}_k\}q_k(t)$$

- Free vibration & no damping:

$$q_k(t) = A_k e^{j2\pi f_k t}$$

Problem Formulation

- Analytic signal:

$$\{\boldsymbol{x}(t)\} = \sum_{k=1}^{K}\{\boldsymbol{\psi}_k\}A_k e^{j2\pi f_k t}$$

- Taking Nyquist samples at

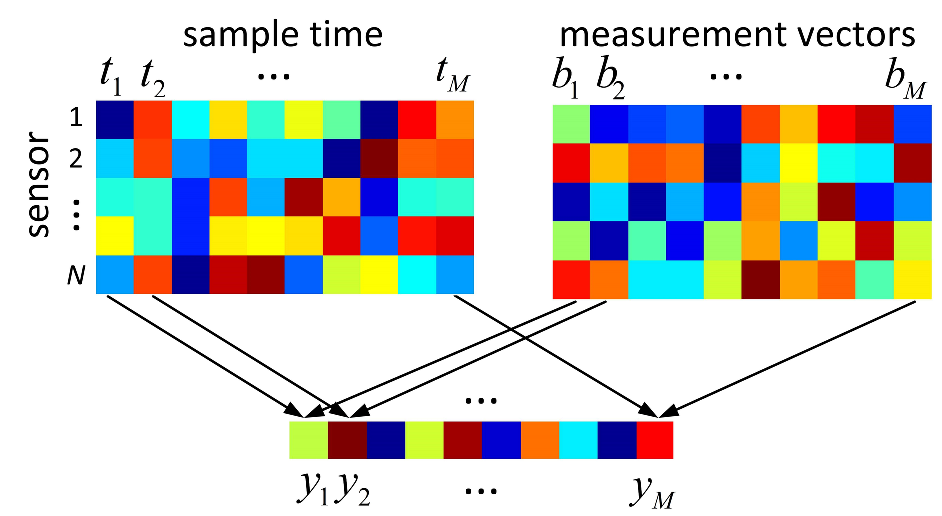
$$T = \{t_1,\ t_2,\ \cdots,\ t_M\} = \{0, T_s,\ \cdots,\ (M-1)T_s\}.$$

- Data matrix:

$$[\mathbf{X}] = [\boldsymbol{x}(t_1),\ \boldsymbol{x}(t_2),\ \cdots,\ \boldsymbol{x}(t_M)] = \sum_{k=1}^{K} A_k\{\boldsymbol{\psi}_k\}\boldsymbol{a}(f_k)^\top \in \mathbb{C}^{N\times M}$$

with $\boldsymbol{a}(f_k) := [e^{j2\pi f_k t_1},\ e^{j2\pi f_k t_2},\ \cdots,\ e^{j2\pi f_k t_M}]^\top$.

Randomized Spatial Compression



$$\begin{aligned} y_m &= \langle [\mathbf{X}](:,m), \boldsymbol{b}_m\rangle \\ &= \langle [\mathbf{X}]\boldsymbol{e}_m, \boldsymbol{b}_m\rangle \\ &= \langle [\mathbf{X}], \boldsymbol{b}_m \boldsymbol{e}_m^H\rangle \\ 1 \le m &\le M, \end{aligned}$$

Atomic Norm Minimization

$$\min_{[\hat{\mathbf{X}}]} \|[\hat{\mathbf{X}}]\|_{\mathcal{A}}$$
$$\text{s.t.}\ \ y_m = \langle [\hat{\mathbf{X}}], \boldsymbol{b}_m \boldsymbol{e}_m^H\rangle,\quad 1 \le m \le M.$$

- Atomic set: $\mathcal{A} = \{\boldsymbol{h}\boldsymbol{a}(f)^\top :\ \|\boldsymbol{h}\|_2 = 1\}$
- Atomic norm:

$$\|[\mathbf{X}]\|_{\mathcal{A}} = \inf\{t > 0 : [\mathbf{X}] \in t\,\mathrm{conv}(\mathcal{A})\} = \inf\left\{\sum_k c_k : [\mathbf{X}] = \sum_k c_k \boldsymbol{h}_k \boldsymbol{a}(f_k)^\top,\ c_k \ge 0\right\}.$$

- Dual polynomial: $\mathcal{Q}(f) = \mathbf{Y}\boldsymbol{a}(f)$

Theoretical Guarantee

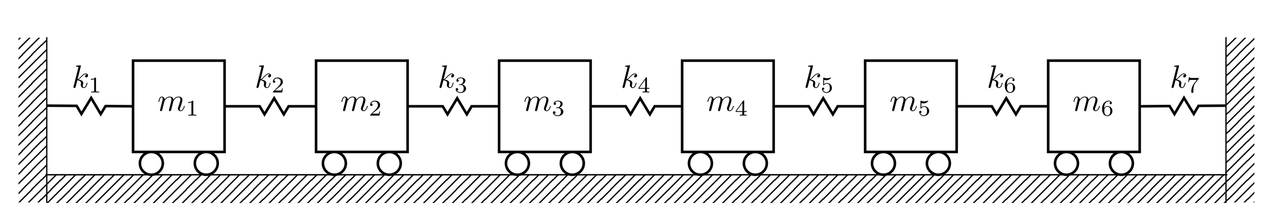
Theorem 1 [Yang, 2016]

Suppose we observe the data matrix $[\mathbf{X}]$ with the above random spatial compression scheme. Assume that the random vectors $\boldsymbol{b}_m$ are i.i.d samples from an distribution with the isotropic and μ-incoherent properties. Assume that the signs $\frac{\{\boldsymbol{\psi}_k\}(n)A_k}{|\{\boldsymbol{\psi}_k\}(n)A_k|}$ are drawn i.i.d. from the uniform distribution on the complex unit circle, and assume the minimum separation $\Delta_f = \min_{k\neq j}|(f_k - f_j)T_s| \ge \frac{4}{M-1}$. Then there exists a numerical constant C such that

$$M \ge C\mu K N \log\left(\frac{MKN}{\delta}\right)\log^2\left(\frac{MN}{\delta}\right)$$

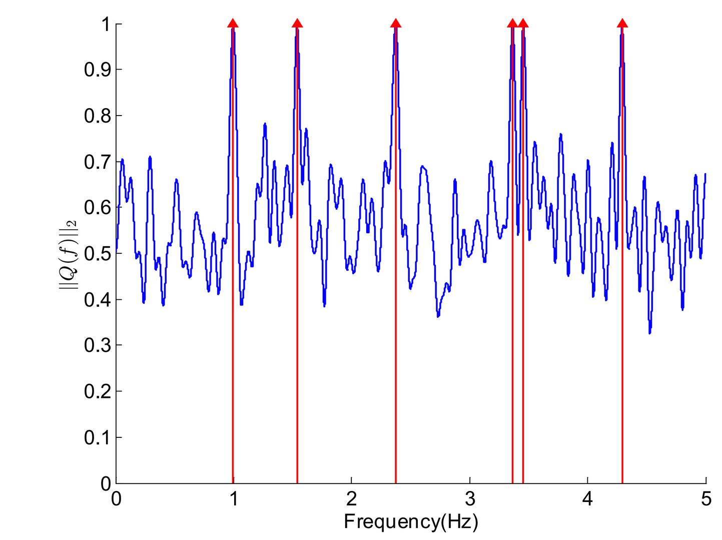
is sufficient to guarantee that we can recover $[\mathbf{X}]$ via ANM and localize the frequencies with probability at least $1-\delta$.

ANM-based Strategy vs. SVD-based Strategy

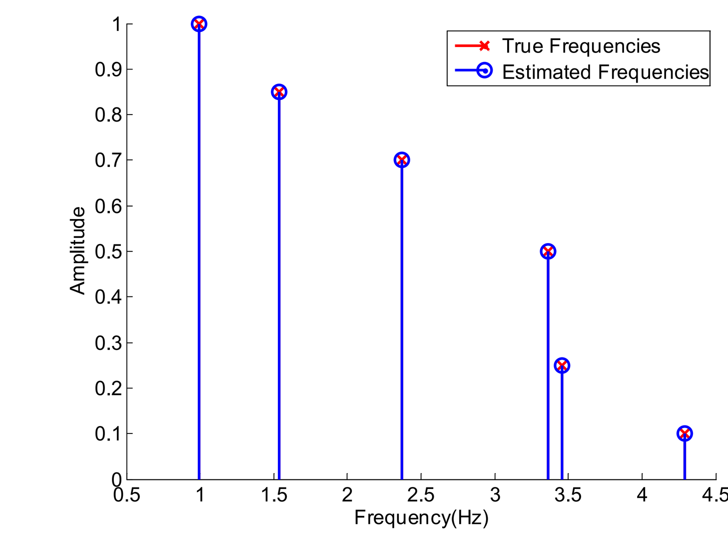


- Mass:
  - Orthogonal: $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 1$ kg.
  - Non-orthogonal: $m_1 = 1,\ m_2 = 2,\ m_3 = 3,\ m_4 = 4,\ m_5 = 5,\ m_6 = 6$ kg.
- Stiffness: $k_1 = k_5 = 200,\ k_2 = k_6 = 150,\ k_3 = 100,\ k_4 = 50,\ k_7 = 200$ N/m.
- $M = 150,\ N = 6.$
- # of measurements: SVD($M \times N$), ANM(M).

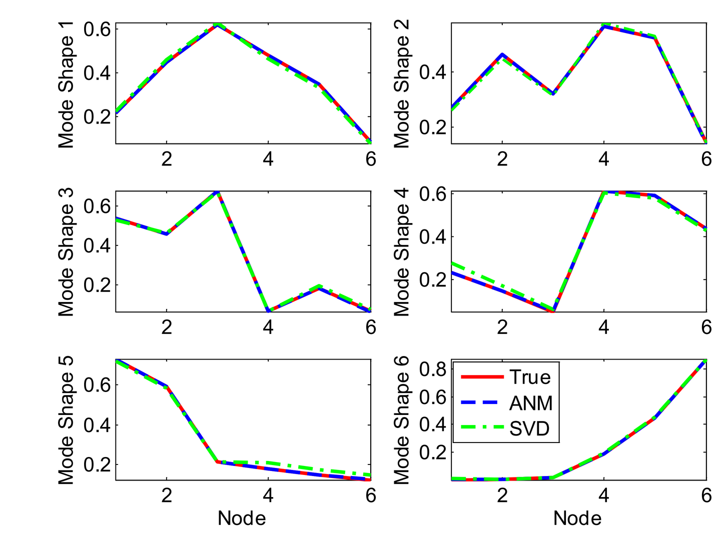
ANM-based Strategy vs. SVD-based Strategy



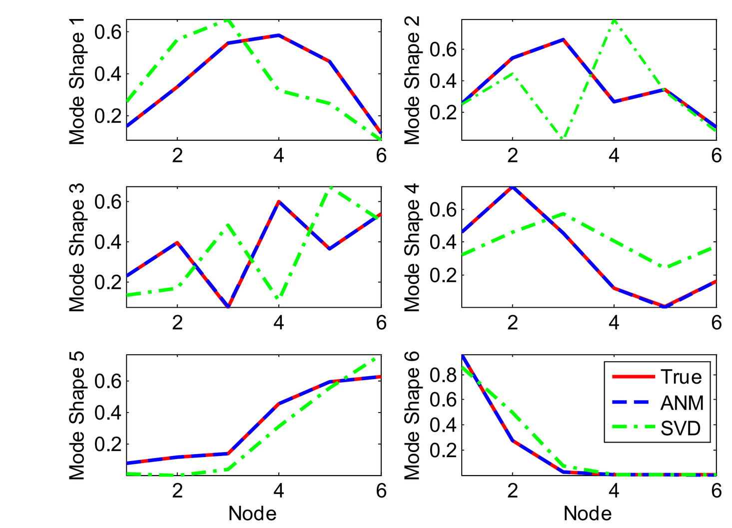
Frequency localization with dual polynomial.



The estimated frequencies.

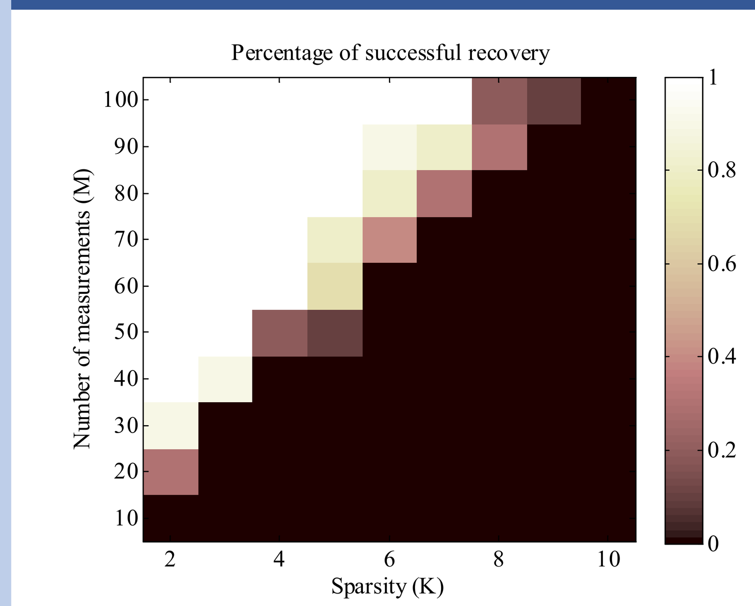


Orthogonal mode shapes.

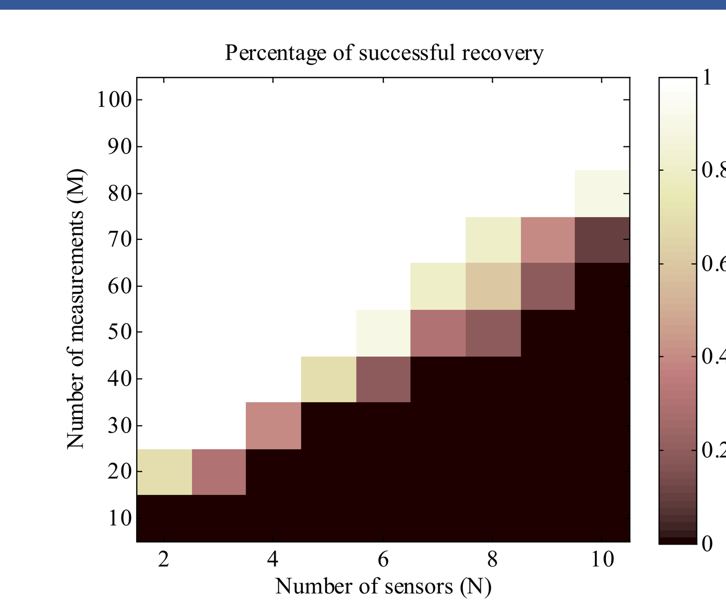


Non-orthogonal mode shapes.

M vs. K and N



number of sensors is set as $N = 5$



number of active modes is set as $K = 3$

Conclusions

- The recovery will be successful with high probability if the number of time samples M is proportional to KN.
- ANM can achieve a better performance in recovering mode shapes when compared with SVD (especially when the mode shapes are not orthogonal).
- In future, we will
  - take noise into consideration
  - work on free vibration with damping and forced vibration

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