SUPER-RESOLUTION DELAY-DOPPLER ESTIMATION FOR SUB-NYQUIST RADAR VIA ATOMIC NORM MINIMIZATION

Feng Xi, Shengyao Chen, and Zhong Liu

Department of Electronic Engineering, Nanjing University of Science and Technology, China

ICASSP2017

CONTRIBUTIONS

- **Purpose：** Achieve gridless delay-Doppler estimation with sub-Nyquist sampled radar echo signal.
- **Key idea：**
  - Regard the sub-Nyquist sampling as a linear mapping which maps a low-rank matrix to the sampling vector;
  - Use Atomic Norm Minimization (ANM) to recover the low-rank matrix from the sub-Nyquist samples;
- **Result:** A two-staged processing is developed:
  - Stage 1: solve a computationally efficient ANM problem;
  - Stage 2: implement the delay-Doppler estimation and pairing.

SUB-NYQUIST RADAR

Signal Model

The received radar echo signal is :

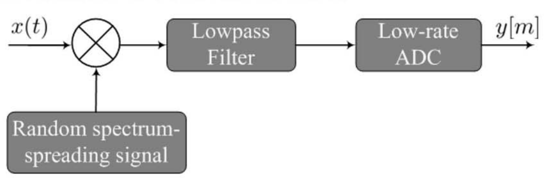
$$x(t)=\sum_{n=0}^{N-1}\sum_{k=1}^{K}\alpha_k g_n(t-nT-\tau_k)e^{j2\pi\nu_k nT}$$

N: the number of transmitted pulses; T: the pulse repetition interval (PRI); $g_n(t)$: the pulsed signal transmitted in the n-th PRI; $\{\tau_k,\nu_k\}$ is the unknown delay-Doppler parameter for the k-th target.

- **Note:** Here we don't require that the radar transmits the same pulse in each PRI.

Sub-Nyquist Sampling

- The radar echo signal is sampled by an Analog-to-Information Converter (AIC), *e.g.*, random demodulator , Xampling, and so on.
- Random demodulator is considered here:



- During the n-th PRI, the sub-Nyquist sample is

$$y_n[m]=\sum_{k=1}^{K}\alpha_k\int_{nT}^{(n+1)T}h(\tau)p(mT_{cs}-\tau)g_n(mT_{cs}-nT-\tau_k-\tau)e^{j2\pi\nu_k nT}d\tau$$

$h(t)$: the lowpass filter ; $p(t)$: the random spectrum-spreading signal;
T_{cs} : the sampling interval of the low-rate ADC.

- Assumption 1: $h(t)$ is an ideal lowpass filter with bandwith B_{cs} .
- Assumption 2: $p(t)$ is a T-periodic signal $p(t)=\sum_{l=-L_p}^{L_p}\rho_l e^{j\pi lt/T}$.
- Assumption 3: $T_{cs}=1/B_{cs}, B/Bcs=M$

Frequency-Domain Representation

The continuous-time Fourier transform of $y_n(t)$ is

$$Y_n(f)=\sum_{l=-L_0}^{L_0}\sum_{k=1}^{K}\rho_l\alpha_k e^{j2\pi\nu_k nT}G_n(f-l/T)e^{-j2\pi(f-l/T)\tau_k}$$

The frequency-domain sampling vector of $Y_n(f)$ can be represented as

$$\tilde{\mathbf{y}}_n=\mathbf{P}\mathbf{G}_n\mathbf{W}(\tau)\mathbf{B}\mathbf{V}^T(\nu)\mathbf{e}_n$$

$$\begin{bmatrix}\rho_{1-L_1} & \rho_{2-L_1} & \cdots & \rho_{1+L_2}\\ \rho_{2-L_1} & \rho_{2-L_1} & \cdots & \rho_{2+L_2}\\ \vdots & \vdots & \ddots & \vdots\\ \rho_{M-L_1} & \rho_{M+1-L_1} & \cdots & \rho_{M+L_2}\end{bmatrix}\begin{bmatrix}G_n(f_{(L_1)}) & 0 & \cdots & 0\\ 0 & G_n(f_{(L_1+1)}) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & G_n(f_{(L_2)})\end{bmatrix}$$

$$\begin{bmatrix}\alpha_1 e^{-j2\pi f_{(L_1)}\tau_1} & 0 & \cdots & 0\\ 0 & \alpha_2 e^{-j2\pi f_{(L_1)}\tau_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \alpha_K e^{-j2\pi f_{(L_1)}\tau_K}\end{bmatrix}$$

$\mathbf{W}(\tau)=[\mathbf{w}(\tau_1/T),\cdots,\mathbf{w}(\tau_K/T)]$ with $\mathbf{w}(\theta)=[1,e^{j2\pi\theta},\cdots,e^{j2\pi(L-1)\theta}]^T$
$\mathbf{V}(\nu)=[\mathbf{v}(\nu_1 T),\cdots,\mathbf{v}(\nu_K T)]$ with $\mathbf{v}(\theta)=[1,e^{j2\pi\theta},\cdots,e^{j2\pi(N-1)\theta}]^T$

Let $\theta_k=\tau_k/T-1/2, \vartheta_k=-\nu_k T\in(-1/2,1/2]$, the m-th element of $\tilde{\mathbf{y}}_n$ can be represented as

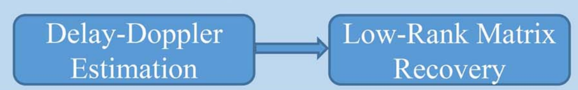
$$\begin{aligned}\tilde{y}_n[m]&=\sum_{k=1}^{K}\beta_k\mathbf{m}_{m,n}^H\mathbf{w}(\theta_k)\mathbf{v}^H(\vartheta_k)\mathbf{e}_n\\ &=\mathrm{Tr}\Big(\mathbf{e}_n\mathbf{m}_{m,n}^H\sum_{k=1}^{K}\beta_k\mathbf{w}(\theta_k)\mathbf{v}^H(\vartheta_k)\Big)\\ &=\langle\mathbf{X},\mathbf{m}_{m,n}\mathbf{e}_n^H\rangle,\end{aligned}$$

where $\mathbf{X}=\sum_{k=1}^{K}\beta_k\mathbf{w}(\theta_k)\mathbf{v}^H(\vartheta_k)$ is a low-rank matrix.

Let $\tilde{\mathbf{y}}=[\tilde{\mathbf{y}}_0^T,\tilde{\mathbf{y}}_1^T,\cdots,\tilde{\mathbf{y}}_{N-1}^T]^T$, the process of sub-Nyquist sampling can be expressed as

$$\tilde{\mathbf{y}}=\mathcal{B}(\mathbf{X})$$

$\mathcal{B}(\cdot)$: the linear operator mapping the matrix $\mathbf{X}$ to $\tilde{\mathbf{y}}$.



Method & Algorithm

Low-Rank Matrix Recovery via ANM

The set of atoms is defined as

$$\mathcal{A}\triangleq\{\mathrm{A}(\boldsymbol{\theta},\phi)=e^{j\phi}\mathbf{w}(\theta)\mathbf{v}^H(\vartheta):\theta,\vartheta\in\mathbb{T},\phi\in\mathbb{S}\},$$

where $\mathbb{T}\triangleq(-1/2,1/2]$ and $\mathbb{S}\triangleq(0,2\pi]$.

The atomic l_0 norm is defined as

$$\|\mathbf{X}\|_{\mathcal{A},0}=\inf\Big\{\mathcal{K}:\mathbf{X}=\sum_{k=1}^{\mathcal{K}}c_k\mathrm{A}(\boldsymbol{\theta}_k,\phi_k),c_k>0\Big\}$$

$\|\mathbf{X}\|_{\mathcal{A},0}$ equals the optimal value of the following rank minimization problem:

$$\min_{\mathbf{u}_1\in\mathbb{C}^L,\mathbf{u}_2\in\mathbb{C}^N}\ \mathrm{rank}(\mathbf{M}),\quad \text{s.t.}\ \mathbf{M}=\begin{bmatrix}\mathcal{T}(\mathbf{u}_1) & \mathbf{X}\\ \mathbf{X}^H & \mathcal{T}(\mathbf{u}_2)\end{bmatrix}\geq 0,$$

where $\mathcal{T}(\mathbf{u})$ denotes the Toeplitz matrix with $\mathbf{u}^T$ as its first row.

With relaxation, the low-rank matrix recovery can be cast as:

$$\min_{\mathbf{u}_1\in\mathbb{C}^L,\mathbf{u}_2\in\mathbb{C}^N,\mathbf{X}}\ \mathrm{trace}(\mathcal{T}(\mathbf{u}_1))+\mathrm{trace}(\mathcal{T}(\mathbf{u}_2)),$$
$$\text{s.t.}\ \mathbf{M}=\begin{bmatrix}\mathcal{T}(\mathbf{u}_1) & \mathbf{X}\\ \mathbf{X}^H & \mathcal{T}(\mathbf{u}_2)\end{bmatrix}\geq 0,$$
$$\tilde{\mathbf{y}}=\mathcal{B}(\mathbf{X}).$$

Delay-Doppler Estimation and Pairing Algorithm

- **Input**: $\mathbf{u}_1^*,\mathbf{u}_2^*,\mathbf{X}^*,K^*=\mathrm{rank}(\mathbf{M})$.
- **Step1**: Perform Vandermonde decomposition $\mathcal{T}(\mathbf{u}_1^*)=\mathbf{W}_1\mathbf{\Sigma}_1\mathbf{W}_1^H$ and $\mathcal{T}(\mathbf{u}_2^*)=\mathbf{V}_2\mathbf{\Sigma}_2\mathbf{V}_2^H$, acquire two sets of parameters $\{\theta_1^*,\theta_2^*,\cdots,\theta_{K_1}^*\}$ and $\{\vartheta_1^*,\vartheta_2^*,\cdots,\vartheta_{K_2}^*\}$.
- **Step 2**: Compute a $K_1\times K_2$ dimensional matrix $\mathbf{O}=\mathbf{\Sigma}_1^{-1/2}\mathbf{W}_1^\dagger\mathbf{X}\mathbf{V}_2^\dagger\mathbf{\Sigma}_2^{-1/2}$.
- **Step 3**: Select the positions of the K^* largest elements in the matrix $\mathbf{O}$ to determine the K^* delay-Doppler pairs.
- **Note:** There exists a matrix $\mathbf{O}\in\mathbb{C}^{K_1\times K_2}$ satisfying $\mathbf{X}=\mathbf{W}_1\mathbf{\Sigma}_1^{1/2}\mathbf{O}\mathbf{\Sigma}_2^{1/2}\mathbf{V}_2^H$. The matrix $\mathbf{O}$ determines the pairing relationship between the two sets of parameters.

Simulation Results

A LFM signal with bandwidth 20MHz and pulse-width $2\mu s$ is transmitted. The receiver samples the radar echoes at $1/4$ Nyquist rate, i.e., $B_{cs}=5$MHz. Other parameters are PRI=$4\mu s$, $N=50$. The performance of ANM-based method is compared with Doppler focusing (in *O. Bar-Ilan and Y. C. Eldar "Sub-Nyquist radar via Doppler focusing," IEEE T-SP, 2014*) and Cramér-Rao bound (CRB).

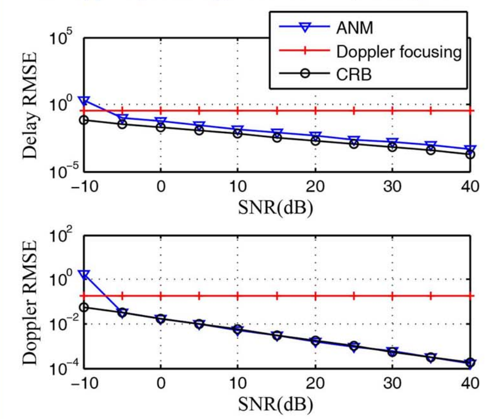


Fig 1. Two delay-Doppler pairs, *i.e.*, $K=2$.

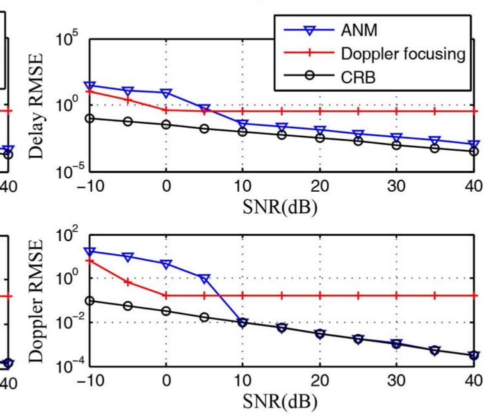


Fig 2. Four delay-Doppler pairs, *i.e.*, $K=4$.

Contact Information: xifeng@njust.edu.cn (Feng Xi)