Sequential Joint Signal Detection and Signal-to-Noise Ratio Estimation

M. Fauß*, K. G. Nagananda†, A. M. Zoubir*, and H. V. Poor‡

*Signal Processing Group, Technische Universität Darmstadt, Darmstadt, Germany - {fauss,zoubir}@spg.tu-darmstadt.de
† Dept. of Electronics and Communications Engineering, PES University, Bangalore, India - kgnagananda@pes.edu
‡ Dept. of Electrical Engineering, Princeton University, Princeton, NJ, USA - poor@princeton.edu

Motivation

- Detecting a signal of unknown power in white noise of unknown power is a fundamental problem in signal processing.
- In order to **detect** the signal, both noise and signal power have to be **estimated** either directly or indirectly.
- The **classic approach** is to formulate the problem either as a detection problem under uncertainty or an estimation problem with a subsequent detection step.
- Such hierarchical procedures can typically **not be guaranteed to meet constraints** on both the detection and estimation performance and often require a **large number of samples** to work reliably.
- The **proposed approach** is to solve the detection and estimation problem **jointly** and **sequentially** such that both performance constraints are met and the expected number of required samples is minimized.

Problem Formulation

Signal model:

$$x[n] = s[n] + w[n], \quad n = 1, 2, \dots,$$

$s[n]$: signal → zero mean i.i.d. Gaussian with unknown variance $\sigma_s^2 \geq 0$
$w[n]$: noise → zero mean i.i.d. Gaussian with unknown variance $\sigma_w^2 > 0$

Signal-to-noise ratio (SNR):

$$\theta = \frac{\sigma_s^2}{\sigma_w^2}, \qquad \theta_{\text{min}} : \text{minimum SNR for reliable detection}$$

Hypothesis test:

$$\begin{cases} H_0 : \theta = 0 & \text{(signal absent)} \\ H_1 : \theta \geq \theta_{\text{min}} & \text{(signal present)} \end{cases}$$

Constrained optimization problem:

$$\min_{\psi,\delta,\hat{\theta}} \mathbb{E}_{\theta^*}[N] \quad \text{subject to} \quad P_0(\delta_N = 1) \leq \alpha \tag{1}$$

$$P_\theta(\delta_N = 0) \leq \beta(\theta) \quad \forall \theta \geq \theta_{\text{min}} \tag{2}$$

$$\mathbb{E}_\theta\left[(\hat{\theta}_N - \theta)^2\right] \leq \gamma(\theta) \quad \forall \theta \geq \theta_{\text{min}} \tag{3}$$

N : number of samples at stopping time
$\hat{\theta}_n$: estimator for θ after the nth sample
θ^* : nominal SNR value
ψ_n, δ_n stopping and decision rule after the nth sample → $\psi_n, \delta_n \in \{0, 1\}$

Assumption:

- Knowledge of a sequence of noise-only realizations $\tilde{w}_1, \tilde{w}_2, \dots$.
- Either recorded before performing the test, or generated on a secondary sensor.

Solution Methodology

1. Transform the two sequences $x[n]$ and $\tilde{w}[n]$ to a single sequence of Bernoulli random variables whose success probability is determined by the true SNR.
2. Perform sequential joint detection and estimation on the Bernoulli sequence.

Transformation to a Bernoulli Sequence

Birnbaum's procedure [1]: Sequentially transforms the two Gaussian sequences $x[n]$ and $\tilde{w}[n]$ into a sequence of i.i.d. Bernoulli random variables

$$b[m], \; m \geq 1 \quad \text{with success probability} \quad \rho = \frac{1}{\theta + 2}.$$

Reformulated hypothesis test:

$$\begin{cases} H_0 : \rho = 0.5 & \text{(signal absent)} \\ H_1 : \rho \leq \rho_{\text{max}} & \text{(signal present)} \end{cases}$$

Joint Detection and Estimation

- Reformulate the optimization problem in terms of ρ instead of θ.
- Relax constraints to hold on discrete set of SNR values $\mathcal{P} = \{\rho_1, \dots, \rho_K\}$.
- The constrained problem can be solved via its Lagrange dual [2]
  - minimize with respect to the primal variables → dynamic programming
  - maximize with respect to the dual variables → convex optimization

Lagrange multipliers: (1) → λ_0, (2) → λ_k, (3) → μ_k, $k = 1, \dots, K$

Sufficient test statistic: observed number of 0s and 1s in $b[m]$ → m_0, m_1

Likelihood-ratios under P_0 and P_{ρ_k} with respect to P_{ρ^*}:

$$Z_0^{m_0,m_1} = \left(\frac{0.5}{1-\rho^*}\right)^{m_0} \left(\frac{0.5}{\rho^*}\right)^{m_1}, \quad Z_k^{m_0,m_1} = \left(\frac{1-\rho_k}{1-\rho^*}\right)^{m_0} \left(\frac{\rho_k}{\rho^*}\right)^{m_1}$$

Optimal decision rule:

$$\delta^*_{m_0,m_1} = \begin{cases} 1, & \lambda_0 Z_0^{m_0,m_1} \leq E_\lambda^{m_0,m_1} \\ 0, & \lambda_0 Z_0^{m_0,m_1} > E_\lambda^{m_0,m_1} \end{cases}$$

Optimal stopping rule:

$$\psi^*_{m_0,m_1} = \begin{cases} 1, & G_{m_0,m_1} = R_{m_0,m_1} \\ 0, & G_{m_0,m_1} > R_{m_0,m_1} \end{cases}$$

Optimal estimator:

$$\tilde{\theta}^*_{m_0,m_1} = \frac{E_{\mu,1}^{m_0,m_1}}{E_{\mu,0}^{m_0,m_1}}$$

where

$$E_\lambda^{m_0,m_1} = \sum_{k=1}^K \lambda_k Z_k^{m_0,m_1}, \quad E_{\mu,i}^{m_0,m_1} = \sum_{k=1}^K \rho_k^{-i} \mu_k Z_k^{m_0,m_1}$$

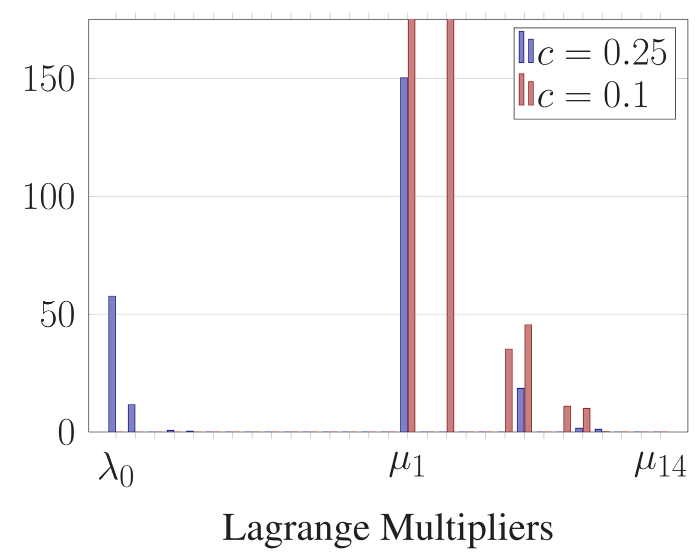
$$G_{m_0,m_1} = \min\left[\lambda_0 Z_0^{m_0,m_1}, E_\lambda^{m_0,m_1}\right] + E_{\mu,2}^{m_0,m_1} - \frac{(E_{\mu,1}^{m_0,m_1})^2}{E_{\mu,0}^{m_0,m_1}}$$

$$R_{m_0,m_1} = \min\left[G_{m_0,m_1}, 1 + \rho^* R_{m_0,m_1+1} + (1-\rho^*) R_{m_0+1,m_1}\right]$$

Experimental Results

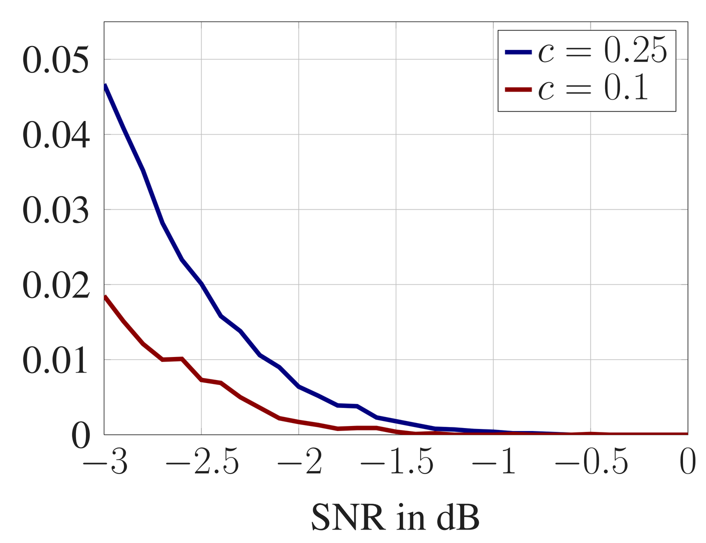
SNR grid: $\mathcal{P} = \{-3\,\text{dB}, -2\,\text{dB}, \dots, 10\,\text{dB}\}$, $\theta^* = 3\,\text{dB}$
Error Prob. constrains: $\alpha = \beta(\theta) = 0.05$
MSE constraints: relative MSE bounded by constant c → $\gamma(\theta) = c\theta^2$
Monte Carlo runs: 10 000

Optimal Lagrange Multipliers



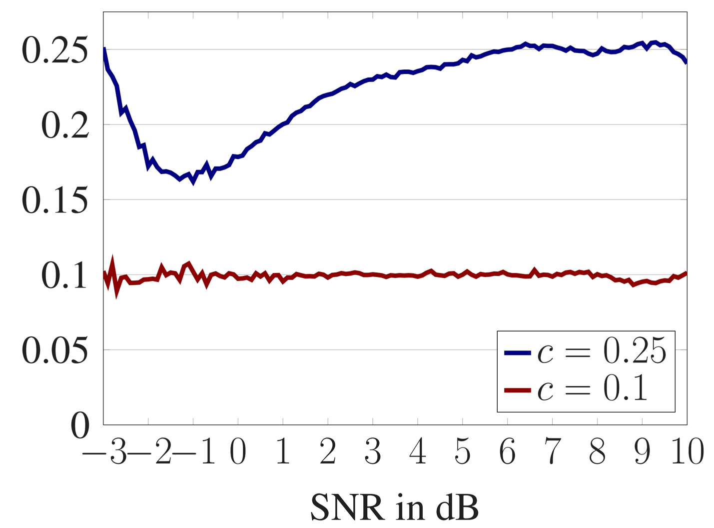
- Solution is sparse → most of the performance constraints are inactive.
- Few constrains are sufficient to bound the performance over large intervals.

Type II Error Probabilities



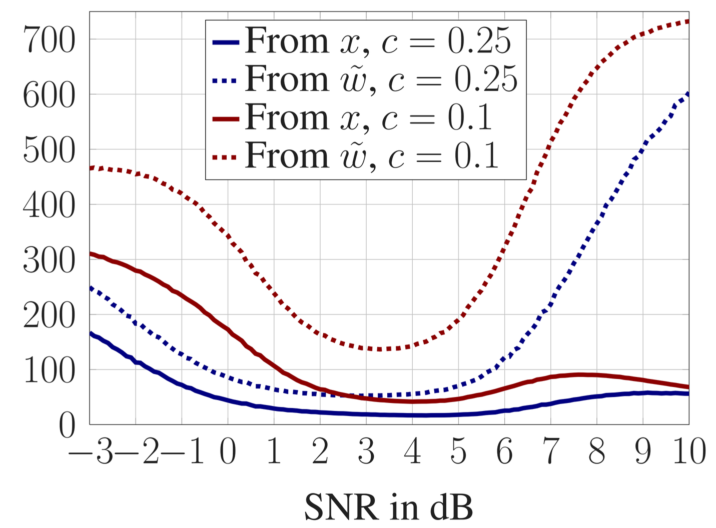
- Constraints are met for all SNR values in the feasible interval.
- For most SNR values the detection constraint are inactive.

Relative MSE



- Constraints are met for all SNR values in the feasible interval.
- For $c = 0.1$, MSE constraints dominate, i.e., hold with equality.

Average Sample Number



- ASNs for both cases are high for low SNRs and decrease for higher SNRs.
- Number of samples drawn from $x[n]$ stays almost constant for large SNRs.

References

[1] A. Birnbaum, "Sequential tests for variance ratios and components of variance," *Annals of Mathematical Statistics*, vol. 29, no. 2, pp. 504–514, 1958.

[2] M. Fauß and A. M. Zoubir, "A linear programming approach to sequential hypothesis testing," *Sequential Analysis*, vol. 34, no. 2, pp. 235–263, 2015.

This research was supported in part by the U. S. National Science Foundation under Grants CNS-1456793 and ECCS-1343210.