Overview of Signals and Systems

Chapter Intended Learning Outcomes:

(i) Get basic concepts of signals and systems

(ii) Realize that signals and systems arise in our daily life

What is Signal?

- Anything that conveys information, e.g.,
    - Speech
    - Electrocardiogram (ECG)
    - Radar pulse
    - DNA sequence
    - Stock price
    - Code division multiple access (CDMA) signal
    - Image
    - Video

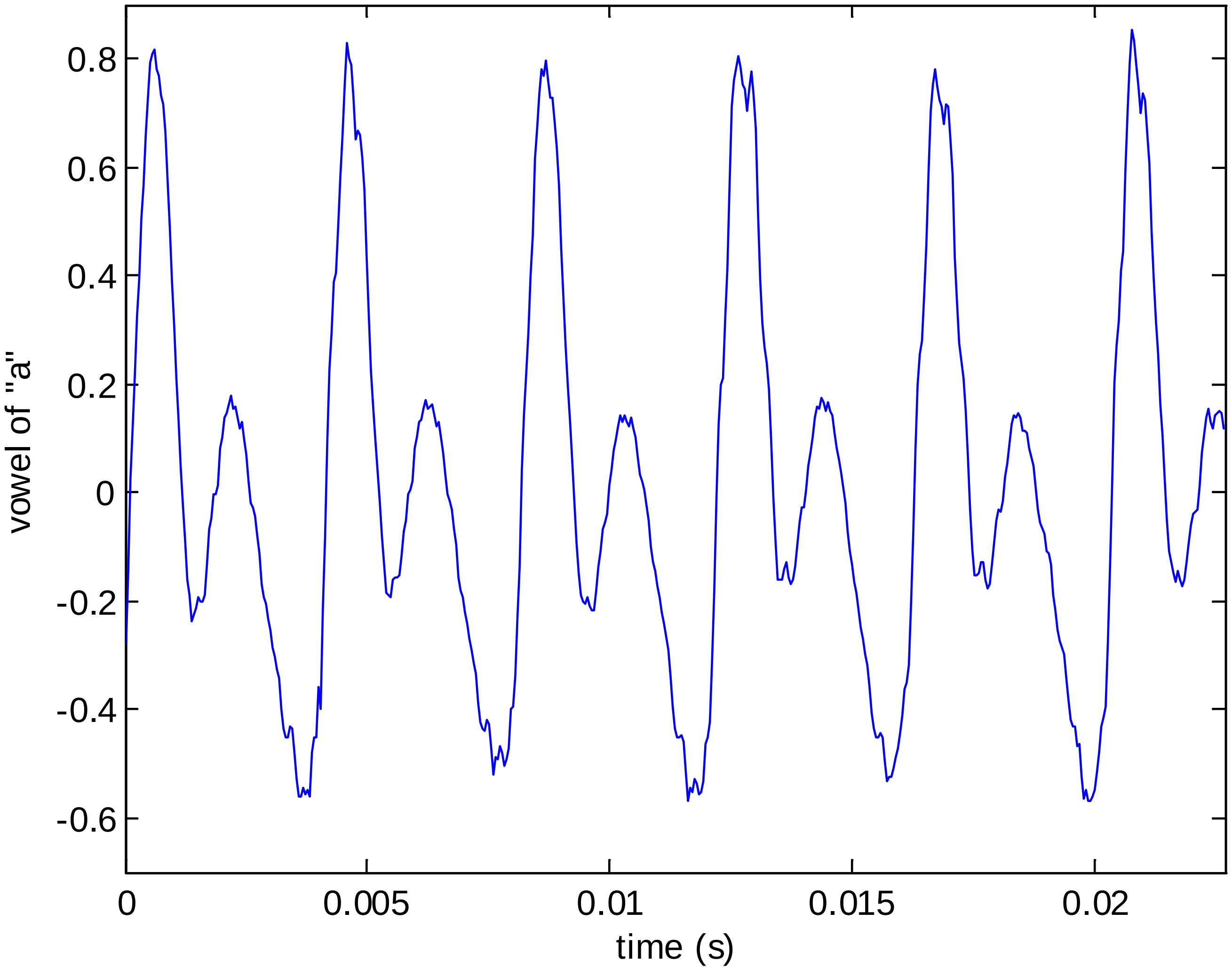


Fig.1.1: Speech

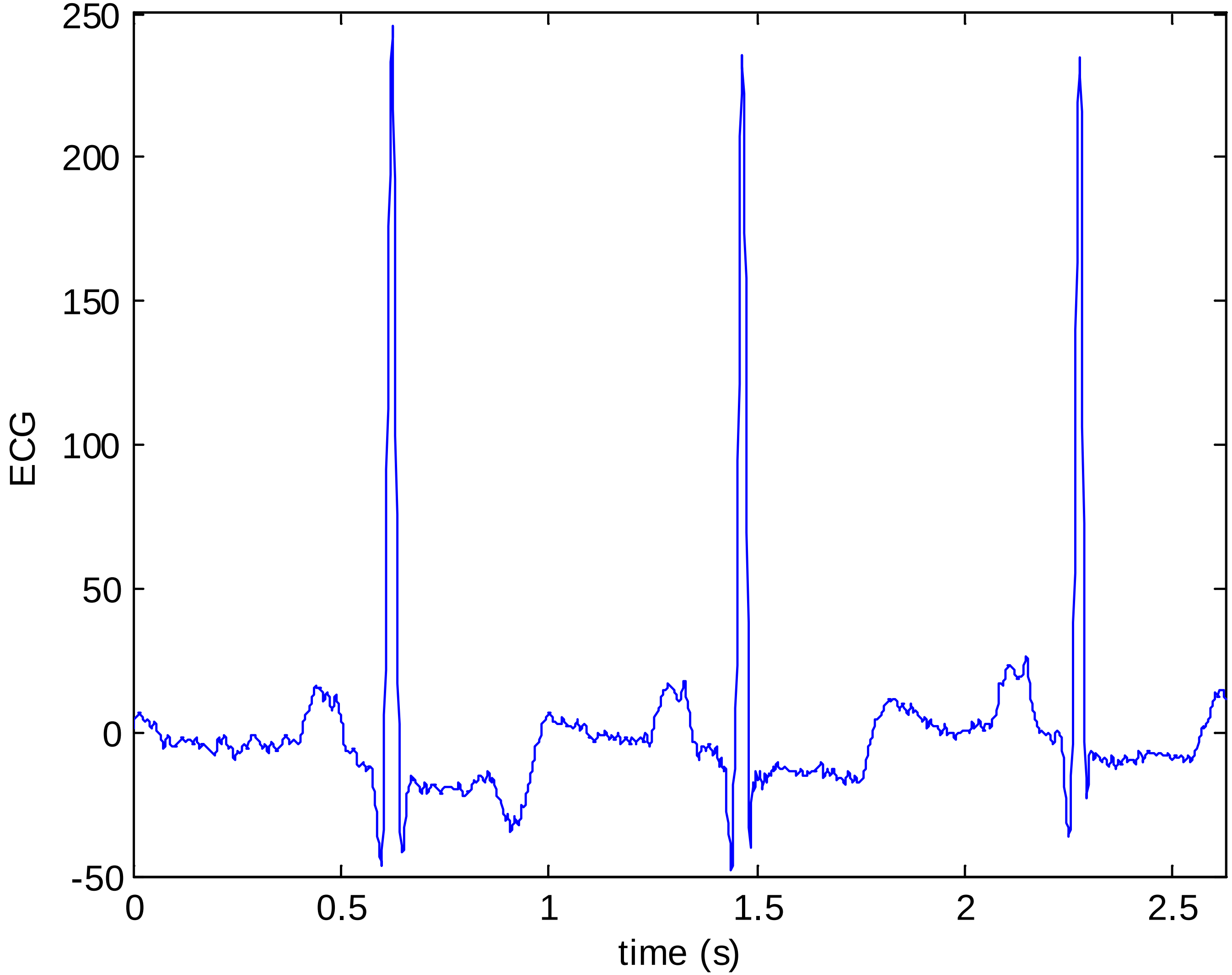


Fig.1.2: ECG

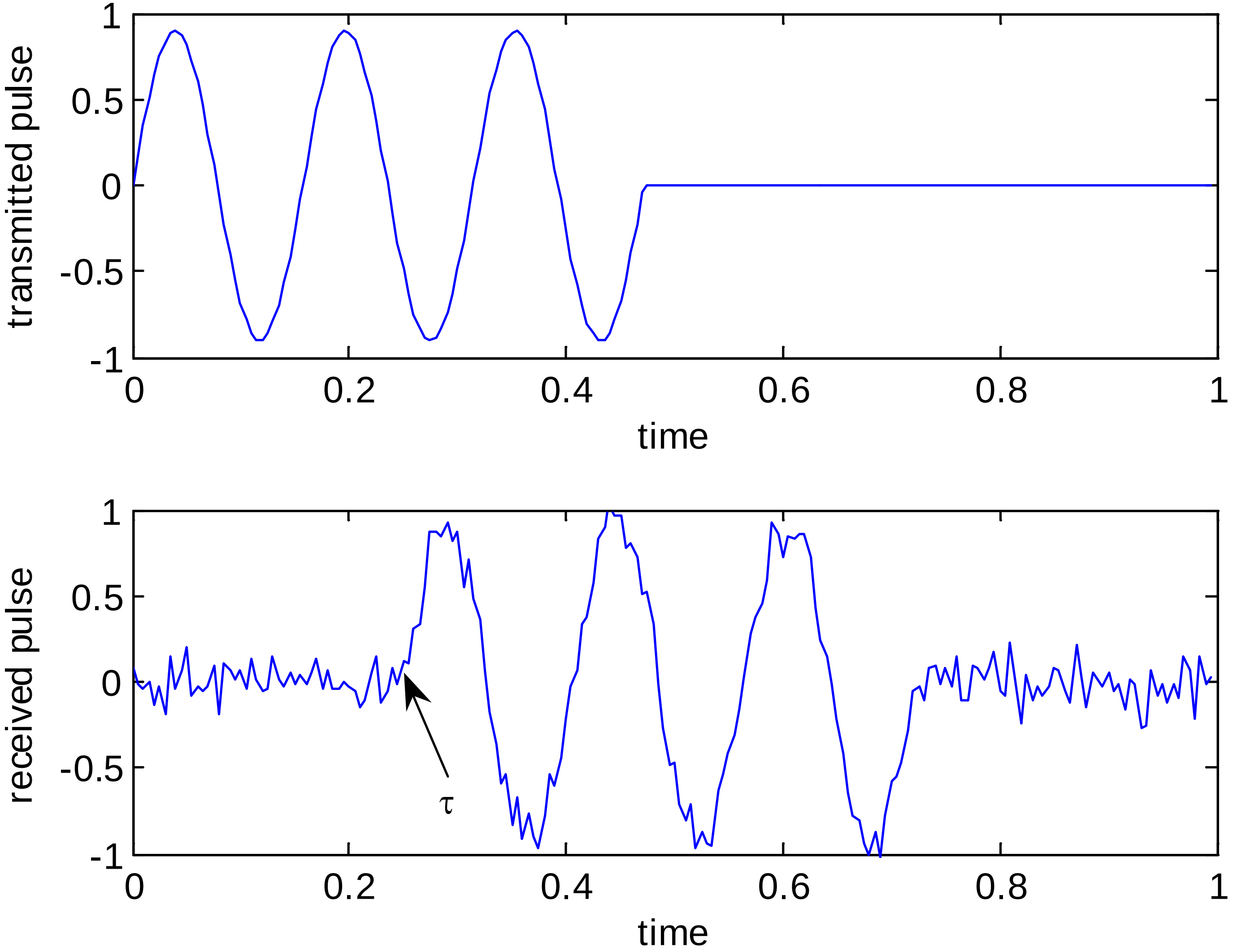


Fig.1.3: Transmitted & received radar waveforms: $s(t)$ & $r(t)$

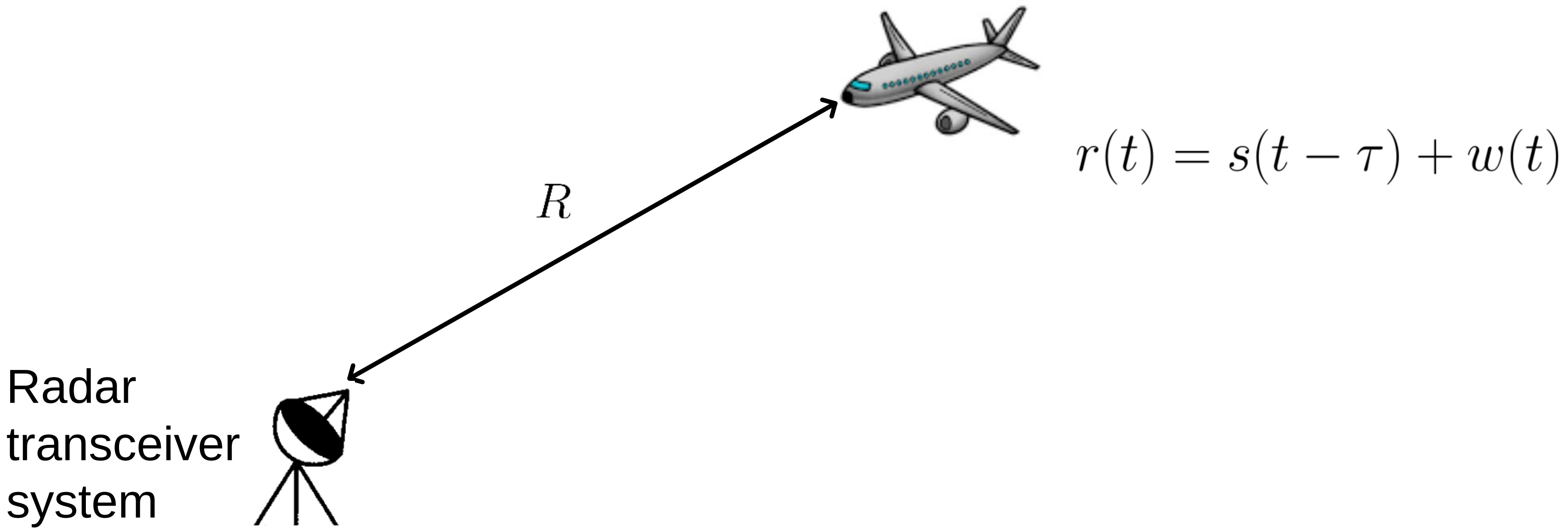


Fig.1.4: Radar ranging

Given the signal propagation speed, denoted by c, the time delay τ is related to R as:

$$\tau = \frac{2R}{c} \tag{1.1}$$

Hence radar pulse contains the object range information.

- Can be a function of one, two or three independent variables, e.g., speech is 1-D signal, function of time; image is 2-D, function of space; wind is 3-D, function of latitude, longitude and elevation.

- 3 types of signals that are functions of time:
  - Continuous-time (analog) $x(t)$: defined on a continuous range of time t, amplitude can be any value.
  - Discrete-time $x(nT)$ (sampled): defined only at discrete instants of time $t = \cdots - T, 0, T, 2T, \cdots$, amplitude can be any value.
  - Digital (quantized) $x_Q(nT)$: both time and amplitude are discrete, i.e., it is defined only at $t = \cdots - T, 0, T, 2T, \cdots$ and amplitude is confined to a finite set of numbers.

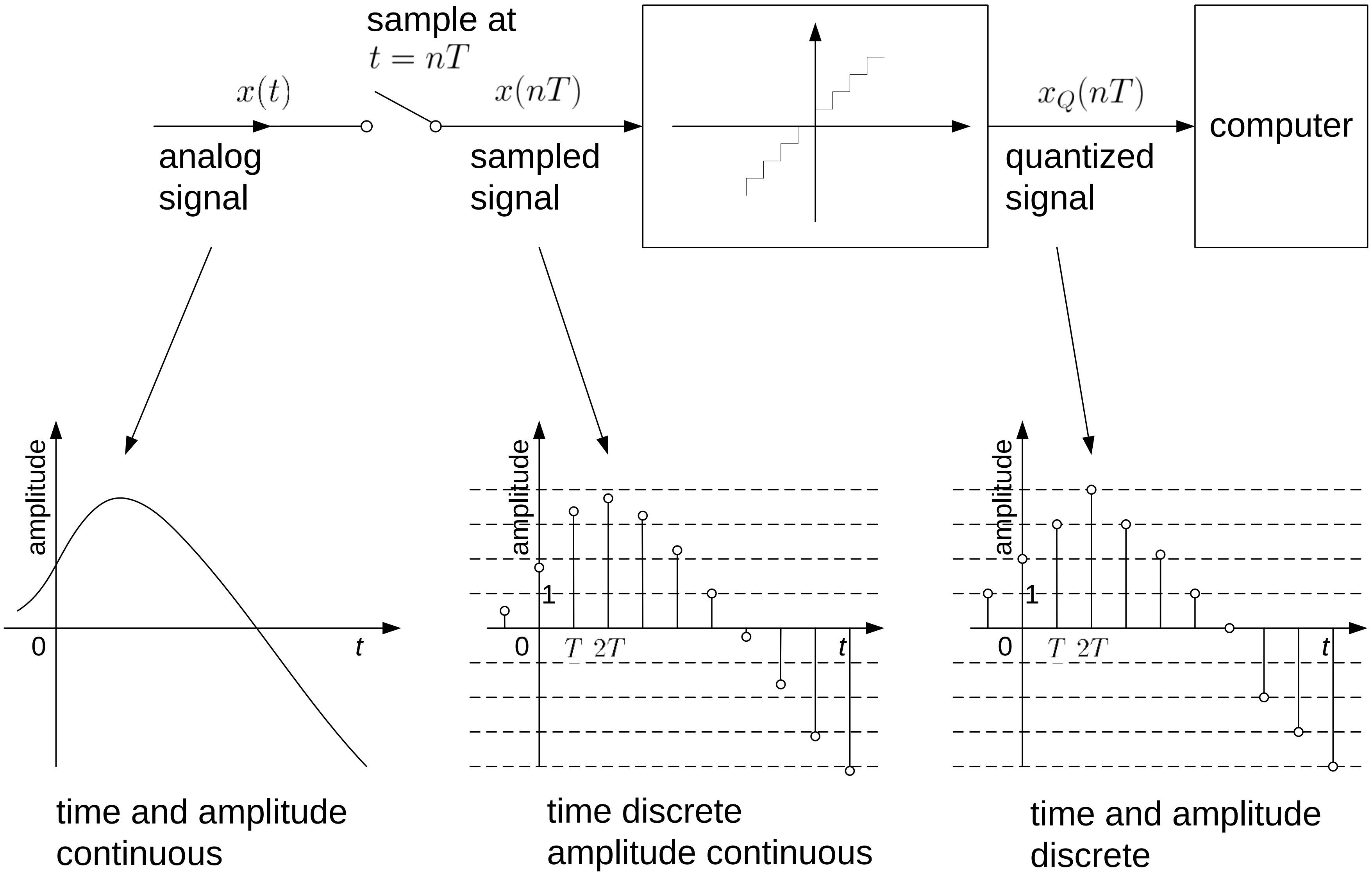


Fig. 1.5: Relationships between $x(t)$, $x(nT)$ and $x_Q(nT)$

$x(nT)$ at $n=0$ is close to 2 and $x_Q(0)=2$.

$x(nT) \in (3,4)$ at $n=1$ and $x_Q(T)=3$.

Using 4-bit representation, $x_Q(0)=0010$ and $x_Q(T)=0011$, and in general, the value of $x_Q(nT)$ is restricted to be an integer between -8 and 7 according to the two's complement representation.

In this course, we focus on continuous-time and discrete-time signals. Discrete-time signal is also commonly represented by $x[n]$ with $n=\cdots-1,0,1,\cdots$ being the time index (You can just consider normalizing T in $x(nT)$ to be 1).

The digital signal can be considered as discrete-time if the quantizer has very high resolution.

What is System?

- Mathematical model or abstraction of a physical process that relates input to output:

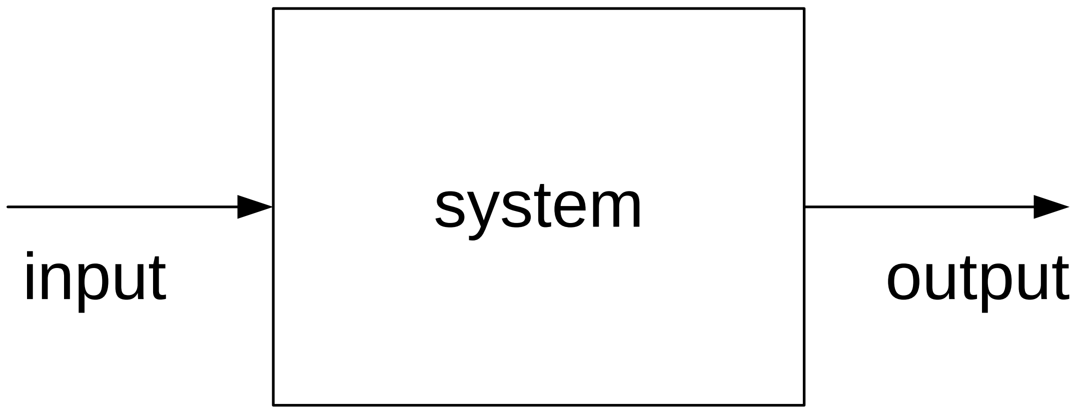


Fig.1.6: System with input and output

- It operates on an input to produce an output, e.g.:
  - Grading system: inputs are coursework and examination marks, output is grade.

- Squaring system: input is 5, then the output is 25.
- Amplifier: input is $\cos(\omega t)$, then output is $10\cos(\omega t)$.
- Communication system: input to mobile phone is voice, output from mobile phone is CDMA signal.
- Noise reduction system: input is a noisy speech, output is a noise-reduced speech.
- Feature extraction system: input is $\cos(\omega t)$, output is ω.

- An analog system deals with continuous-time input and output while a discrete-time system deals discrete-time input and output.

- A system can be realized in hardware or software via an algorithm.

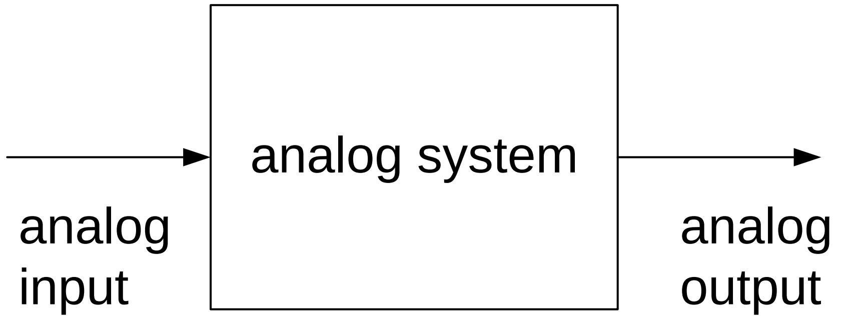


Fig.1.7: Continuous-time system

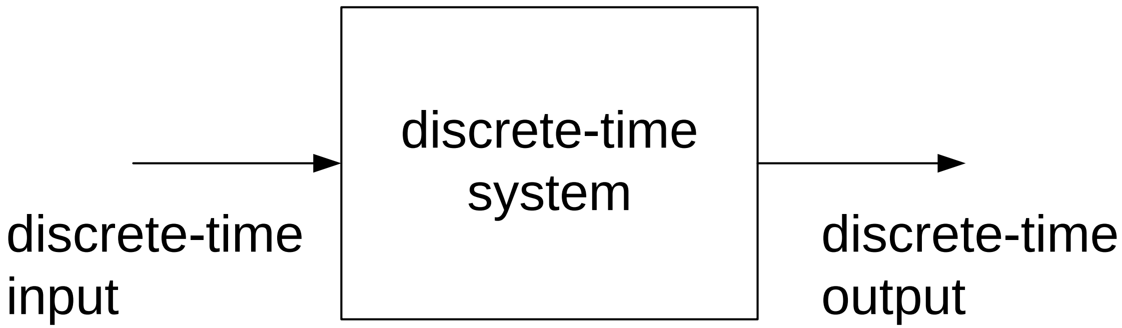


Fig.1.8: Discrete-time system

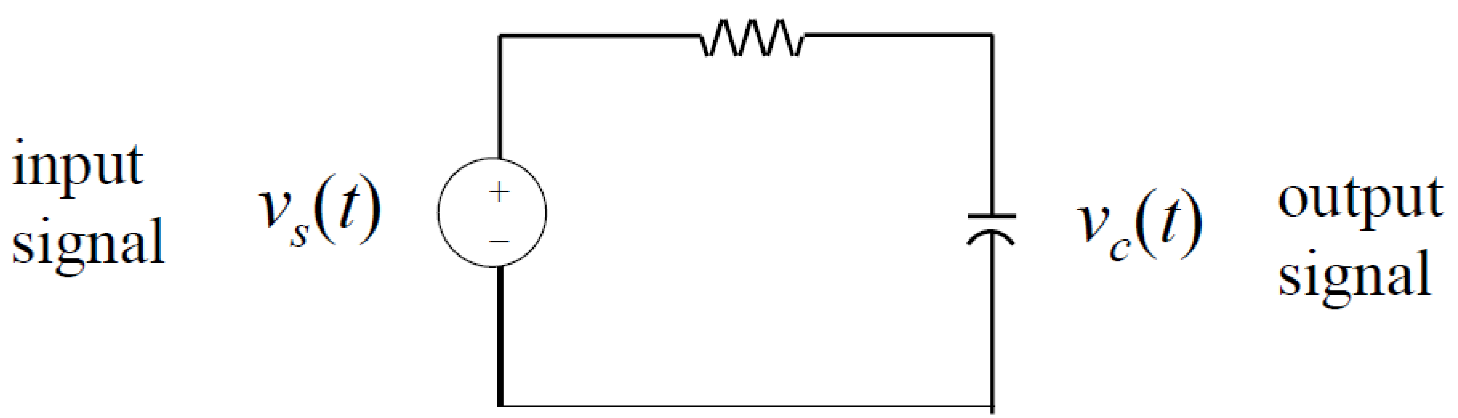


Fig.1.9: Hardware system of resistor-capacitor circuit

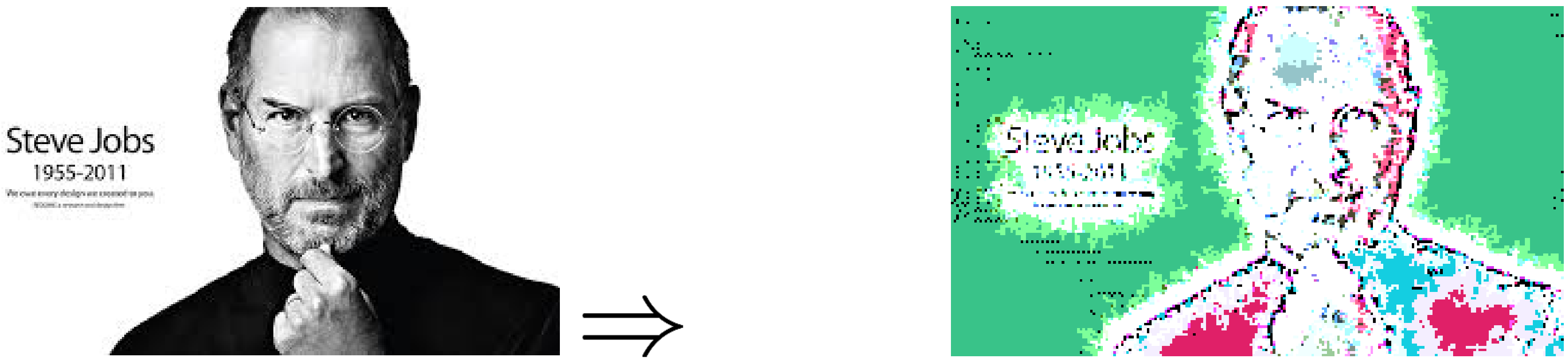


Fig.1.10: Pop-art production using an algorithm

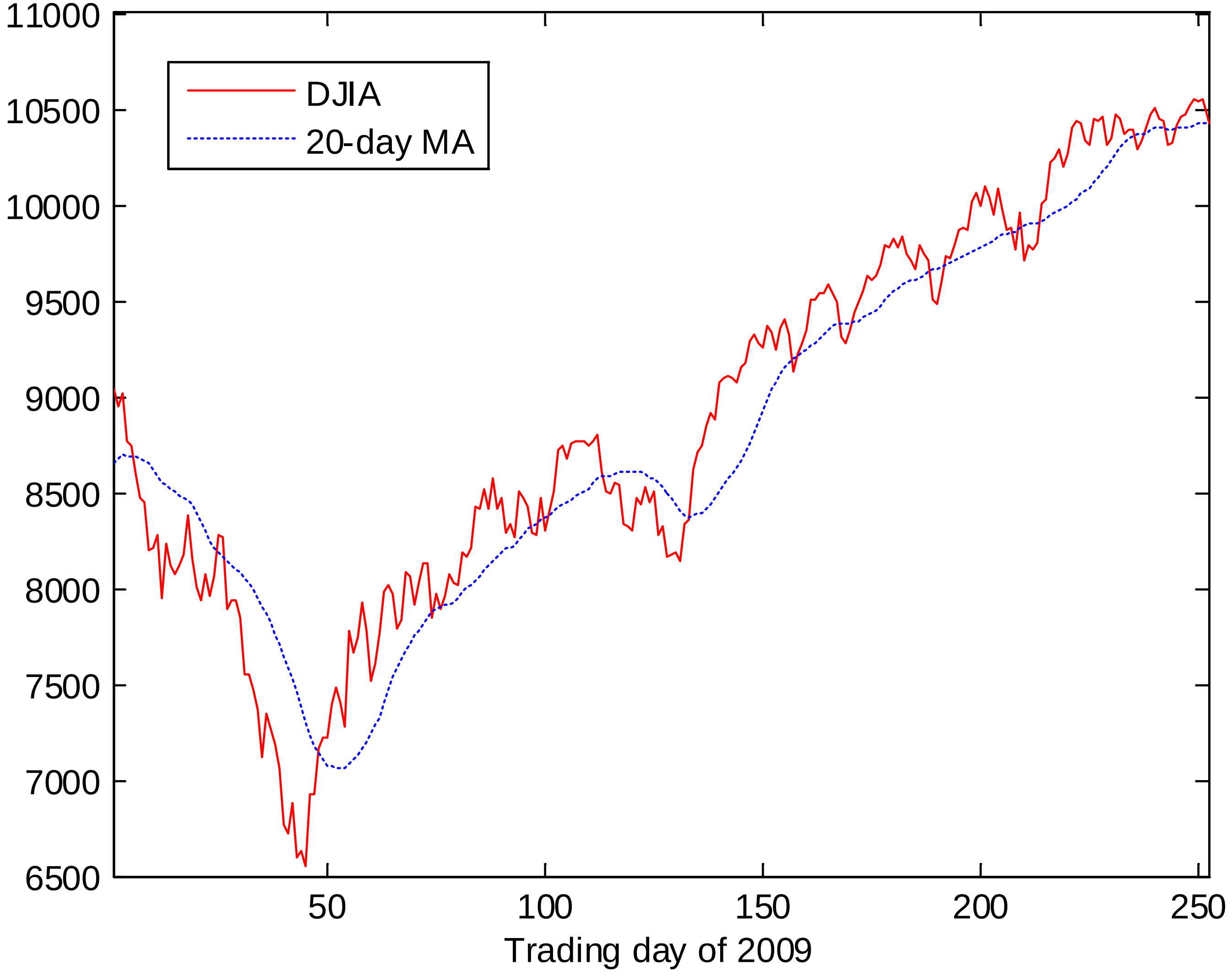


Fig.1.11: Software system for moving average of Dow Jones

What will You Learn?

- Signal representation and characterization, which includes generating signals, classifying signal types and properties, performing operations on signals.
- System classification and analysis, which includes analysis of system stability and causality, understanding the importance of impulse response in linear time-invariant (LTI) systems.
- Transform tools include Fourier series and Fourier transform as well as their applications in signal and LTI system analysis, e.g.: a periodic continuous-time signal $x(t)$ can be represented as sum of complex exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \qquad t \in (-\infty, \infty) \qquad (1.2)$$

Why Important?

- Signals and systems arise in our daily life, studying it will lay a good foundation for you in other relevant/higher-level courses and to solve real-world problems:
  - Generate signals which meet certain specifications, e.g., synthesized speech and music.
  - Design systems which produce desired outputs, e.g., a system which suppresses noise in the measured data
  - New signal representation for efficient data processing, e.g., David Donoho proposed sparse representation and obtained the Shaw Prize 2013.
    https://www.youtube.com/watch?v=5wv4grOMgIU

How to Study?

Make sure you have a clear concept and then practice.

Signals in Time Domain

Chapter Intended Learning Outcomes:

(i) Classify different types of signals

(ii) Perform basic operations on signals

(iii) Recognize basic continuous-time and discrete-time signals and understand their properties

(iv) Generate and visualize discrete-time signals using MATLAB

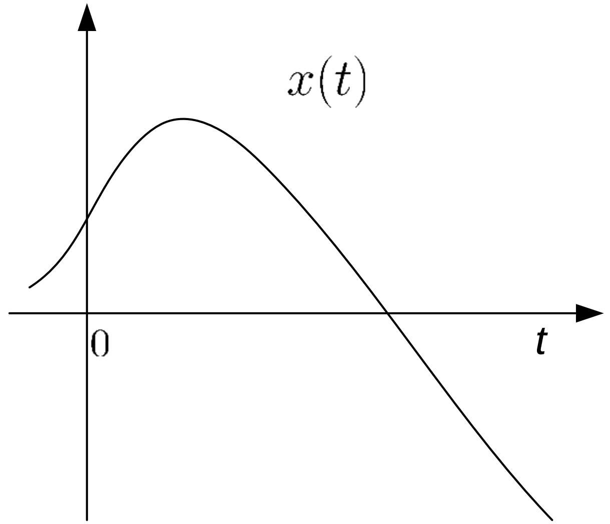
Classification of Signals

There are many ways of classifying signals. Common examples are provided as follows.

Continuous-Time versus Discrete-Time

$x(t)$: take a value at every instant of time t.

$x[n]$: defined only at particular instants of time n.



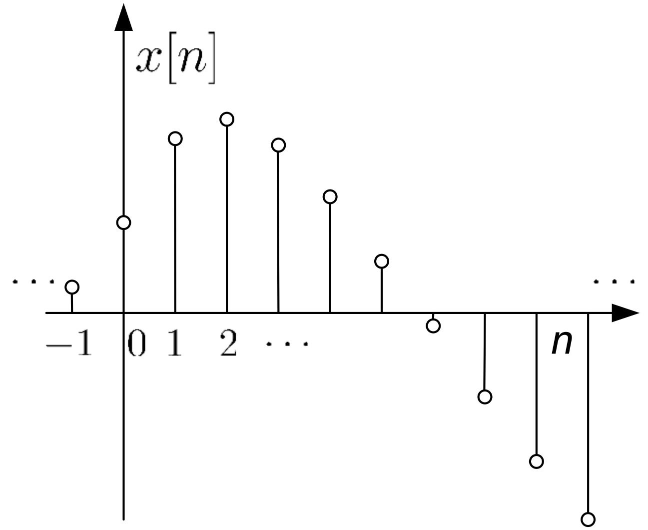


Fig. 2.1: Continuous-time versus discrete-time signals

Real versus Complex

Real-valued signal means that $x(t)$ or $x[n]$ is real for all t or n.

Complex-valued signal means that $x(t)$ or $x[n]$ can be decomposed as:

$$x(t) = \Re\{x(t)\} + j\Im\{x(t)\} \text{ or } x[n] = \Re\{x[n]\} + j\Im\{x[n]\} \quad (2.1)$$

where $\Re\{\}$ and $\Im\{\}$ represent the real and imaginary parts, respectively, while the latter is nonzero, and $j = \sqrt{-1}$.

Note that for a complex number x, we can also use magnitude $|x|$ and phase $\angle(x)$ for its representation:

$$|x| = \sqrt{(\Re\{x\})^2 + (\Im\{x\})^2} \quad (2.2)$$

and

$$\angle(x) = \tan^{-1}\left(\frac{\Im\{x\}}{\Re\{x\}}\right) \quad (2.3)$$

The magnitude can also be computed as:

$$|x| = \sqrt{x \cdot x^*} \tag{2.4}$$

where

$$x^* = \Re\{x\} - j\Im\{x\} \tag{2.5}$$

is the complex conjugate of x.

Furthermore, we see that complex signals include real signals.

<u>Example 2.1</u>

Determine if the following signals are real or complex.

(a) $x(t) = 1 + 2t + 3t^2$

(b) $x(t) = jt$

(a) It is real-valued signal as $x(t)$ is real for all t.

(b) It is complex as $x(t)$ has nonzero imaginary component.

Periodic versus Aperiodic

$x(t)$ is said to be periodic if there exists $T > 0$ such that

$$x(t) = x(t + T) \tag{2.6}$$

for all t. The smallest T for which (2.6) holds is called the fundamental period.

$x[n]$ is said to be periodic if there exists a positive integer N such that

$$x[n] = x[n + N] \tag{2.7}$$

for all n. The smallest N for which (2.7) holds is called the fundamental period.

If a signal is not periodic, then it is aperiodic.

Example 2.2

Determine if the following signals are periodic or not. If it is periodic, compute the fundamental period.

(a) $x(t) = 1 + 2t + 3t^2$

(b) $x(t) = \cos(10\pi t)$

(c) $x(t) = \begin{cases} \cos(100t), & t \in [1, 100] \\ 0, & \text{otherwise} \end{cases}$

(d) $x[n] = \cos\left(\frac{2\pi n}{3}\right)$

(e) $x[n] = \cos\left(\frac{2n}{3}\right)$

(a) A quadratic function should not be periodic.

(b) As cosine function has a period of 2π, we can write:

$$x(t) = \cos(10\pi t) = \cos(10\pi t + 2\pi) = x(t + 2\pi/(10\pi)) = x(t + 1/5)$$

Hence it is periodic with $T = 1/5$.

(c) $x(t)=x(t+T)$ is not fulfilled for all t and it is aperiodic.

(d) Again, we can write:

$$x[n]=\cos\left(\frac{2\pi n}{3}\right)=\cos\left(\frac{2\pi n}{3}+2\pi\right)=\cos\left(\frac{2\pi(n+3)}{3}\right)=x[n+3]$$

Hence it is periodic with $N=3$.

(e) It is aperiodic because we cannot find an integer N to fulfil (2.3).

Even versus Odd

A signal is called an even function if

$$x_e(t)=x_e(-t) \quad \text{or} \quad x_e[n]=x_e[-n] \tag{2.8}$$

A signal is called an odd function if

$$x_o(t)=-x_o(-t) \quad \text{or} \quad x_o[n]=-x_o[-n] \tag{2.9}$$

Any signal can be represented by a sum of even and odd signals:

$$x(t) = x_e(t) + x_o(t) \quad \text{or} \quad x[n] = x_e[n] + x_o[n] \tag{2.10}$$

where

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \quad \text{and} \quad x_o(t) = \frac{1}{2}[x(t) - x(-t)] \tag{2.11}$$

or

$$x_e[n] = \frac{1}{2}[x[n] + x[-n]] \quad \text{and} \quad x_o[n] = \frac{1}{2}[x[n] - x[-n]] \tag{2.12}$$

<u>Example 2.3</u>

Determine if the following signals are even or odd.

(a) $x(t) = \cos(\Omega\pi t)$

(b) $x(t) = \sin(\Omega\pi t)$

(c) $x(t) = \sin(\Omega\pi t + \theta)$

(d) $x[n] = 1 + 2n - 3n^2$

(a) It is even because $\cos(\Omega\pi t) = \cos(-\Omega\pi t)$.
(b) It is odd because $\sin(\Omega\pi t) = -\sin(-\Omega\pi t)$.

(c) It is neither odd nor even because:

$$\sin(\Omega\pi t + \theta) = \sin(\theta)\cos(\Omega\pi t) + \cos(\theta)\sin(\Omega\pi t)$$

which is a linear combination of even and odd functions.

(d) It is neither odd nor even. Applying (2.12) yields:

$$x_e[n] = \frac{1}{2}\left[(1+2n-3n^2) + (1+2(-n)-3(-n)^2)\right] = \frac{1}{2}\left[2-6n^2\right] = 1-3n^2$$

$$x_o[n] = \frac{1}{2}\left[(1+2n-3n^2) - (1+2(-n)-3(-n)^2)\right] = \frac{1}{2}\left[4n\right] = 2n$$

which are the even and odd components. Note that the same result can be easily obtained by inspection.

Energy versus Power

Energy of $x(t)$ or $x[n]$ is defined as:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \text{or} \quad \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{2.13}$$

If the signal energy is infinite, it is meaningful to use power of $x(t)$ or $x[n]$ as the measure, which is defined as:

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \quad \text{or} \quad \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{2.14}$$

Signal power is the time average of signal energy.

Note that for real signal, $|x(t)|^2 = x^2(t)$, while $|x(t)|^2 = x(t)x^*(t)$ for complex signal.

A signal is energy signal if $0 < E_x < \infty$, indicating its $P_x = 0$.

A signal is power signal if $0 < P_x < \infty$, indicating its $E_x = \infty$.

Example 2.4

Determine if the following signals are energy or power signals and then compute their energies or powers.

(a) $x(t) = \begin{cases} -2, & t \in [0, 10] \\ 0, & \text{otherwise} \end{cases}$

(b) $x(t) = A\cos(\omega t + \theta)$

(c) $x[n] = 10e^{j2n}$

(a) $x^2(t) = 4$ only for $t \in [0, 10]$ and is zero otherwise. Thus it is an energy signal. Using (2.13), we get:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{10} 4dt = 40$$

(b) From Example 2.2, we know that $x(t)$ is periodic with $T = 2\pi/\omega$. We can just use one period in (2.14):

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T}\int_0^T x^2(t)dt \\ &= \frac{\omega}{2\pi}\int_0^T A^2\cos^2(\omega t+\theta)dt = \frac{A^2\omega}{2\pi}\int_0^{2\pi/\omega}\frac{1}{2}\left[1+\cos(2\omega t+2\theta)\right]dt \\ &= \frac{\omega}{2\pi}\cdot\frac{A^2}{2}\cdot\frac{2\pi}{\omega} = \frac{A^2}{2} < \infty \end{aligned}$$

Hence it is a periodic signal with power $A^2/2$.

(c) $|x[n]|^2 = 10e^{j2n}\cdot 10e^{-j2n} = 100$. Summing $|x[n]|^2$ from $n=-\infty$ to $n=\infty$ is infinity and thus it is a power signal with P_x:

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2 = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}100 = 100$$

Basic Signal Operations

Three basic operators on signals are described as follows.

Time Shift

Shift the signal to left or right:

$$x(t) \longrightarrow x(t-t_0) \quad \text{or} \quad x[n] \longrightarrow x[n-n_0] \tag{2.15}$$

If t_0 or n_0 is positive, then it corresponds to time delay while it is a time advance for negative t_0 or n_0

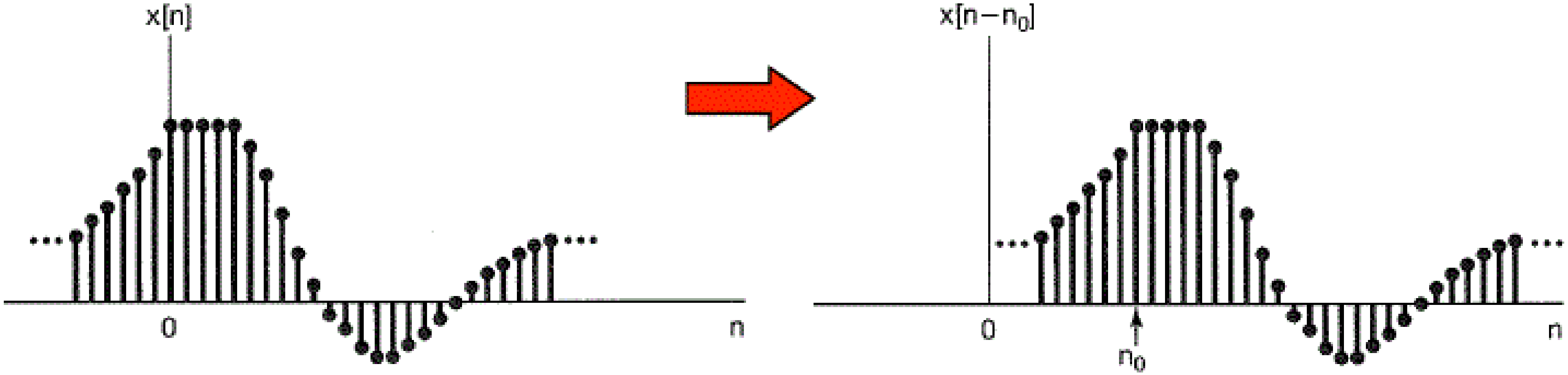


Fig. 2.2: Illustration of time shift

Time Reversal

Flip the signal around the vertical axis:

$$x(t) \longrightarrow x(-t) \quad \text{or} \quad x[n] \longrightarrow x[-n] \tag{2.16}$$

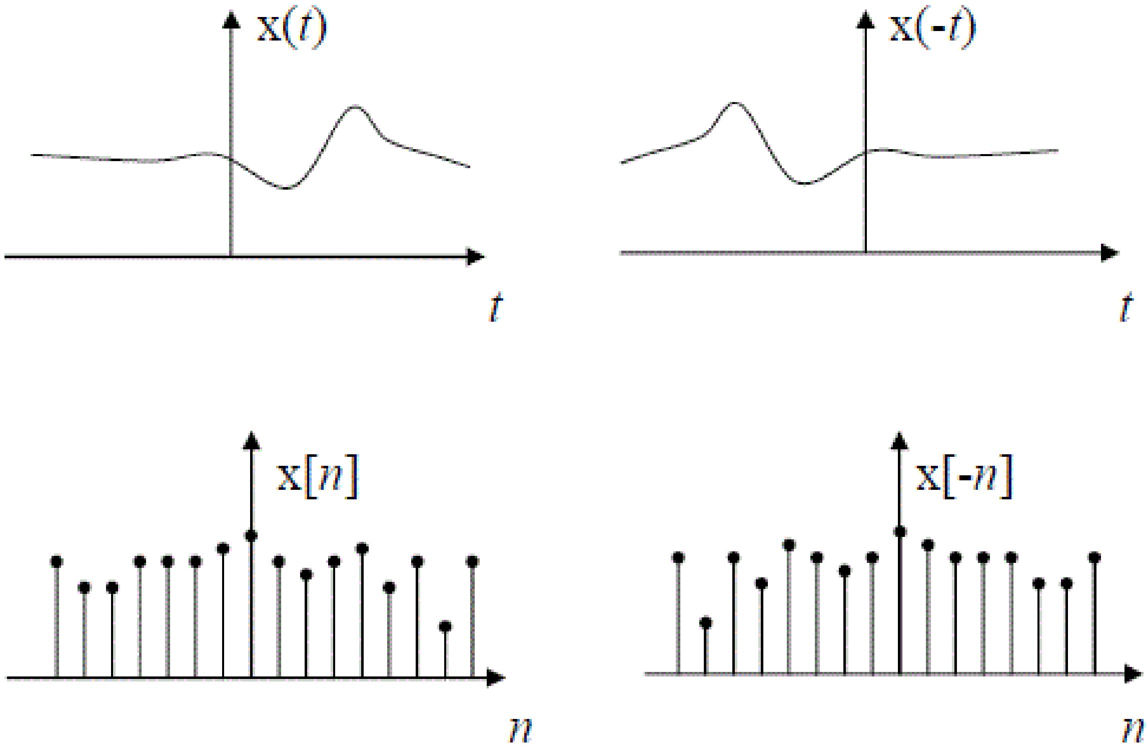


Fig. 2.3: Illustration of time reversal

Time Scaling

Linearly stretch or compress the signal:

$$x(t) \longrightarrow x(ct) \qquad (2.17)$$

where $c < 1$ means stretch and $c > 1$ means compression. We do not discuss $x[n]$ as it is not defined for all time instants.

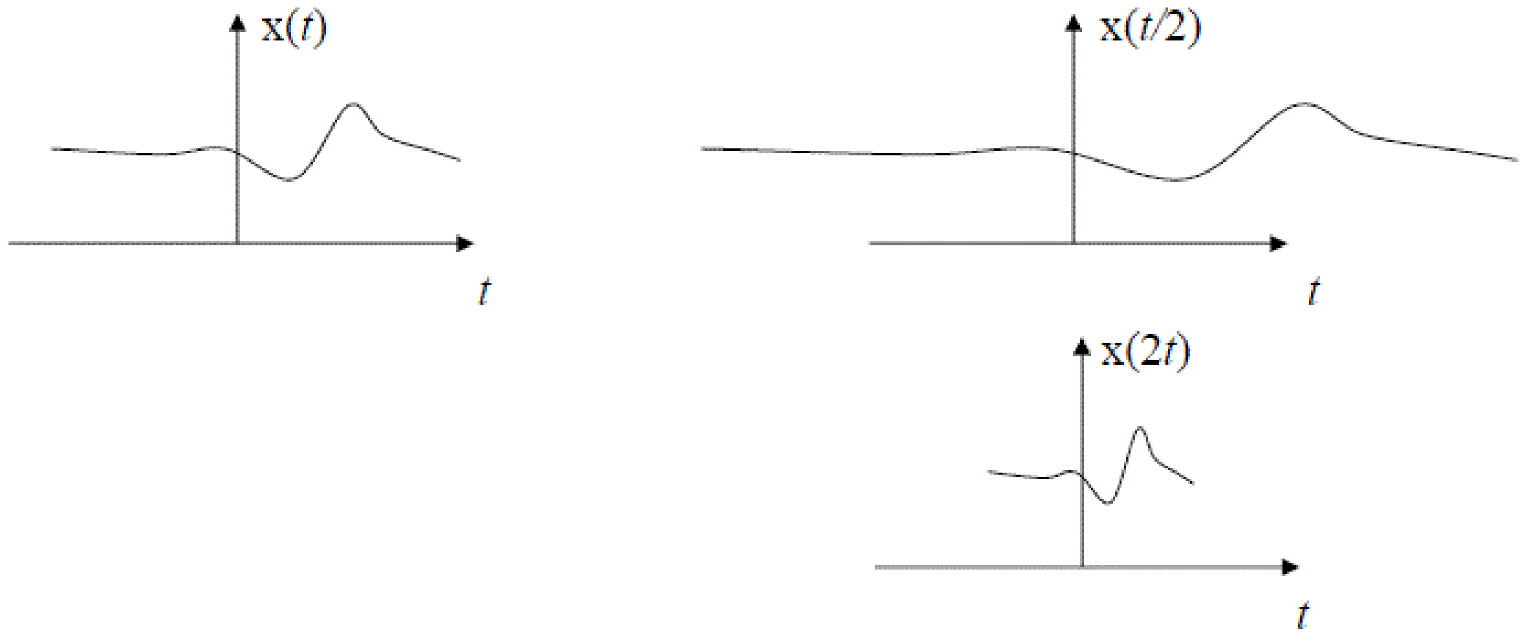


Fig. 2.4: Illustration of time scaling

Basic Continuous-Time Signals

Typical examples of continuous-time signals are described as follows.

Unit Impulse

The unit impulse $\delta(t)$ has the following characteristics:

$$\delta(t) = 0, \quad t \neq 0 \tag{2.18}$$

and

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \tag{2.19}$$

(2.18) and (2.19) indicate that $\delta(t)$ has a very large value or impulse at $t = 0$. That is, $\delta(t)$ is not well defined at $t = 0$.

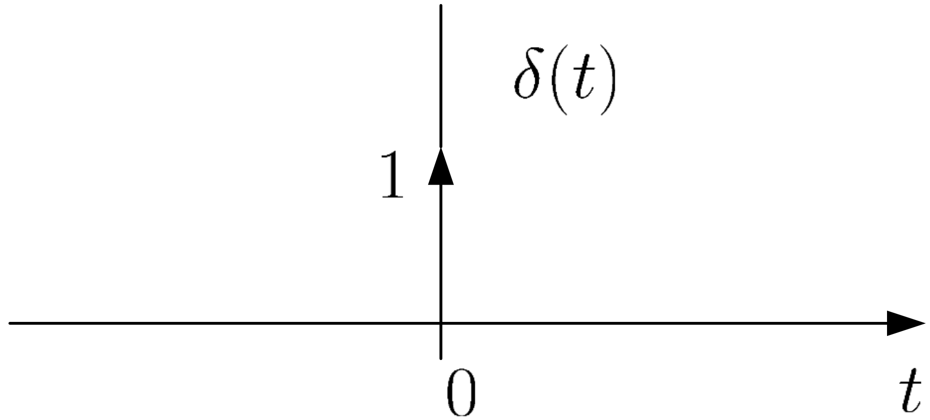


Fig. 2.5: Graphical representation of $\delta(t)$

From (2.18), multiplying a continuous-time signal $x(t)$ by an impulse $\delta(t-t_0)$ gives:

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0) \quad (2.20)$$

That is, we only need to care about the value of $x(t)$ at the impulse location, namely, $t=t_0$.

We may consider $\delta(t)$ as the building block of any continuous-time signal, described by the sifting property:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \tag{2.21}$$

That is, imagining $x(t)$ as a sum of infinite impulse functions and each with amplitude $x(\tau)$.

Unit Step

The unit step function $u(t)$ has the form of:

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{2.22}$$

As there is a sudden change from 0 to 1 at $t = 0$, $u(0)$ is not well defined.

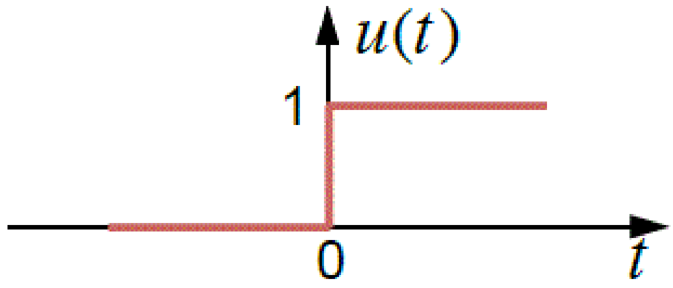


Fig. 2.6: Graphical representation of $u(t)$

$u(t)$ can be expressed in terms of $\delta(t)$ as:

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau = \int_{0}^{\infty} \delta(t-\tau)d\tau \qquad (2.23)$$

Conversely, we can use $u(t)$ to represent $\delta(t)$:

$$\delta(t) = \frac{du(t)}{dt} \qquad (2.24)$$

Sinusoid

It is a sine wave of the form:

$$x(t) = A\cos(\omega t + \phi) \quad (2.25)$$

which is characterized by three parameters, amplitude $A > 0$, radian frequency ω and phase $\phi \in [0, 2\pi)$.

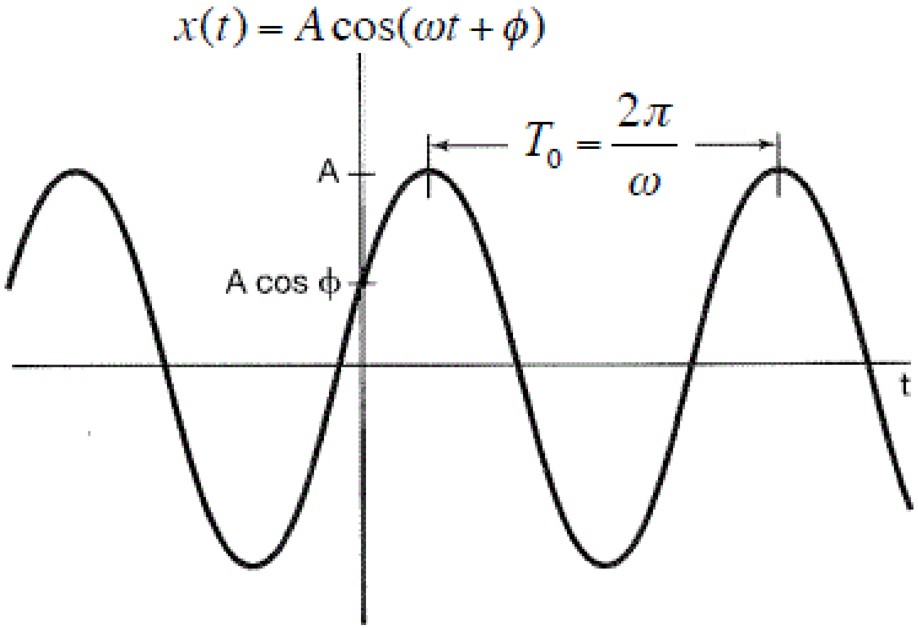


Fig. 2.7: Sinusoid

Rate of oscillation increases with ω.

Apart from ω, $f = \omega/(2\pi)$, the frequency in Hz can be used.

Fundamental period T_0 is determined as:

$$x(t) = x(t+T_0) = A\cos(\omega(t+T_0)+\phi) = A\cos(\omega t + 2\pi + \phi)$$

$$\Rightarrow \omega T_0 = 2\pi \Rightarrow T_0 = \frac{2\pi}{\omega} = \frac{1}{f} \tag{2.26}$$

For the complex-valued case, it has the form of:

$$x(t) = Ae^{j(\omega t+\phi)} \tag{2.27}$$

Using the Euler formula:

$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \tag{2.28}$$

It is seen that the real part of (2.27) is (2.25), while the imaginary part is $A\sin(\omega t + \phi)$ which is also a sinusoid.

A complex sinusoid is also periodic with radian frequency ω.

According to (2.28), we can obtain:

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2} \tag{2.29}$$

and

$$\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j} \tag{2.30}$$

Note that the general form of (2.25) or (2.27) is:

$$x(t) = \sum_{l=1}^{L} A_l \cos(\omega_l t + \phi_l) \quad \text{or} \quad x(t) = \sum_{l=1}^{L} A_l e^{j(\omega_l t + \phi_l)} \tag{2.31}$$

which is a sum of L tones.

Exponential

For the real-valued case, it has the form:

$$x(t) = Ae^{at} \tag{2.32}$$

where A and a are real numbers.

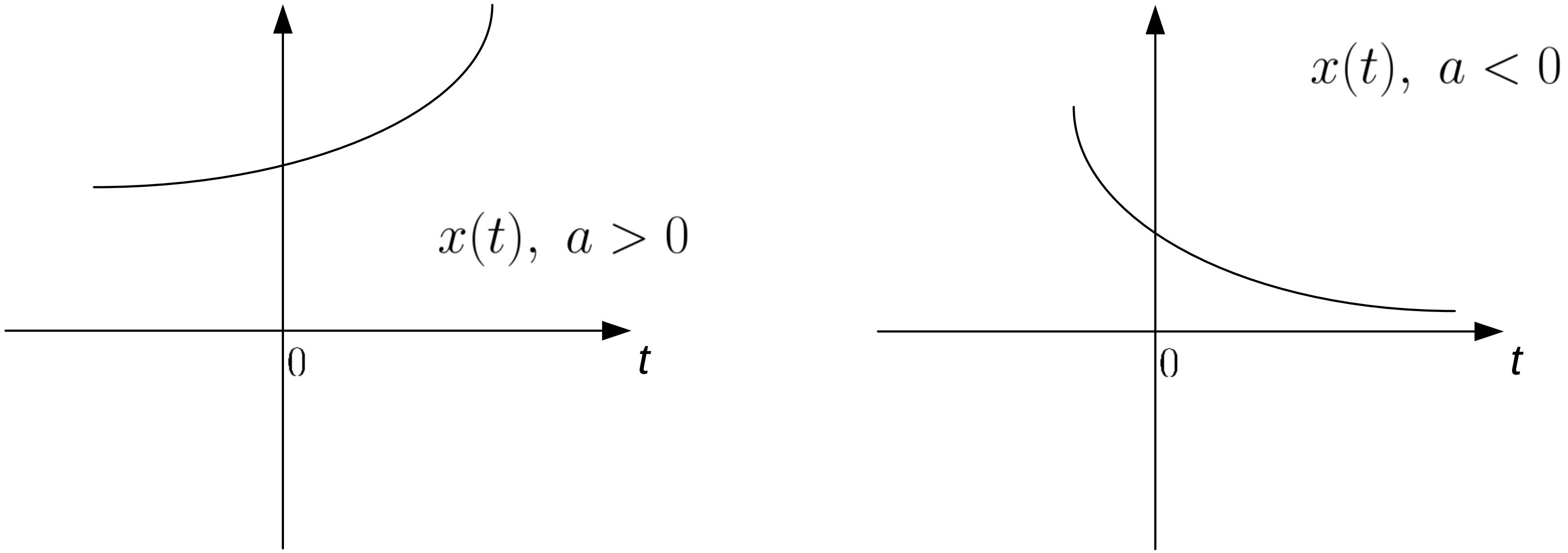


Fig. 2.8: Real exponential with $A > 0$

With complex A and a, (2.32) also represents complex case.

Basic Discrete-Time Signals

Typical examples of discrete-time signals are described as follows.

Unit Impulse (or Sample)

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \qquad (2.33)$$

which is similar to $\delta(t)$ but $\delta[n]$ is simpler because it is well defined for all n while $\delta(t)$ is not defined at $t = 0$.

Unit Step

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \qquad (2.34)$$

which is similar to $u(t)$ but $u[n]$ is well defined for all n while $u(t)$ is not defined $t = 0$.

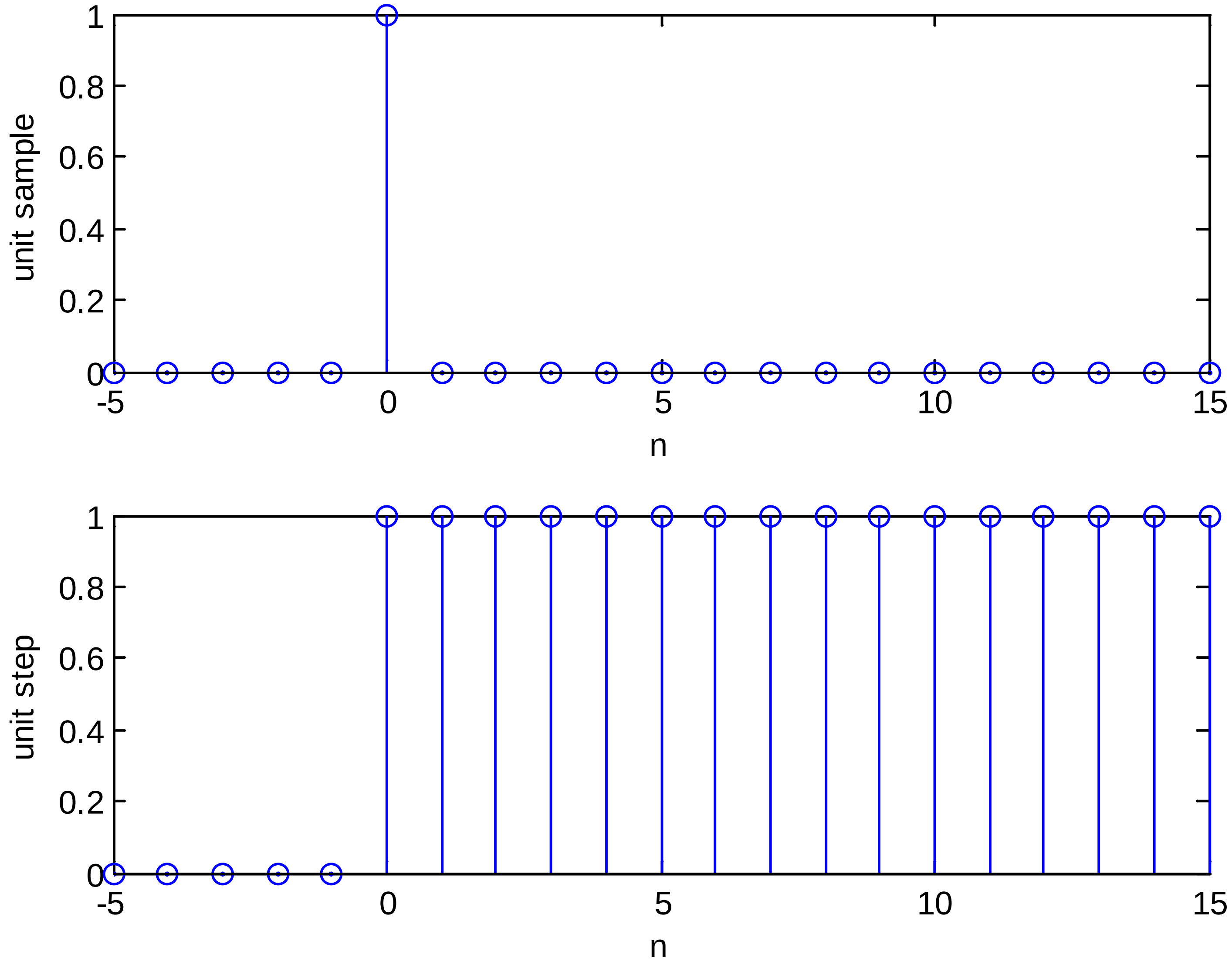


Fig. 2.9: Unit sample $\delta[n]$ and unit step $u[n]$

$\delta[n]$ is an important function because it serves as the building block of any discrete-time signal $x[n]$:

$$x[n] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$
$$= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad (2.35)$$

For example, $u[n]$ can be expressed in terms of $\delta[n]$ as:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \qquad (2.36)$$

Conversely, we can use $u[n]$ to represent $\delta[n]$:

$$\delta[n] = u[n] - u[n-1] \qquad (2.37)$$

which are analogous to (2.21), (2.23)-(2.24).

Sinusoid

For real-valued case, it has the form of:

$$x[n] = A\cos(\omega n + \theta) \tag{2.38}$$

which is characterized by three parameters, amplitude $A > 0$, radian frequency ω and phase $\phi \in [0, 2\pi)$.

(2.38) can be extended to the complex model as:

$$x[n] = Ae^{j(\omega n + \phi)} \tag{2.39}$$

The general form of (2.38) or (2.39) is:

$$x[n] = \sum_{l=1}^{L} A_l \cos(\omega_l n + \phi_l) \quad \text{or} \quad x[n] = \sum_{l=1}^{L} A_l e^{j(\omega_l n + \phi_l)} \tag{2.40}$$

which is a sum of L sinusoids.

Exponential

For the real-valued case, it has the form:

$$x[n] = Ae^{an} \tag{2.41}$$

where A and a are real numbers.

(2.41) can also represent complex exponential by using complex A and a.

Introduction to MATLAB

MATLAB stands for "Matrix Laboratory".

Interactive matrix-based software for numerical and symbolic computation in scientific and engineering applications.

Its user interface is relatively simple to use, e.g., we can use the `help` command to understand the usage and syntax of each MATLAB function.

Together with the availability of numerous toolboxes, there are many useful and powerful commands for various disciplines.

MathWorks offers MATLAB to C conversion utility.

Similar packages include Maple and Mathematica.

Example 2.5

Use MATLAB to generate a discrete-time sinusoid of the form:

$$x[n] = A\cos(\omega n + \theta), \quad n = 0, 1, \cdots, N-1$$

with $A = 1$, $\omega = 0.3$, $\theta = 1$ and $N = 21$, which has a duration of 21 samples.

We can generate $x[n]$ by using the following MATLAB code:

```
N=21;                  %number of samples is 21
A=1;                   %tone amplitude is 1
w=0.3;                 %frequency is 0.3
p=1;                   %phase is 1
for n=1:N
x(n)=A*cos(w*(n-1)+p); %time index should be >0
end
```

Note that x is a vector and its index should be at least 1.

Alternatively, we can also use:

```
N=21;                     %number of samples is 21
A=1;                      %tone amplitude is 1
w=0.3;                    %frequency is 0.3
p=1;                      %phase is 1
n=0:N-1;                  %define time index vector
x=A.*cos(w.*n+p);         %first time index is also 1
```

Both give

```
x =
Columns 1 through 7
0.5403 0.2675 -0.0292 -0.3233 -0.5885 -0.8011 -0.9422
Columns 8 through 14
-0.9991 -0.9668 -0.8481 -0.6536 -0.4008 -0.1122 0.1865
Columns 15 through 21
0.4685 0.7087 0.8855 0.9833 0.9932 0.9144 0.7539
```

Which approach is better? Why?

To plot $x[n]$, we can either use the commands `stem(x)` and `plot(x.)`

If the time index is not specified, the default start time is $n = 1$.

Nevertheless, it is easy to include the time index vector in the plotting command.

e.g., Using `stem` to plot $x[n]$ with the correct time index:

```
n=0:N-1;                %n is vector of time index
stem(n,x)               %plot x versus n
```

Similarly, `plot(n,x)` can be employed to show $x[n]$.

The MATLAB programs for this example are provided as `ex2_5.m` and `ex2_5_2.m`.

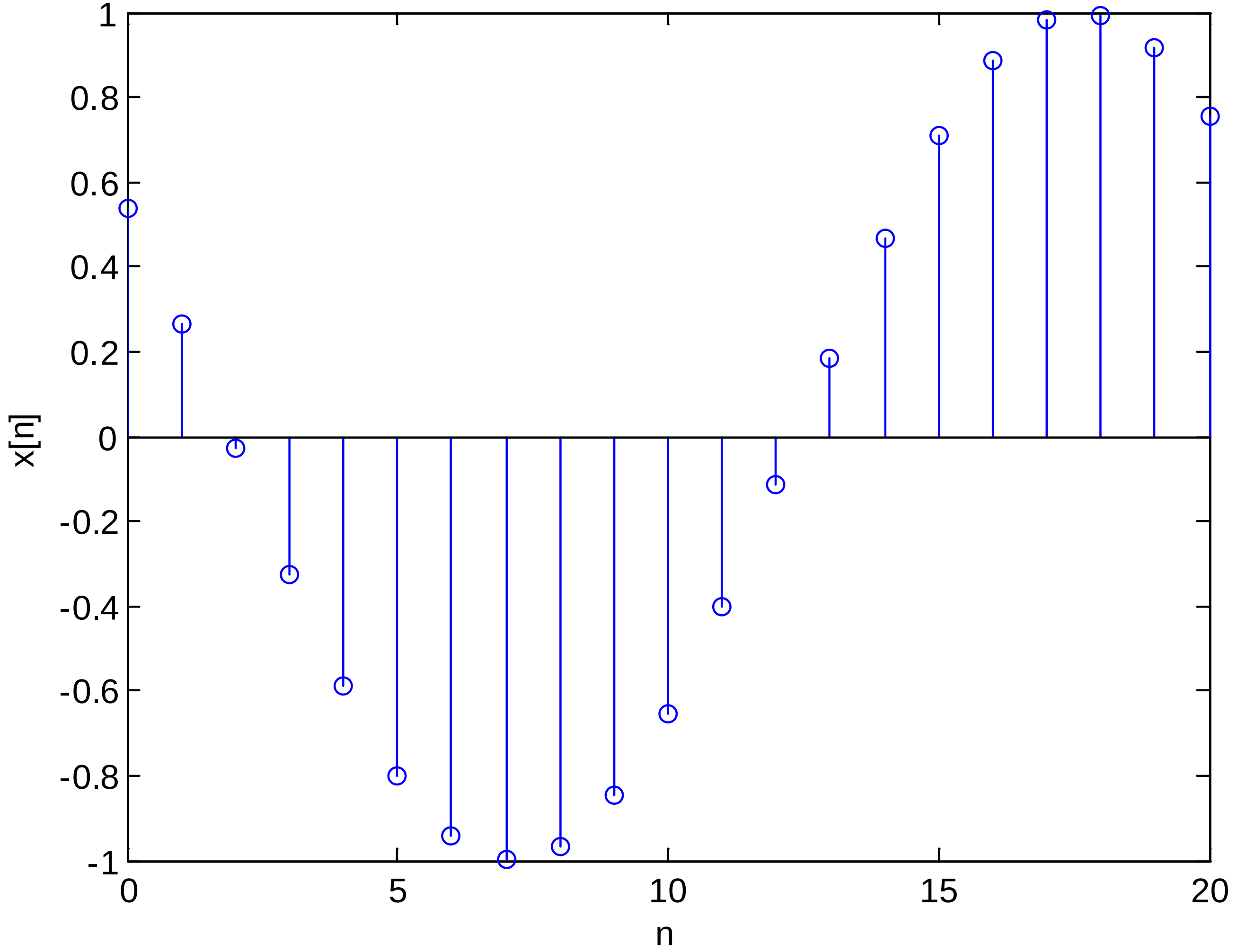


Fig. 2.10: Plot of discrete-time sinusoid using stem

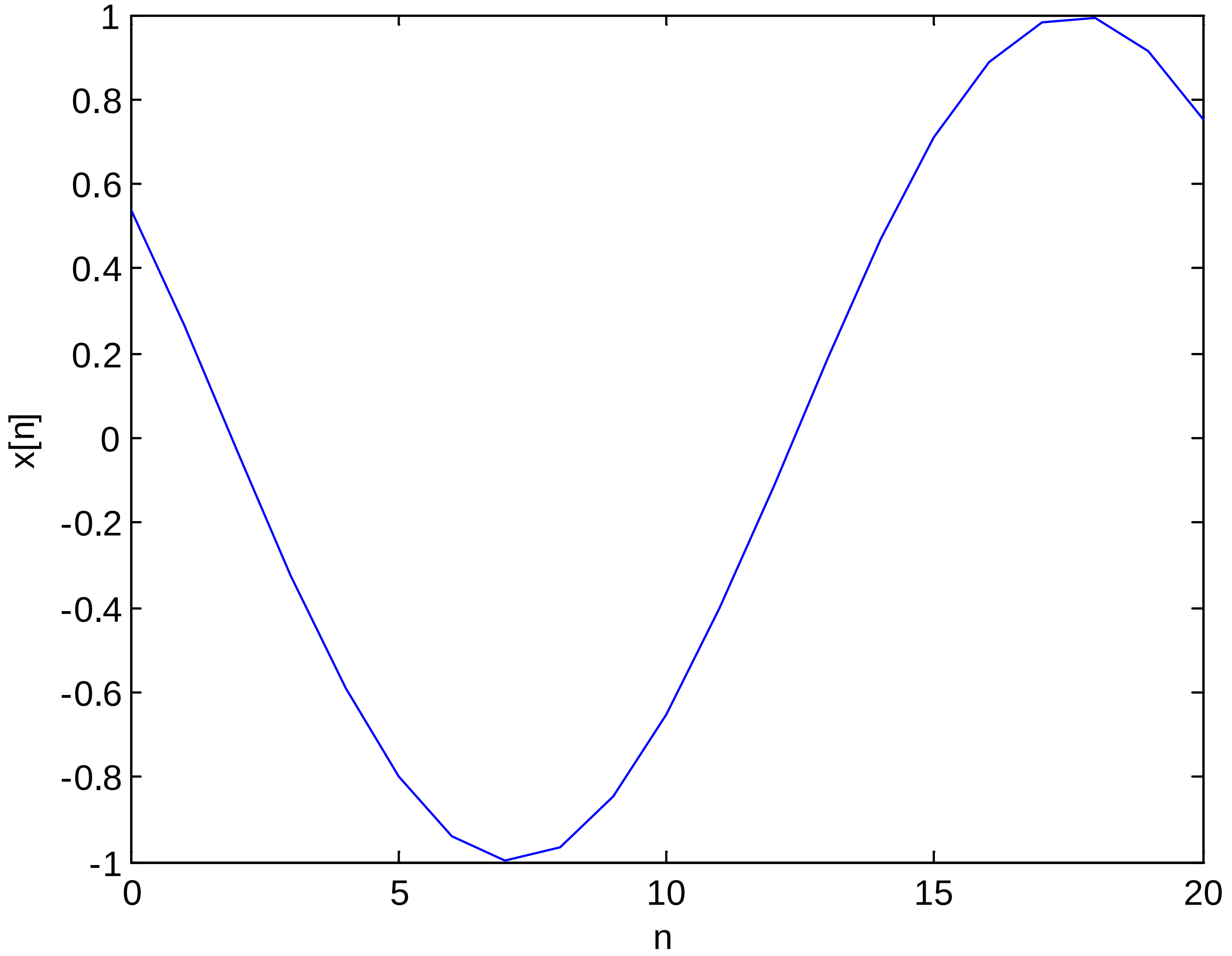


Fig. 2.11: Plot of discrete-time sinusoid using plot

Systems in Time Domain

Chapter Intended Learning Outcomes:

(i) Classify different types of systems

(ii) Understand the property of convolution and its relationship with linear time-invariant system

(iii) Understand the relationship between differential equation, difference equation and linear time-invariant system

(iv) Perform basic operations in systems

System Overview

It can be classified as continuous-time and discrete-time:

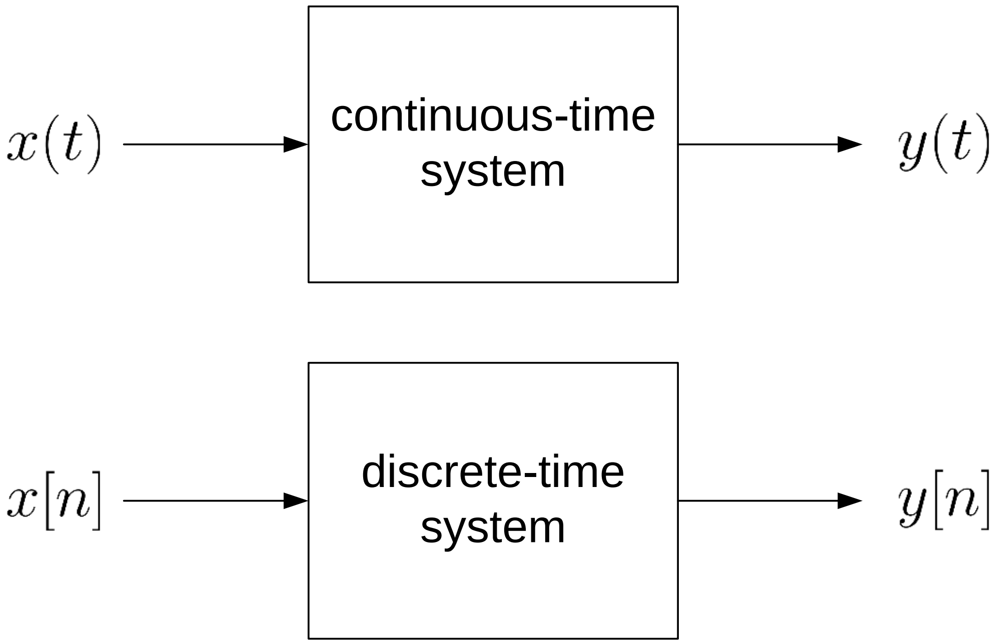


Fig. 3.1: Continuous-time and discrete-time systems

In a continuous-time (discrete-time) system, the input and output are continuous-time (discrete-time) signals.

A system is an operator $\mathcal{T}$ which maps input into output:

$$y(t) = \mathcal{T}\{x(t)\} \quad \text{or} \quad y[n] = \mathcal{T}\{x[n]\} \tag{3.1}$$

Systems can be connected/combined to form a bigger/overall system, e.g., break down a big task into several sub-tasks and each system handles one sub-task.

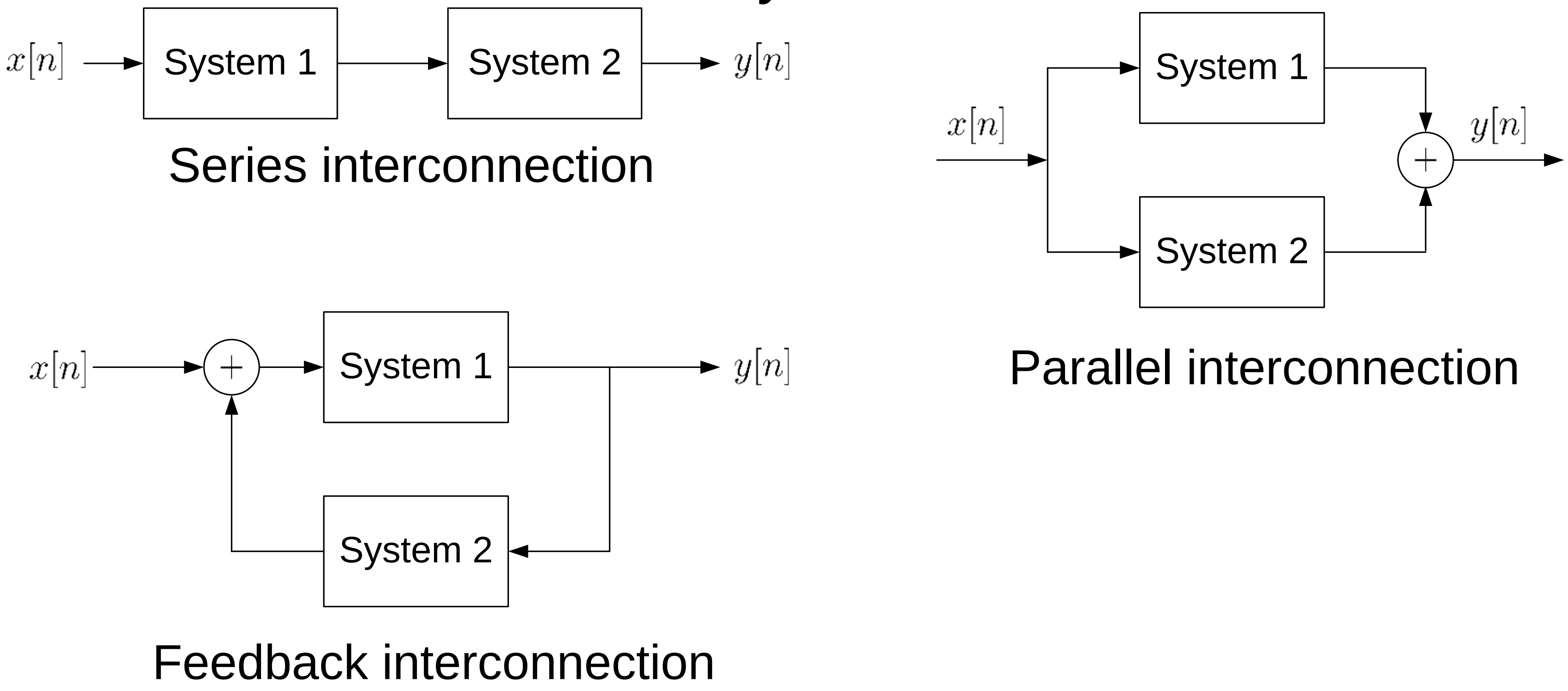


Fig. 3.2: Examples of system interconnections

Basic System Properties

Memoryless, invertibility, causality, stability, linearity, and time-invariance, are described as follows.

Memoryless

A system is memoryless if its output at a given time is dependent only on the input at that same time, i.e., $y(t)$ at time t depends only on $x(t)$ at time t; $y[n]$ at time n depends only on $x[n]$ at time n.

A memoryless system does not have memory to store any input values because it just operates on the current input.

If a system is not memoryless, we can call it a system with memory.

Example 3.1

Determine if the following systems are memoryless or not

(a) $y(t) = x^2(t)$

(b) $y[n] = x[n] + x[n-2]$

(a) The system is memoryless because the output at time t depends only on the input at time t.

(b) The system is not memoryless because $y[n]$ also depends only on $x[n-2]$, which is a previous input, and thus it needs memory to store $x[n-2]$ when processing the input at time n.

Invertibility

A system is invertible if distinct inputs lead to distinct outputs, or if an inverse system exists.

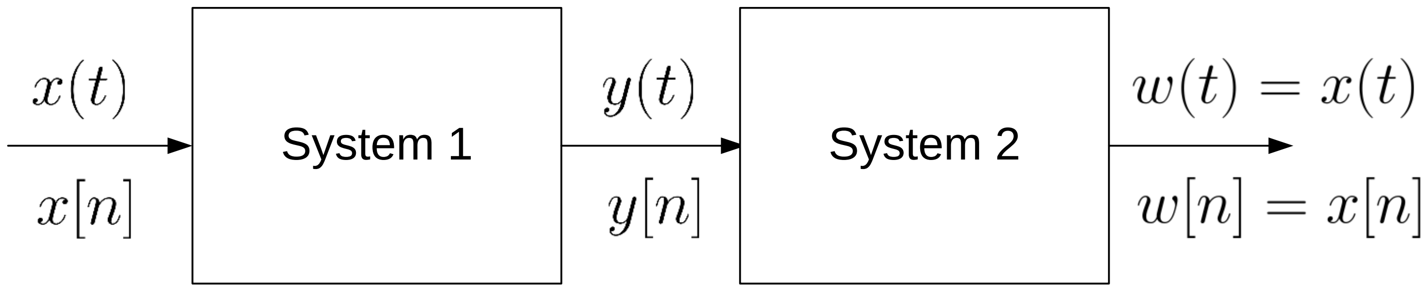


Fig. 3.3: Invertible system

That is, if we can get back the input $x(t)$ or $x[n]$ by passing the output $y(t)$ or $y[n]$ through another system, then the system is invertible, otherwise it is non-invertible.

Example 3.2

Determine if the following systems are invertible or not

(a) $y(t) = 2x(t)$

(b) $y(t) = x^2(t)$

(c) $y[n] = \sum_{k=-\infty}^{n} x[k]$

(d) $y[n] = 0$

(a) The system is invertible because we can pass $y(t)$ using another system to produce $w(t) = 0.5y(t) = x(t)$.

(b) The system is not invertible because the sign information is lost in the system output. Even employing the square root function, there are two possibilities: $w(t) = \sqrt{y(t)}$ or $w(t) = -\sqrt{y(t)}$.

(c) $$y[n] = \sum_{k=-\infty}^{n} x[k] = [\cdots x[n-2] + x[n-1]] + x[n] = y[n-1] + x[n]$$

If we pass $y[n]$ using another system, $w[n] = y[n] - y[n-1] = x[n]$ can be obtained and hence the system is invertible.

(d) Any inputs will give the same output of zero and hence the system is not invertible.

Linearity

A system is linear if it obeys principle of superposition.

Given two pairs of inputs and outputs, linearity implies:

$$\mathcal{T}\{ax_1(t) + bx_2(t)\} = a\mathcal{T}\{x_1(t)\} + b\mathcal{T}\{x_2(t)\} = ay_1(t) + by_2(t) \quad (3.2)$$

and

$$\mathcal{T}\{ax_1[n] + bx_2[n]\} = a\mathcal{T}\{x_1[n]\} + b\mathcal{T}\{x_2[n]\} = ay_1[n] + by_2[n] \quad (3.3)$$

where $|a| < \infty$ and $|b| < \infty$.

If the system does not satisfy superposition, it is non-linear.

Example 3.3

Determine whether the following system with input $x[n]$ and output $y[n]$, is linear or not:

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \cdots + x[n-1] + x[n]$$

A standard approach to determine the linearity of a system is given as follows. Let

$$y_i[n] = \mathcal{T}\{x_i[n]\}, \quad i = 1, 2, 3$$

with

$$x_3[n] = ax_1[n] + bx_2[n]$$

If $y_3[n] = ay_1[n] + by_2[n]$, then the system is linear. Otherwise, the system is non-linear. This also applies to continuous-time system.

Assigning $x_3[n] = ax_1[n] + bx_2[n]$, we have:

$$\begin{aligned} y_3[n] &= \sum_{k=-\infty}^{n} x_3[k] \\ &= \sum_{k=-\infty}^{n} (ax_1[k] + bx_2[k]) \\ &= a\sum_{k=-\infty}^{n} x_1[k] + b\sum_{k=-\infty}^{n} x_2[k] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Note that the outputs for $x_1[n]$ and $x_2[n]$ are $y_1[n] = \sum_{k=-\infty}^{n} x_1[k]$ and $y_2[n] = \sum_{k=-\infty}^{n} x_2[k]$.

As a result, the system is linear.

Example 3.4

Determine whether the following system with input $x[n]$ and output $y[n]$, is linear or not:

$$y[n] = 3x^2[n] + 2x[n-3]$$

The system outputs for $x_1[n]$ and $x_2[n]$ are $y_1[n] = 3x_1^2[n] + 2x_1[n-3]$ and $y_2[n] = 3x_2^2[n] + 2x_2[n-3]$. Assigning $x_3[n] = ax_1[n] + bx_2[n]$, its system output is then:

$$\begin{aligned} y_3[n] &= 3x_3^2[n] + 2x_3[n-3] \\ &= 3\left(ax_1[n] + bx_2[n]\right)^2 + 2ax_1[n-3] + 2bx_2[n-3] \\ &= 3\left(a^2x_1^2[n] + b^2x_2^2[n] + 2abx_1[n]x_2[n]\right) + 2ax_1[n-3] + 2bx_2[n-3] \\ &\neq a\left(3x_1^2[n] + 2x_1[n-3]\right) + b\left(3x_2^2[n] + 2x_2[n-3]\right) \\ &= ay_1[n] + by_2[n] \end{aligned}$$

As a result, the system is non-linear.

Time-Invariance

A system is time-invariant if a time-shift of input causes a corresponding shift in output:

$$\text{if} \quad y(t) = \mathcal{T}\{x(t)\} \quad \text{then} \quad y(t-t_0) = \mathcal{T}\{x(t-t_0)\} \tag{3.4}$$

and

$$\text{if} \quad y[n] = \mathcal{T}\{x[n]\} \quad \text{then} \quad y[n-n_0] = \mathcal{T}\{x[n-n_0]\} \tag{3.5}$$

That is, the system response is independent of time.

Example 3.5

Determine whether the following system with input $x[n]$ and output $y[n]$, is time-invariant or not.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

A standard approach to determine the time-invariance of a system is given as follows.

Let $y_1[n] = \mathcal{T}\{x_1[n]\}$ where $x_1[n] = x[n-n_0]$. If $y_1[n] = y[n-n_0]$, then the system is time-invariant. Otherwise, the system is time-variant. This also applies to continuous-time system.

From the given input-output relationship, $y[n-n_0]$ is:

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

Let $x_1[n] = x[n-n_0]$, its system output is:

$$\begin{aligned} y_1[n] &= \sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x[k-n_0] = \sum_{l=-\infty}^{n-n_0} x[l], \quad l = k - n_0 \\ &= y[n-n_0] \end{aligned}$$

As a result, the system is time-invariant.

Example 3.6

Determine whether the following system with input $x[n]$ and output $y[n]$, is time-invariant or not:

$$y[n] = 3x[3n]$$

From the given input-output relationship, $y[n-n_0]$ is of the form:

$$y[n-n_0] = 3x[3(n-n_0)] = 3x[3n-3n_0]$$

Let $x_1[n] = x[n-n_0]$, its system output is:

$$y_1[n] = 3x_1[3n] = 3x[3n-n_0] \neq y[n-n_0]$$

As a result, the system is time-variant.

Causality

A system is causal if the output $y(t)$ (or $y[n]$) at time t (or n) depends on input $x(t)$ (or $x[n]$) up to time t (or n).

In casual system, output does not depend on future input.

On the other hand, in a non-causal system, the output depends on future input.

Example 3.7

Determine if the following systems are causal or not

(a) $y(t) = x^2(t)$

(b) $y[n] = x[n] + x[n+2]$

(c) $y[n] = \sum_{k=-\infty}^{n} x[k]$

(a) The system is causal because it does not depend on future input.

(b) The system is not causal because it depends on future input, namely, $x[n+2]$.

(c) $$y[n] = \sum_{k=-\infty}^{n} x[k] = \cdots x[n-2] + x[n-1] + x[n]$$

We see that the output $y[n]$ at time n depends on input $x[n]$ up to time n. Hence the system is causal.

Stablity

A system is stable if every bounded input $x(t)$ or $x[n]$ produces a bounded output $y(t)$ or $y[n]$ for all time t or n. That is:

$$|y(t)| < B \quad \text{if} \quad |x(t)| < A, \quad |A| < \infty, \ |B| < \infty \tag{3.6}$$

and

$$|y[n]| < B \quad \text{if} \quad |x[n]| < A, \quad |A| < \infty, \ |B| < \infty \tag{3.7}$$

If a bounded input produces an unbounded output, then the system is unstable.

<u>Example 3.8</u>

Determine if the following systems are stable or not

(a) $y(t) = x^2(t)$

(b) $y[n] = x[n] + x[n+2]$

(c) $y[n] = \dfrac{1}{x[n]}$

(a) If $x(t)$ is bounded, say, $|x(t)| < A$ for all t, we easily get

$$|y(t)| < A^2$$

Hence the system is stable.

(b) The system is stable because:

$$|y[n]| = |x[n] + x[n+2]| \leq |x[n]| + |x[n+2]| < 2A$$

for a bounded input with $|x[n]| < A$ for all n.

(c) The system is not stable. It is because for a bounded input, namely, $x[n] = 0$, the output is unbounded.

Linear Time-Invariant System Characterization

In this course, we will mainly study systems which are both linear and time-invariant.

Apart from being fundamental, many practical applications correspond to linear time-invariant (LTI) system.

Impulse Response

The impulse response ($h(t)$ or $h[n]$) is the output of a LTI system when the input is the unit impulse ($\delta(t)$ or $\delta[n]$):

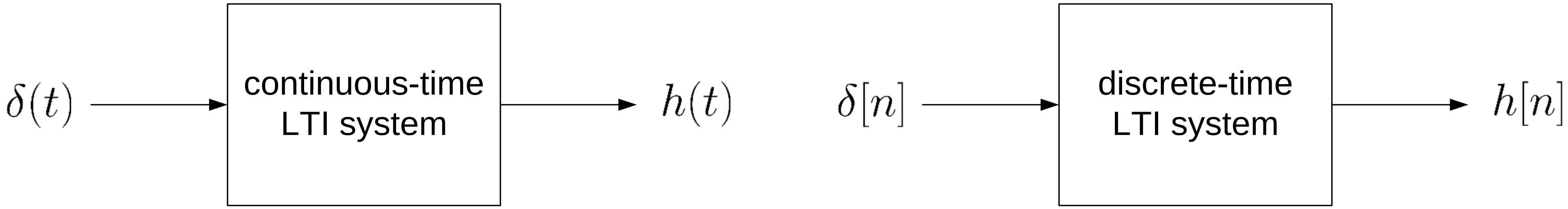


Fig. 3.4: Impulse response

For a continuous-time system, the impulse response is also continuous-time signal.

For a discrete-time system, the impulse response is also discrete-time signal.

A LTI system can be characterized by its impulse response, which indicates the system functionality.

Convolution

Convolution is used to describe the relationship between input, output and impulse response of a LTI in time domain.

We start with considering the discrete-time impulse response $h[n] = \mathcal{T}\{\delta[n]\}$ of a LTI system.

Recall (2.35) that a discrete-time signal is a linear combination of impulses with different time-shifts:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \tag{3.8}$$

Consider $x[n]$ as the system input with $y[n]$ being the output:

$$\begin{aligned} y[n] &= \mathcal{T}\{x[n]\} = \mathcal{T}\left\{\sum_{m=-\infty}^{\infty} x[m]\delta[n-m]\right\} \\ &= \sum_{m=-\infty}^{\infty} x[m]\mathcal{T}\{\delta[n-m]\} \end{aligned} \tag{3.9}$$

due to the linearity property of (3.3).

Furthermore, using time-invariance property yields:

$$h[n-m] = \mathcal{T}\{\delta[n-m]\} \tag{3.10}$$

Substituting (3.10) into (3.9), we obtain:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] \otimes h[n] \tag{3.11}$$

which is called the convolution of $x[n]$ and $h[n]$, and $\otimes$ denotes the convolution operator.

According to (3.11), $h[n]$ completely characterizes the LTI system because for any input $x[n]$, the output $y[n]$ can be computed with the use of $h[n]$ via convolution where only multiplication and addition are involved.

There are three properties in convolution:

- Commutative

$$\begin{aligned} x[n] \otimes h[n] &= h[n] \otimes x[n] \\ &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \end{aligned} \quad (3.12)$$

- Associative

$$x[n] \otimes (h_1[n] \otimes h_2[n]) = (x[n] \otimes h_1[n]) \otimes h_2[n] \quad (3.13)$$

Combining (3.12) and (3.13) yields:

$$\begin{aligned} y[n] &= x[n] \otimes h_1[n] \otimes h_2[n] \\ &= x[n] \otimes h_2[n] \otimes h_1[n] \\ &= x[n] \otimes (h_1[n] \otimes h_2[n]) \end{aligned} \quad (3.14)$$

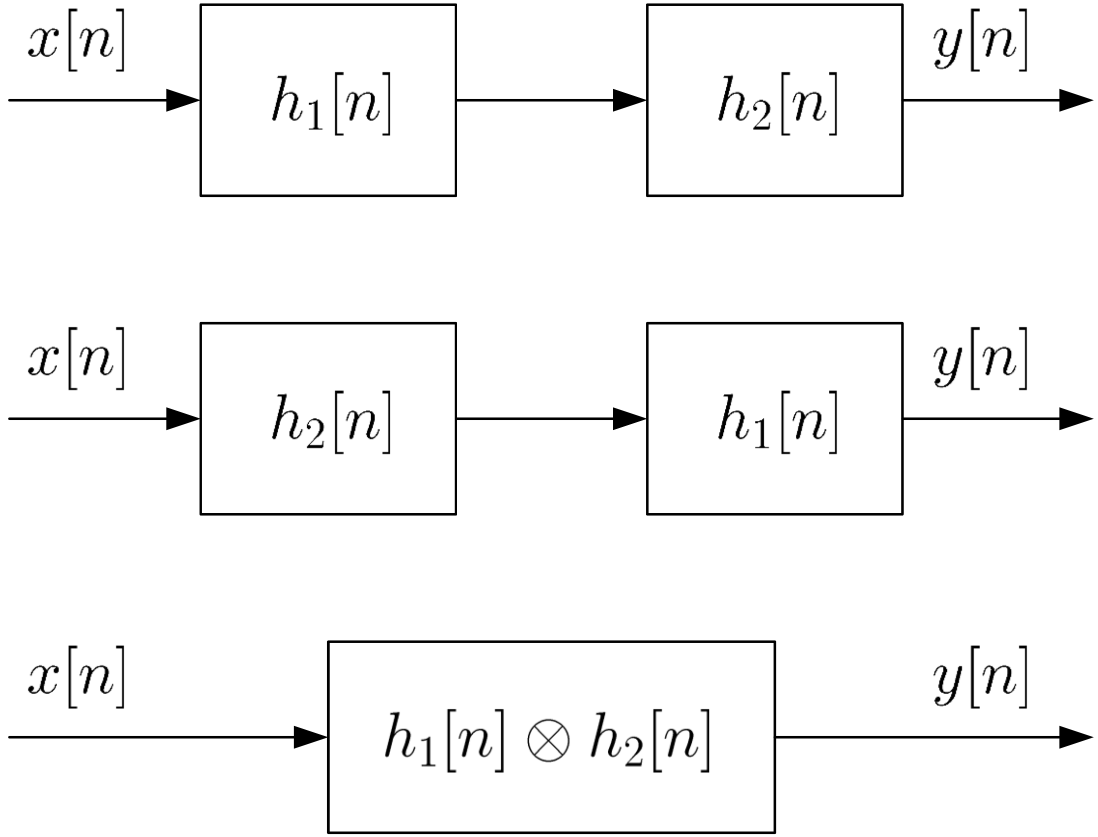


Fig.3.5: Cascade interconnection and convolution

- Distributive

$$y[n] = x[n] \otimes (h_1[n] + h_2[n]) = x[n] \otimes h_1[n] + x[n] \otimes h_2[n] \quad (3.15)$$

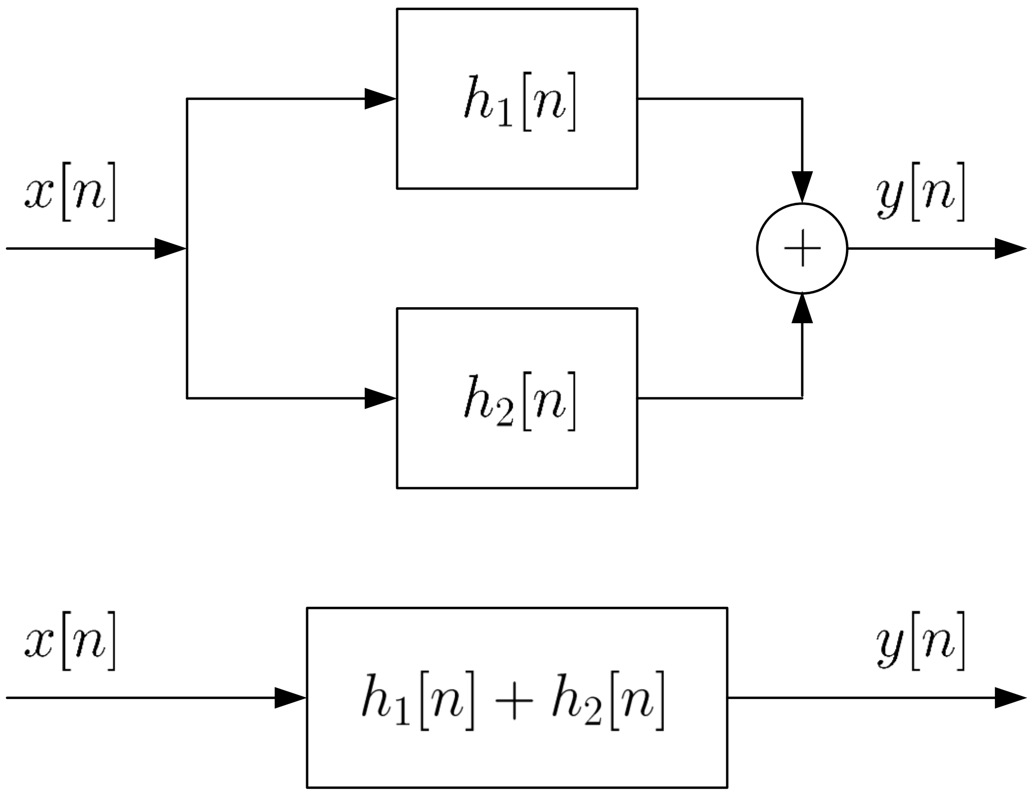


Fig.3.6: Parallel interconnection and convolution

Example 3.9

Determine the function of a LTI discrete-time system if its impulse response is $h[n] = 0.5\delta[n] + 0.5\delta[n-1]$.

Using (3.11) and (3.8), we have:

$$
\begin{aligned}
y[n] &= x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \sum_{m=-\infty}^{\infty} x[m]\left(0.5\delta[n-m] + 0.5\delta[n-1-m]\right) \\
&= 0.5 \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] + 0.5 \sum_{m=-\infty}^{\infty} x[m]\delta[n-1-m] \\
&= 0.5(x[n] + x[n-1])
\end{aligned}
$$

The system computes the mean value of two input samples, current value and past value.

Similarly, for the continuous-time case, we start with considering $h(t) = \mathcal{T}\{\delta(t)\}$ of a LTI system.

Recall (2.21) that a continuous-time signal is considered as a linear combination of impulses with different time-shifts:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \qquad (3.16)$$

Analogous to the development in (3.9)-(3.11), we get:

$$\begin{aligned} y(t) &= \mathcal{T}\{x(t)\} = \mathcal{T}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\} \\ &= \int_{-\infty}^{\infty} x(\tau)\mathcal{T}\left\{\delta(t-\tau)\right\}d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t)\otimes h(t) \end{aligned} \qquad (3.17)$$

Equation (3.17) is the convolution for the continuous-time case. However, the computation is more complicated than the discrete-time convolution because integration is needed.

Again, we see that $h(t)$ characterizes the input-output relationship of LTI system.

Same as the discrete-time case, $h(t)$ specifies the system functionality and satisfies the commutative, associative as well as distributive properties.

Example 3.10

Determine the function of a LTI continuous-time system if its impulse response is $h(t) = \delta(t) + \delta(t-1)$.

Using (3.17) and (2.19)-(2.20), we obtain:

$$\begin{aligned}
y(t) &= x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)[\delta(t-\tau) + \delta(t-1-\tau)]d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)\delta(t-1-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau + \int_{-\infty}^{\infty} x(t-1)\delta(t-1-\tau)d\tau \\
&= x(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau + x(t-1)\int_{-\infty}^{\infty} \delta(t-1-\tau)d\tau \\
&= x(t) + x(t-1)
\end{aligned}$$

The system computes sum of inputs at two time instants, one at current time and the other at current time minus 1

Example 3.11

Determine the function of a LTI continuous-time system if its impulse response is $h(t) = 0.1[u(t) - u(t-10)]$.

Using (3.17) and the commutative property, we get:

$$
\begin{aligned}
y(t) &= h(t) \otimes x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \\
&= 0.1 \int_{-\infty}^{\infty} [u(\tau) - u(\tau - 10)] x(t-\tau) d\tau \\
&= \frac{1}{10} \int_{0}^{10} x(t-\tau) d\tau
\end{aligned}
$$

Note that $[u(\tau) - u(\tau - 10)]$ is a rectangular pulse for $\tau \in (0, 10)$.

The system computes average input value from the current time minus 10 to current time.

For LTI systems, we can also use the impulse response to check the system causality and stability.

A LTI system is causal if its impulse response satisfies:

$$h(t) = 0, \quad t < 0 \tag{3.18}$$

$$h[n] = 0, \quad n < 0 \tag{3.19}$$

A LTI system is stable if its impulse response satisfies:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty \tag{3.20}$$

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{3.21}$$

Example 3.12

Show that for a LTI discrete-time system, the causality definition in (3.19) is identical to the universal definition, i.e., $y[n]$ at time n depends on $x[n]$ up to time n.

Expanding the convolution formula in (3.12):

$$
\begin{aligned}
y[n] = x[n] \otimes h[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\
&= \cdots h[-2]x[n+2] + h[-1]x[n+1] + \\
&\quad h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + \cdots
\end{aligned}
$$

If $y[n]$ does not depend on future inputs $x[n+1], x[n+2], \cdots$, we must have $h[-1] = h[-2] = \cdots = 0$ or $h[n] = 0$ for $n < 0$.

Hence the two definitions regarding causality are identical.

Example 3.13

Compute the output $y[n]$ if the input is $x[n]=u[n]$ and the LTI system impulse response is $h[n]=\delta[n]+0.5\delta[n-1]$. Discuss the stability and causality of system.

Using (3.11), we have:

$$\begin{aligned} y[n] &= x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\ &= \sum_{m=-\infty}^{\infty} u[m]\left(\delta[n-m]+0.5\delta[n-1-m]\right) \\ &= \sum_{m=0}^{\infty} \left(\delta[n-m]+0.5\delta[n-1-m]\right) \\ &= \sum_{m=0}^{\infty} \delta[n-m] + 0.5\sum_{m=0}^{\infty} \delta[n-1-m] = u[n]+0.5u[n-1] \end{aligned}$$

Alternatively, we can first establish the general relationship between $y[n]$ and $x[n]$ with the specific $h[n]$ as in Example 3.9:

$$\begin{aligned} y[n] &= x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\ &= \sum_{m=-\infty}^{\infty} x[m] \left(\delta[n-m] + 0.5\delta[n-1-m]\right) \\ &= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] + 0.5 \sum_{m=-\infty}^{\infty} x[m]\delta[n-1-m] \\ &= x[n] + 0.5x[n-1] \end{aligned}$$

Substituting $x[n] = u[n]$ yields the same $y[n]$.

Since $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{1} |h[n]| = 1.5 < \infty$ and $h[n] = 0$ for $n < 0$ the system is stable and causal.

Example 3.14

Compute the output $y[n]$ if the input is $x[n] = a^n u[n]$ and the LTI system impulse response is $h[n] = u[n] - u[n-10]$. Discuss the stability and causality of system.

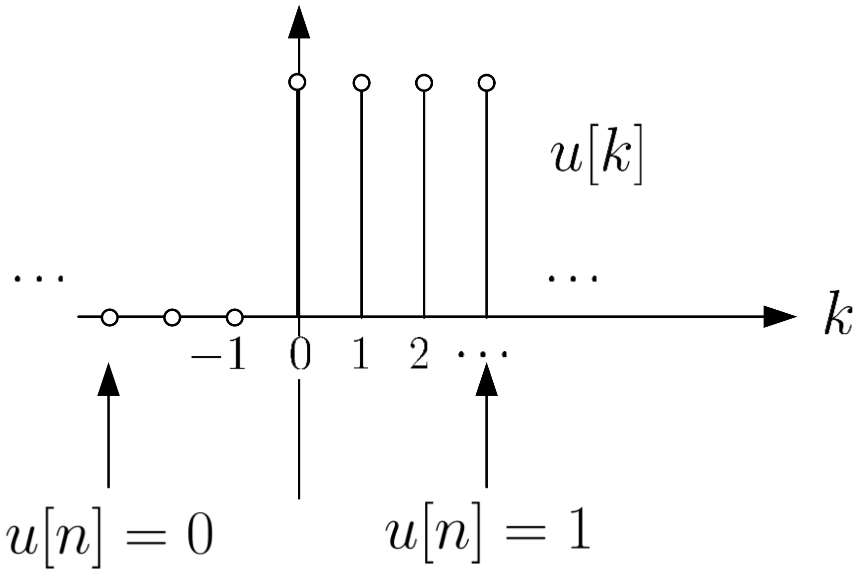
Using (3.11), we have:

$$
\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \sum_{m=-\infty}^{\infty} a^m u[m] \left(u[n-m] - u[n-10-m] \right) \\
&= \sum_{m=0}^{\infty} a^m \left(u[n-m] - u[n-10-m] \right) \\
&= \sum_{m=0}^{\infty} a^m u[n-m] - \sum_{m=0}^{\infty} a^m u[n-10-m]
\end{aligned}
$$

Let $y_1[n]=\sum_{m=0}^{\infty} a^m u[n-m]$ and $y_2[n]=\sum_{m=0}^{\infty} a^m u[n-10-m]$ such that $y[n]=y_1[n]-y_2[n]$. By employing a change of variable, $y_1[n]$ is expressed as

$$y_1[n] = \sum_{m=0}^{\infty} a^m u[n-m] = \sum_{k=n}^{-\infty} a^{n-k} u[k], \quad k=n-m$$

$$= \sum_{k=-\infty}^{n} a^{n-k} u[k]$$



Since $u[k]=0$ for $k<0$, $y_1[n]=0$ for $n<0$. For $n \geq 0$, $y_1[n]$ is:

$$y_1[n] = \sum_{k=0}^{n} a^{n-k} = 1 + a + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$$

where the geometric sum formula is applied:

$$\alpha + \alpha r + \cdots + \alpha r^{N-1} = \alpha \frac{1-r^N}{1-r}$$

That is,

$$y_1[n] = \frac{1-a^{n+1}}{1-a} u[n]$$

Similarly, $y_2[n]$ is:

$$y_2[n] = \sum_{m=0}^{\infty} a^m u[n-10-m]$$
$$= \sum_{k=-\infty}^{n-10} a^{n-10-k} u[k], \quad k = n-10-m$$

Since $u[k]=0$ for $k<0$, $y_2[n]=0$ for $n<10$. For $n \geq 10$, $y_2[n]$ is:

$$y_2[n] = \sum_{k=0}^{n-10} a^{n-10-k} = 1 + a + \cdots + a^{n-10} = \frac{1-a^{n-9}}{1-a}$$

That is,

$$y_2[n] = \frac{1-a^{n-9}}{1-a} u[n-10]$$

Combining the results, we have:

$$y[n] = \frac{1-a^{n+1}}{1-a}u[n] - \frac{1-a^{n-9}}{1-a}u[n-10]$$

or

$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-a^{n+1}}{1-a}, & 0 \le n < 10 \\ \dfrac{a^{n-9}\left(1-a^{10}\right)}{1-a}, & 10 \le n \end{cases}$$

Since $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{9} |h[n]| = 10 < \infty$, the system is stable. Moreover, the system is causal because $h[n] = 0$ for $n < 0$.

Example 3.15

Determine $y[n] = x[n] \otimes h[n]$ where $x[n]$ and $h[n]$ are

$$x[n] = \begin{cases} n^2 + 1, & 0 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h[n] = \begin{cases} n + 1, & 0 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Here, the lengths of both $x[n]$ and $h[n]$ are finite. More precisely, $x[0] = 1$, $x[1] = 2$, $x[2] = 5$, $x[3] = 10$, $h[0] = 1$, $h[1] = 2$, $h[2] = 3$ and $h[3] = 4$ while all other $x[n]$ and $h[n]$ have zero values.

We still use (3.11) but now it reduces to a finite summation:

$$
\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3]
\end{aligned}
$$

By considering the non-zero values of $h[n]$, we obtain:

$$
y[n] = \begin{cases} 1, & n=0 \\ 4, & n=1 \\ 12, & n=2 \\ 30, & n=3 \\ 43, & n=4 \\ 50, & n=5 \\ 40, & n=6 \\ 0, & \text{otherwise} \end{cases}
$$

Alternatively, for finite-length discrete-time signals, we can use the MATLAB command conv to compute the convolution of finite-length sequences:

```
n=0:3;
x=n.^2+1;
h=n+1;
y=conv(x,h)
```

The results are

```
y = 1     4     12     30     43     50     40
```

As the default starting time indices in both h and x are 1, we need to determine the appropriate time index for y

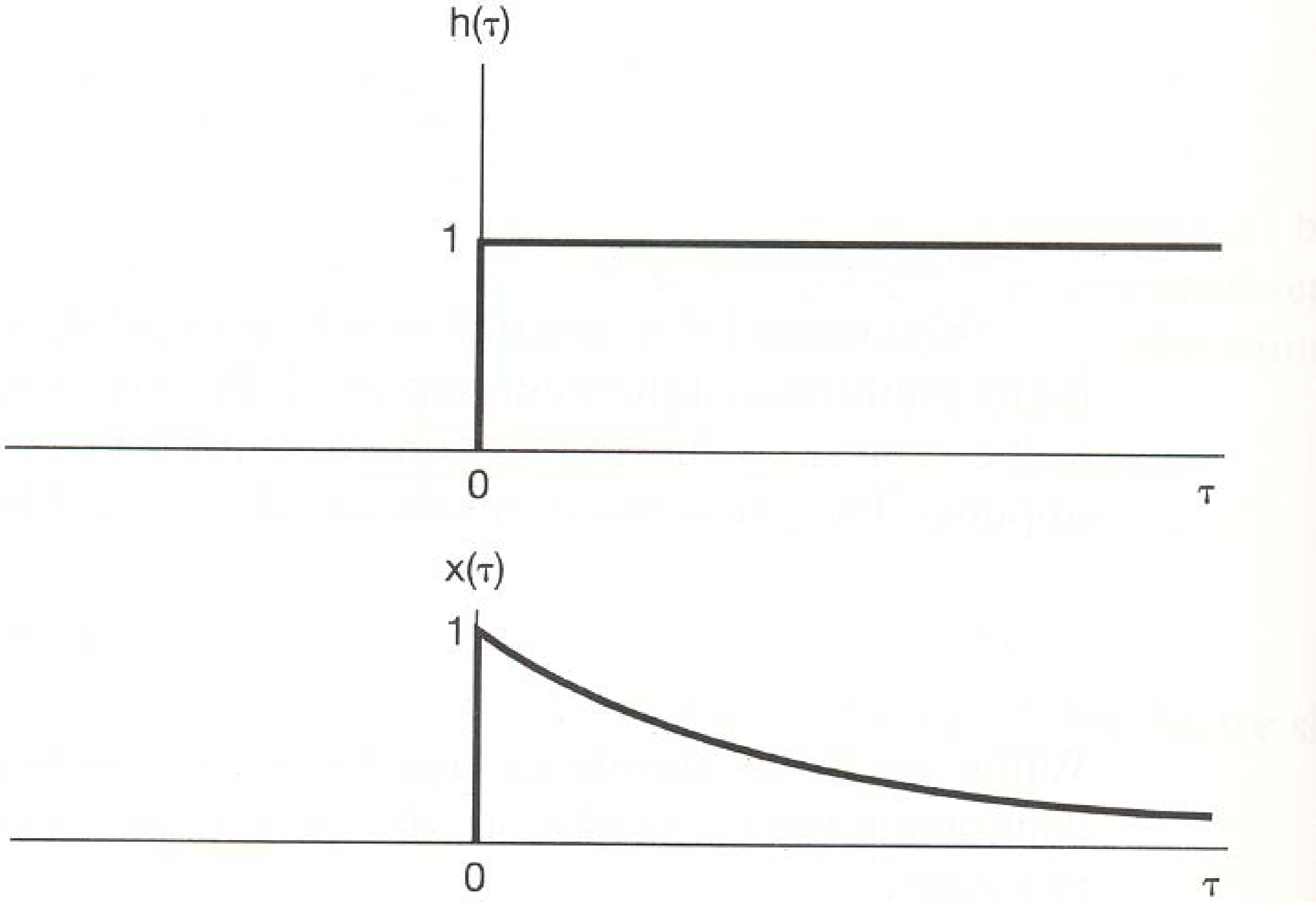
The correct index can be obtained by computing one value of $y[n]$ using (3.11). For simplicity, we may compute $y[0]$:

$$
\begin{aligned}
y[0] &= \sum_{m=-\infty}^{\infty} x[m]h[-m] \\
&= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + \cdots \\
&= x[0]h[0] \\
&= 1
\end{aligned}
$$

In general, if the lengths of $x[n]$ and $h[n]$ are M and N, respectively, the length of $y[n] = x[n] \otimes h[n]$ is $(M + N - 1)$.

Example 3.16

Compute the output $y(t)$ if the input is $x(t) = e^{-at}u(t)$ with $a > 0$ and the LTI system impulse response is $h(t) = u(t)$. Discuss the stability and causality of system.



Using (3.17), we have:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau \\
&= \int_{0}^{\infty} e^{-a\tau}u(t-\tau)d\tau, \quad \lambda = t-\tau \\
&= \int_{t}^{-\infty} e^{-a(t-\lambda)}u(\lambda)\cdot -d\lambda \\
&= e^{-at}\int_{-\infty}^{t} e^{a\lambda}u(\lambda)d\lambda
\end{aligned}
$$

Similar to Example 3.14, when $t<0$, the integral will only involve the zero part of $u(\lambda)$ because $u(\lambda)=0$ for $\lambda<0$. Hence

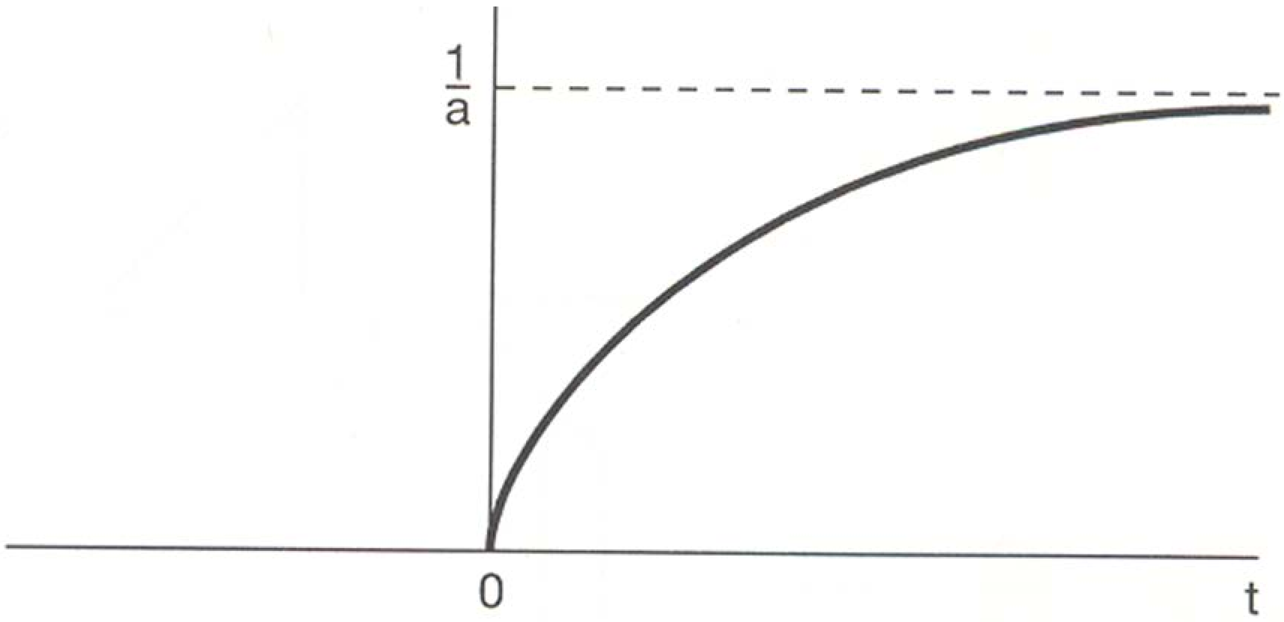
$$
y(t) = e^{-at}\int_{-\infty}^{t} e^{a\lambda}u(\lambda)d\lambda = 0, \quad t<0
$$

When $t>0$, the integral will involve the non-zero part of $u(\lambda)$ because $u(\lambda)=1$ for $0<\lambda\leq t$. The output is then:

$$
\begin{aligned}
y(t) &= e^{-at}\int_{-\infty}^{t} e^{a\lambda}u(\lambda)d\lambda = e^{-at}\int_{0}^{t} e^{a\lambda}d\lambda \\
&= e^{-at}\cdot\frac{1}{a}e^{a\lambda}\Big|_0^t = e^{-at}\cdot\frac{1}{a}\left(e^{at}-1\right) = \frac{1}{a}\left(1-e^{-at}\right)
\end{aligned}
$$

We can combine the results for $t<0$ and $t>0$ to yield:

$$
y(t) = \frac{1}{a}\left(1-e^{-at}\right)u(t)
$$



Linear Constant Coefficient Difference Equation

For a LTI discrete-time system, its input $x[n]$ and output $y[n]$ are related via a N th-order linear constant coefficient difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{3.22}$$

which is useful to check whether a system is both linear and time-invariant or not.

Example 3.17

Determine if the following input-output relationships correspond to LTI systems.

(a) $y[n] = 0.1y[n-1] + x[n] + x[n-1]$

(b) $y[n] = x[n+1] + x[n]$

(c) $y[n] = 1/x[n]$

(a) It corresponds to a LTI system with $N = M = 1$, $a_0 = 1$, $a_1 = -0.1$ and $b_0 = b_1 = 1$

(b) We reorganize the equation as:

$$y[n] = x[n+1] + x[n] \Rightarrow y[n-1] = x[n] + x[n-1]$$

which agrees with (3.22) when $N = M = 1$, $a_0 = 0$ and $a_1 = b_0 = b_1 = 1$. Hence it also corresponds to a LTI system.

(c) It does not correspond to a LTI system because $x[n]$ and $y[n]$ are not linear in the equation.

Note that if a system cannot be fitted into (3.22), there are three possibilities: linear and time-variant; non-linear and time-invariant; or non-linear and time-variant.

Example 3.18

Compute the impulse response $h[n]$ for a LTI system which is characterized by the following difference equation:

$$y[n] = x[n] - x[n-1]$$

Using (3.12), we have:

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\ &= \cdots + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + \cdots \end{aligned}$$

we can easily deduce that only $h[0]$ and $h[1]$ are nonzero. That is, the impulse response is:

$$h[n] = \delta[n] - \delta[n-1]$$

The difference equation can be used to generate the system output and even the system input.

Assuming that $a_0 \neq 0$, $y[n]$ is computed as:

$$y[n] = \frac{1}{a_0}\left(-\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]\right) \quad (3.23)$$

Assuming that $b_0 \neq 0$, $x[n]$ can be obtained from:

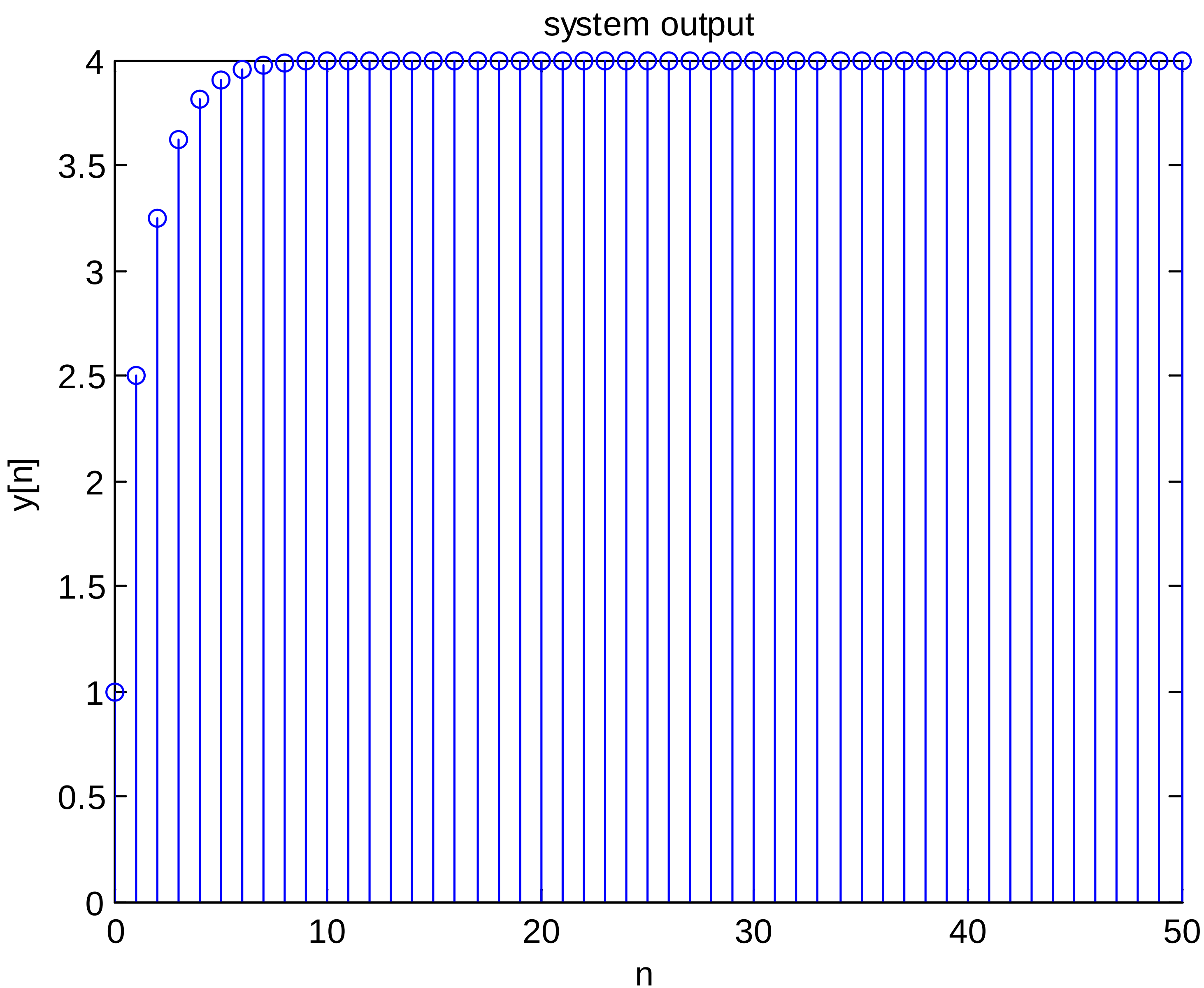
$$x[n] = \frac{1}{b_0}\left(\sum_{k=0}^{N} a_k y[n-k] - \sum_{k=1}^{M} b_k x[n-k]\right) \quad (3.24)$$

Example 3.19

Given a LTI system described by difference equation of $y[n]=0.5y[n-1]+x[n]+x[n-1]$, compute the system output $y[n]$ for $0 \leq n \leq 50$ with an input of $x[n]=u[n]$. It is assumed that $y[-1]=0$.

The MATLAB code is:

```
N=50;             %data length is N+1
y(1)=1;           %compute y[0], only x[n] is nonzero
for n=2:N+1
y(n)=0.5*y(n-1)+2; %compute y[1],y[2],…,y[50]
                   %x[n]=x[n-1]=1 for n>=1
end
n=[0:N];           %set time axis
stem(n,y);
```


system output
y[n]
4
3.5
3
2.5
2
1.5
1
0.5
0
0
10
20
30
40
50
n

Alternatively, we can use the MATLAB command `filter` by rewriting the equation as:

$$y[n] - 0.5y[n-1] = x[n] + x[n-1]$$

The corresponding MATLAB code is:

```
x=ones(1,51);              %define input
a=[1,-0.5];                %define vector of a_k
b=[1,1];                   %define vector of b_k
y=filter(b,a,x);           %produce output
stem(0:length(y)-1,y)
```

The `x` is the input which has a value of 1 for $0 \leq n \leq 50$, while `a` and `b` are vectors which contain $\{a_k\}$ and $\{b_k\}$, respectively.

The MATLAB programs for this example are provided as `ex3_19.m` and `ex3_19_2.m`.

Linear Constant Coefficient Differential Equation

For a LTI continuous-time system, its input $x(t)$ and output $y(t)$ are related via a Nth-order linear constant coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{3.25}$$

which is useful to check whether a system is both linear and time-invariant or not.

Analogous to the discrete-time case, we can use (3.25) to compute system input, output and impulse response.

Fourier Series

Chapter Intended Learning Outcomes:

(i) Represent continuous-time periodic signals using Fourier series

(ii) Understand the properties of Fourier series

(iii) Understand the relationship between Fourier series and linear time-invariant system

Periodic Signal Representation in Frequency Domain

Fourier series can be considered as the frequency domain representation of a continuous-time periodic signal.

Recall (2.6) that $x(t)$ is said to be periodic if there exists $T_p > 0$ such that

$$x(t) = x(t + T_p), \qquad t \in (-\infty, \infty) \tag{4.1}$$

The smallest T_p for which (4.1) holds is called the fundamental period.

Using (2.26), the fundamental frequency is related to T_p as:

$$\Omega_0 = \frac{2\pi}{T_p} \tag{4.2}$$

According to Fourier series, $x(t)$ is represented as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \qquad t \in (-\infty, \infty) \tag{4.3}$$

where

$$a_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-jk\Omega_0 t} dt, \quad k = \cdots - 1, 0, 1, 2, \cdots \tag{4.4}$$

are called Fourier series coefficients. Note that the integration can be done for any period, e.g., $(0, T_p)$, $(-T_p, 0)$.

That is, every periodic signal can be expressed as a sum of harmonically related complex sinusoids with frequencies $\cdots - \Omega_0, 0, \Omega_0, 2\Omega_0, 3\Omega_0, \cdots$, where $2\Omega_0$ is called the first harmonic, $3\Omega_0$ is called the second harmonic, and so on.

This means that $x(t)$ only contains frequencies $\cdots - \Omega_0, 0, \Omega_0, 2\Omega_0, \cdots$ with 0 being the DC component.

Note that the sinusoids are complex-valued with both positive and negative frequencies.

Note also that a_k is generally complex and we can also use magnitude and phase for its representation:

$$|a_k| = \sqrt{(\Re\{a_k\})^2 + (\Im\{a_k\})^2} \tag{4.5}$$

and

$$\angle(a_k) = \tan^{-1}\left(\frac{\Im\{a_k\}}{\Re\{a_k\}}\right) \tag{4.6}$$

From (4.3), $\{a_k\}$ can be used to represent $x(t)$.

Example 4.1

Find the Fourier series coefficients for $x(t)=\cos(10\pi t)+\cos(20\pi t)$.

It is clear that the fundamental frequency of $x(t)$ is $\Omega_0=10\pi$. According to (4.2), the fundamental period is thus equal to $T_p=2\pi/\Omega_0=1/5$, which is validated as follows:

$$\begin{aligned} x\left(t+\frac{1}{5}\right) &= \cos\left(10\pi\left(t+\frac{1}{5}\right)\right)+\cos\left(20\pi\left(t+\frac{1}{5}\right)\right) \\ &= \cos(10\pi t+2\pi)+\cos(20\pi t+4\pi) \\ &= \cos(10\pi t)+\cos(20\pi t) \end{aligned}$$

With the use of Euler formula in (2.29):

$$\cos(u)=\frac{e^{ju}+e^{-ju}}{2}$$

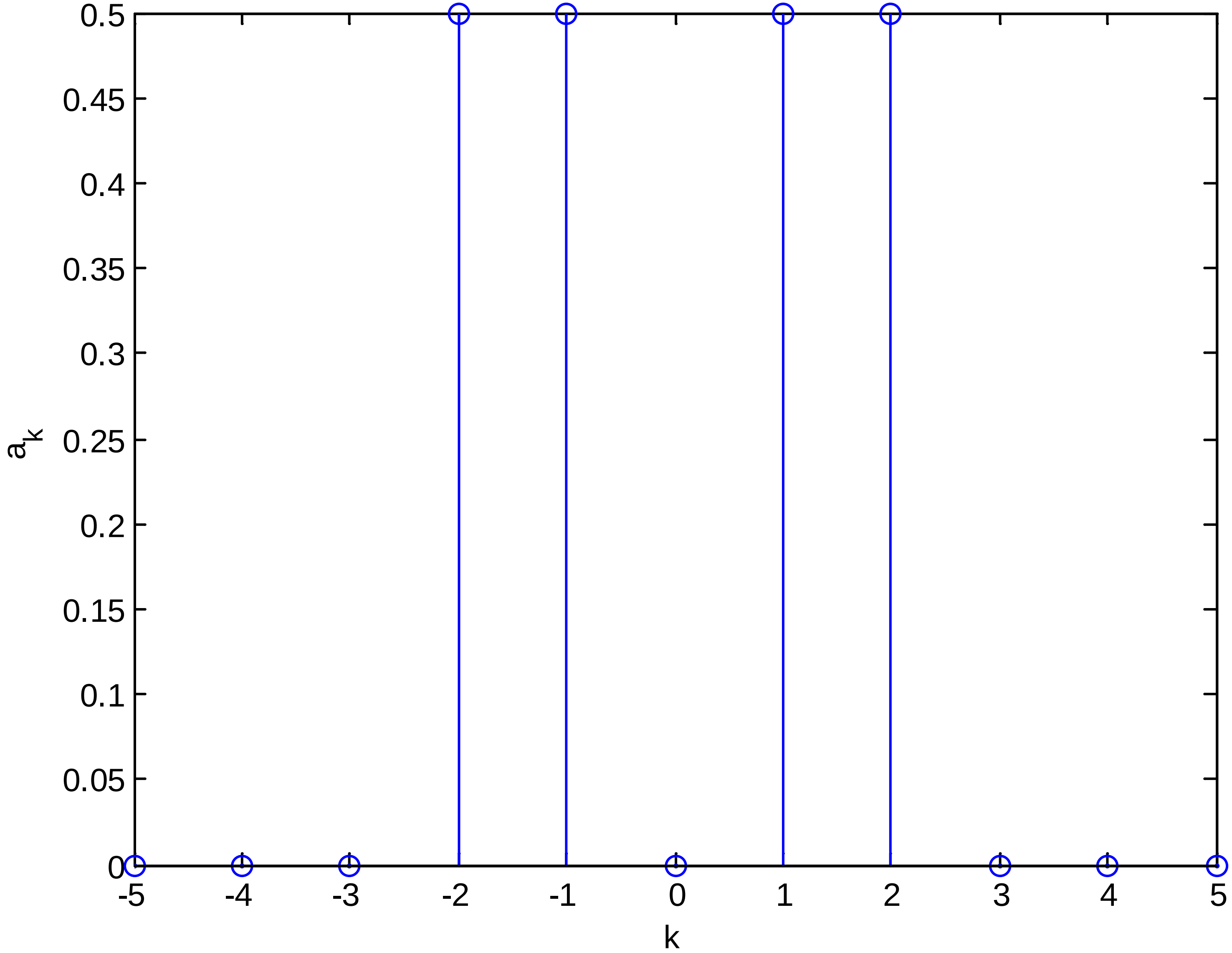
We can express $x(t)$ as:

$$
\begin{aligned}
x(t) &= \cos(10\pi t) + \cos(20\pi t) \\
&= \frac{e^{j10\pi t} + e^{-j10\pi t}}{2} + \frac{e^{j20\pi t} + e^{-j20\pi t}}{2} \\
&= \frac{1}{2}e^{-j20\pi t} + \frac{1}{2}e^{-j10\pi t} + \frac{1}{2}e^{j10\pi t} + \frac{1}{2}e^{j20\pi t}
\end{aligned}
$$

which only contains four frequencies. Comparing with (4.3):

$$
a_k = \begin{cases} 0.5, & k = -2 \\ 0.5, & k = -1 \\ 0.5, & k = 1 \\ 0.5, & k = 2 \\ 0, & \text{otherwise} \end{cases}
$$

Can we use (4.4)? Why?


0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
a_k
-5
-4
-3
-2
-1
0
1
2
3
4
5
k

Example 4.2

Find the Fourier series coefficients for $x(t)=1+\sin(\Omega_0 t)+2\cos(\Omega_0 t)+\cos(3\Omega_0 t+\pi/4)$.

With the use of Euler formulas in (2.29)-(2.30), $x(t)$ can be written as:

$$
\begin{aligned}
x(t) &= 1+\left(1+\frac{1}{2j}\right)e^{j\Omega_0 t}+\left(1-\frac{1}{2j}\right)e^{-j\Omega_0 t}+\frac{1}{2}e^{j\pi/4}e^{3j\Omega_0 t}+\frac{1}{2}e^{-j\pi/4}e^{-3j\Omega_0 t} \\
&= \frac{\sqrt{2}}{4}(1-j)e^{-3j\Omega_0 t}+\left(1+j\frac{1}{2}\right)e^{-j\Omega_0 t}+1+\left(1-j\frac{1}{2}\right)e^{j\Omega_0 t} \\
&\quad +\frac{\sqrt{2}}{4}(1+j)e^{3j\Omega_0 t}
\end{aligned}
$$

Again, comparing with (4.3) yields:

$$a_k = \begin{cases} \frac{\sqrt{2}}{4}(1-j), & k=-3 \\ 1+\frac{j}{2}, & k=-1 \\ 1, & k=0 \\ 1-\frac{j}{2}, & k=1 \\ \frac{\sqrt{2}}{4}(1+j), & k=3 \\ 0, & \text{otherwise} \end{cases}$$

To plot $\{a_k\}$, we may compute $|a_k|$ and $\angle(a_k)$ for all k, e.g.,

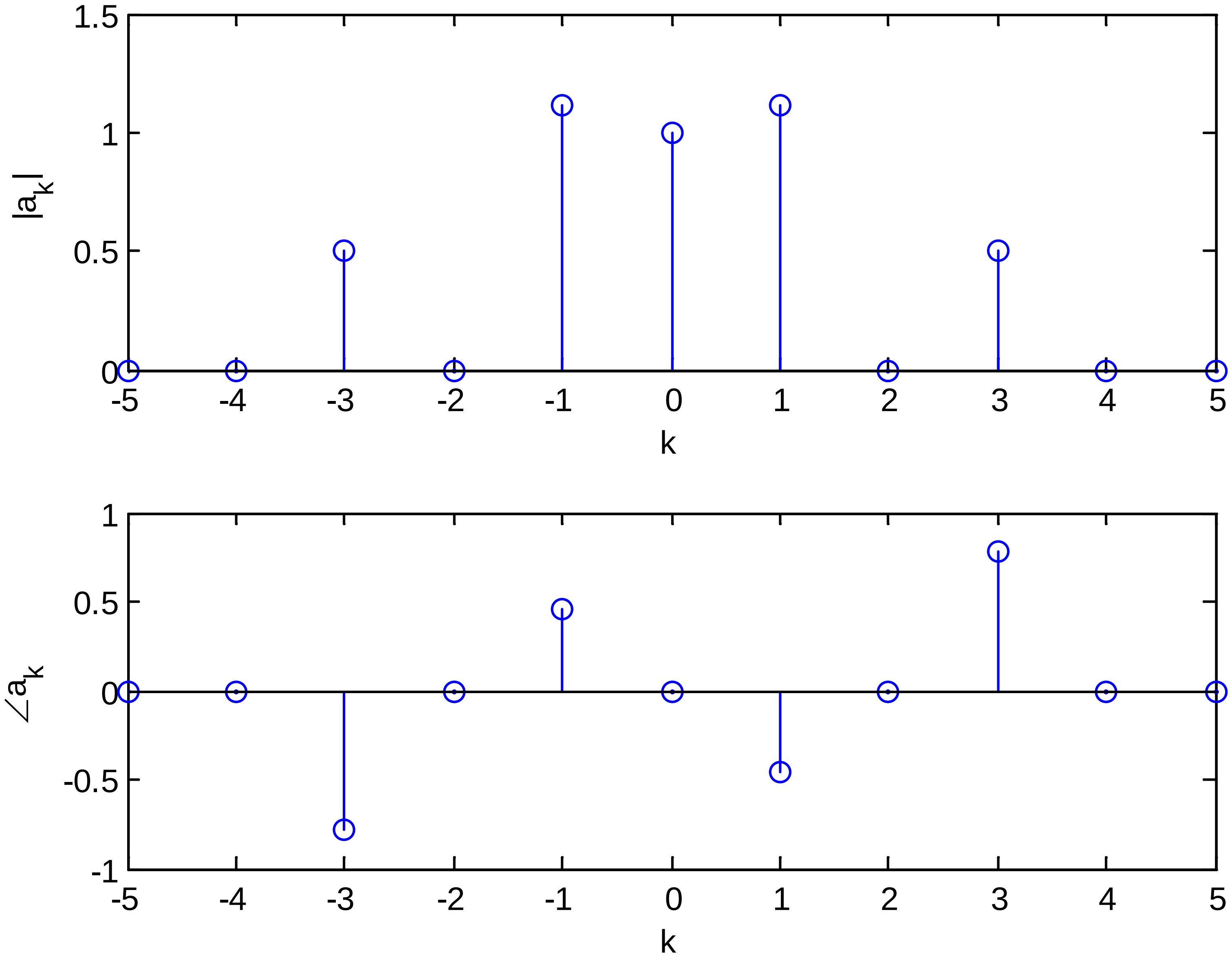
$$|a_{-3}| = \sqrt{\left(\frac{\sqrt{2}}{4}\right)^2 + \left(-\frac{\sqrt{2}}{4}\right)^2} = \frac{1}{2}$$

and

$$\angle(a_{-3}) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

We can also use MATLAB commands abs and angle to compute the magnitude and phase, respectively. After constructing a vector x containing $\{a_k\}$, we can plot $|a_k|$ and $\angle(a_k)$ using:

```
subplot(2,1,1)
stem(n,abs(x))
xlabel('k')
ylabel('|a_k|')
subplot(2,1,2)
stem(n,angle(x))
xlabel('k')
ylabel('\angle{a_k}')
```



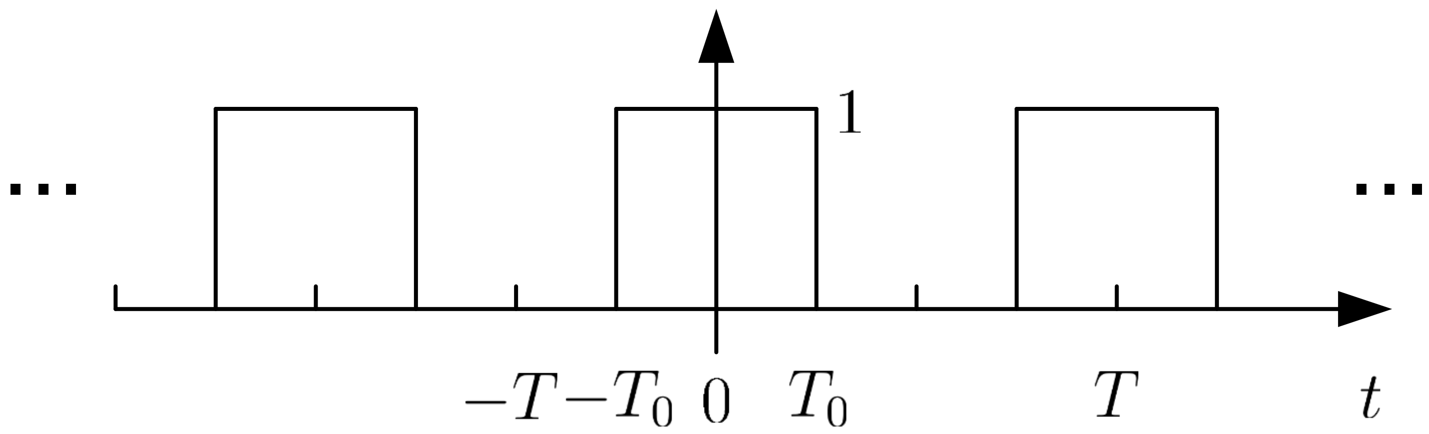
1.5
1
0.5
0
$|a_k|$
-5
-4
-3
-2
-1
0
1
2
3
4
5
k
1
0.5
0
-0.5
-1
$\angle a_k$
-5
-4
-3
-2
-1
0
1
2
3
4
5
k

Example 4.3

Find the Fourier series coefficients for $x(t)$, which is a periodic continuous-time signal of fundamental period T and is a pulse with a width of $2T_0$ in each period. Over the specific period from $-T/2$ to $T/2$, $x(t)$ is:

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & \text{otherwise} \end{cases}$$

with $T > 2T_0$.



Noting that the fundamental frequency is $\Omega_0 = 2\pi/T$ and using (4.4), we get:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\Omega_0 t}dt = \frac{1}{T}\int_{-T_0}^{T_0} e^{-jk\Omega_0 t}dt$$

For $k = 0$:

$$a_0 = \frac{1}{T}\int_{-T_0}^{T_0} 1dt = \frac{2T_0}{T}$$

For $k \neq 0$:

$$a_k = \frac{1}{T}\int_{-T_0}^{T_0} e^{-jk\Omega_0 t}dt = -\frac{1}{jk\Omega_0}e^{jk\Omega_0 t}\Big|_{-T_0}^{T_0} = \frac{\sin(k\Omega_0 T_0)}{k\pi} = \frac{\sin\left(2\pi k T_0/T\right)}{k\pi}$$

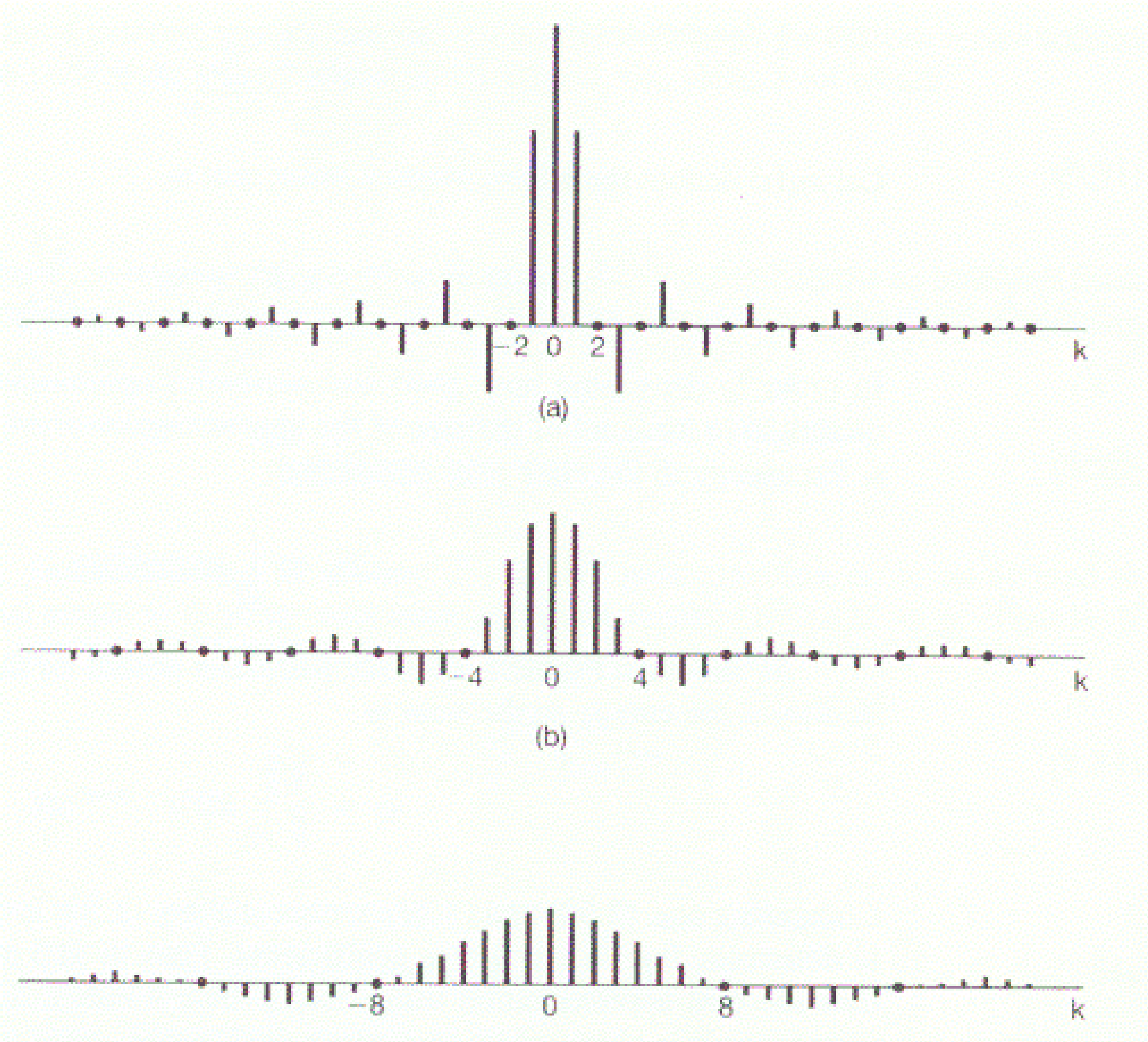
The reason of separating the cases of $k=0$ and $k \neq 0$ is to facilitate the computation of a_0, whose value is not straightforwardly obtained from the general expression which involves "0/0".

Nevertheless, using $\mathrm{L'Hôpital}$'s rule:

$$\lim_{k \to 0} \frac{\sin(2\pi k T_0/T)}{k\pi} = \lim_{k \to 0} \frac{\frac{d \sin(2\pi k T_0/T)}{dk}}{\frac{dk\pi}{dk}} = \lim_{k \to 0} \frac{2\pi T_0/T \cos((2\pi k T_0/T))}{\pi} = \frac{2T_0}{T}$$

An investigation on the values of $\{a_k\}$ with respect to relative pulse width T_0/T is performed as follows.

We see that when T_0/T decreases, $\{a_k\}$ seem to be stretched.



$T = 4T_0$

$T = 8T_0$

$T = 16T_0$

Example 4.4

Find the Fourier series coefficients for the following continuous-time periodic signal $x(t)$:

$$x(t) = \begin{cases} 1.5, & 0 < t < 1 \\ -1.5, & 1 < t < 2 \end{cases}$$

where the fundamental period is $T_p = 2$ and fundamental frequency is $\Omega_0 = \pi$.

Using (4.4) with the period from $t = -1$ to $t = 1$:

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\Omega_0 t}dt \\ &= \frac{1}{2}\int_{-1}^{0}(-1.5)e^{-jk\pi t}dt + \frac{1}{2}\int_{0}^{1} 1.5e^{-jk\pi t}dt \end{aligned}$$

For $k = 0$:

$$a_k = \frac{1}{2}\int_{-1}^{0}(-1.5)dt + \frac{1}{2}\int_{0}^{1} 1.5dt = \frac{1}{2}(-1.5 + 1.5) = 0$$

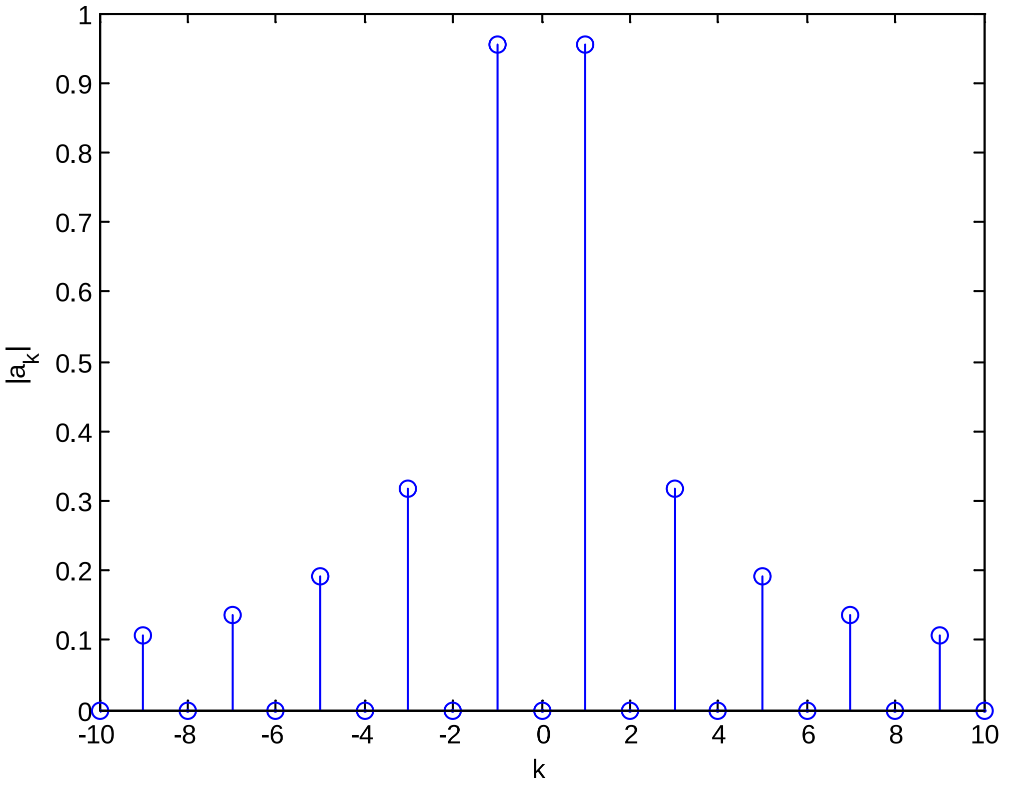
For $k \neq 0$:

$$\begin{aligned} a_k &= \frac{1}{2}\int_{-1}^{0}(-1.5)e^{-jk\pi t}dt + \frac{1}{2}\int_{0}^{1} 1.5e^{-jk\pi t}dt \\ &= \frac{3}{4}\left[\int_{-1}^{0} -e^{-jk\pi t}dt + \int_{0}^{1} e^{-jk\pi t}dt\right] \\ &= \frac{3}{4}\left[-\frac{1}{-jk\pi}\, e^{-jk\pi t}\Big|_{-1}^{0} + \frac{1}{-jk\pi}\, -e^{-jk\pi t}\Big|_{0}^{1}\right] \\ &= \frac{3}{4jk\pi}\left[1 - e^{jk\pi} - e^{-jk\pi} + 1\right] \\ &= \frac{3}{2jk\pi}\left[1 - \cos(k\pi)\right] \end{aligned}$$

MATLAB can be used to validate the answer. First we have:

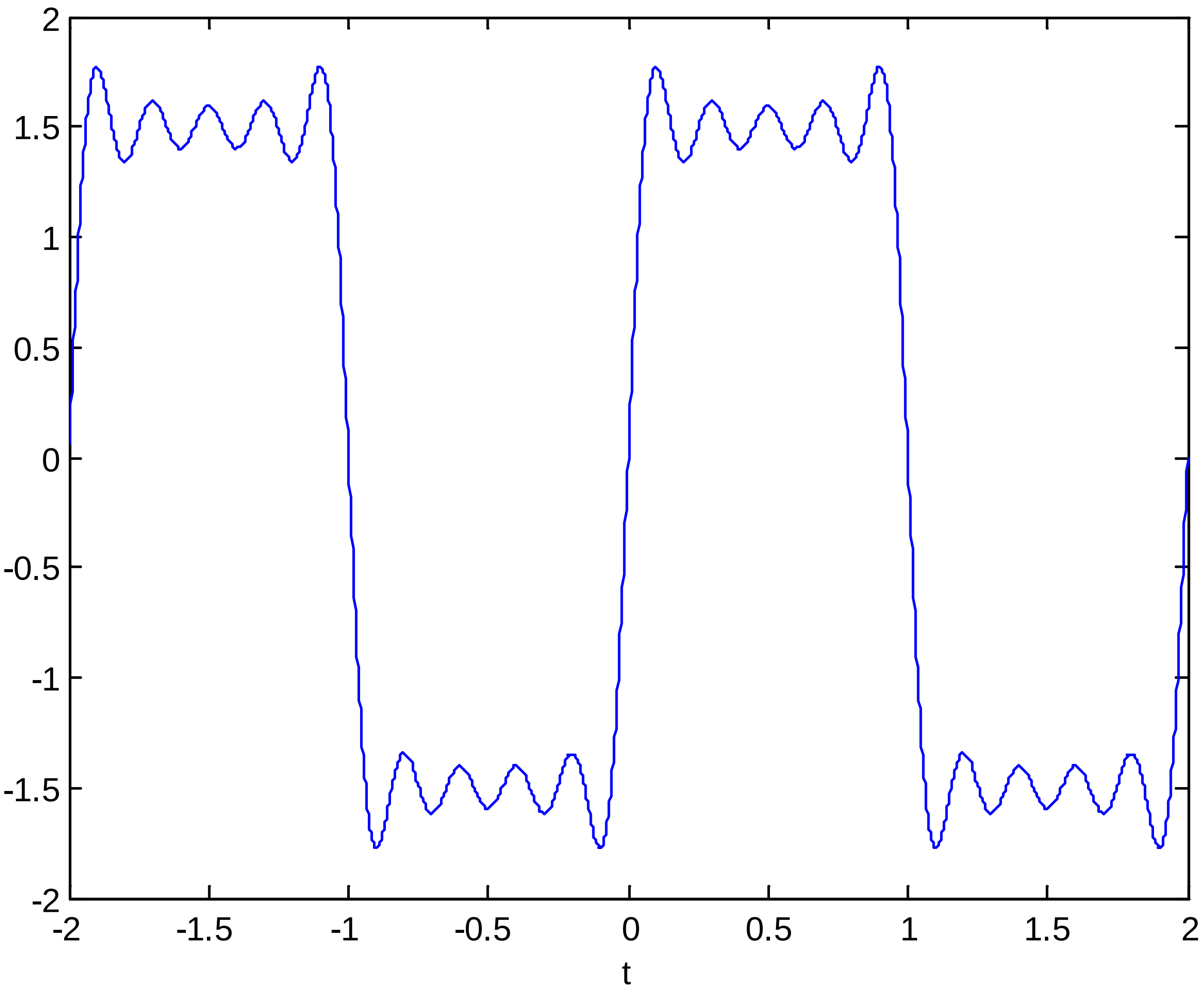
$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \approx \sum_{k=-K}^{K} a_k e^{jk\Omega_0 t}$$

for sufficiently large K because $|a_k|$ is decreasing with k

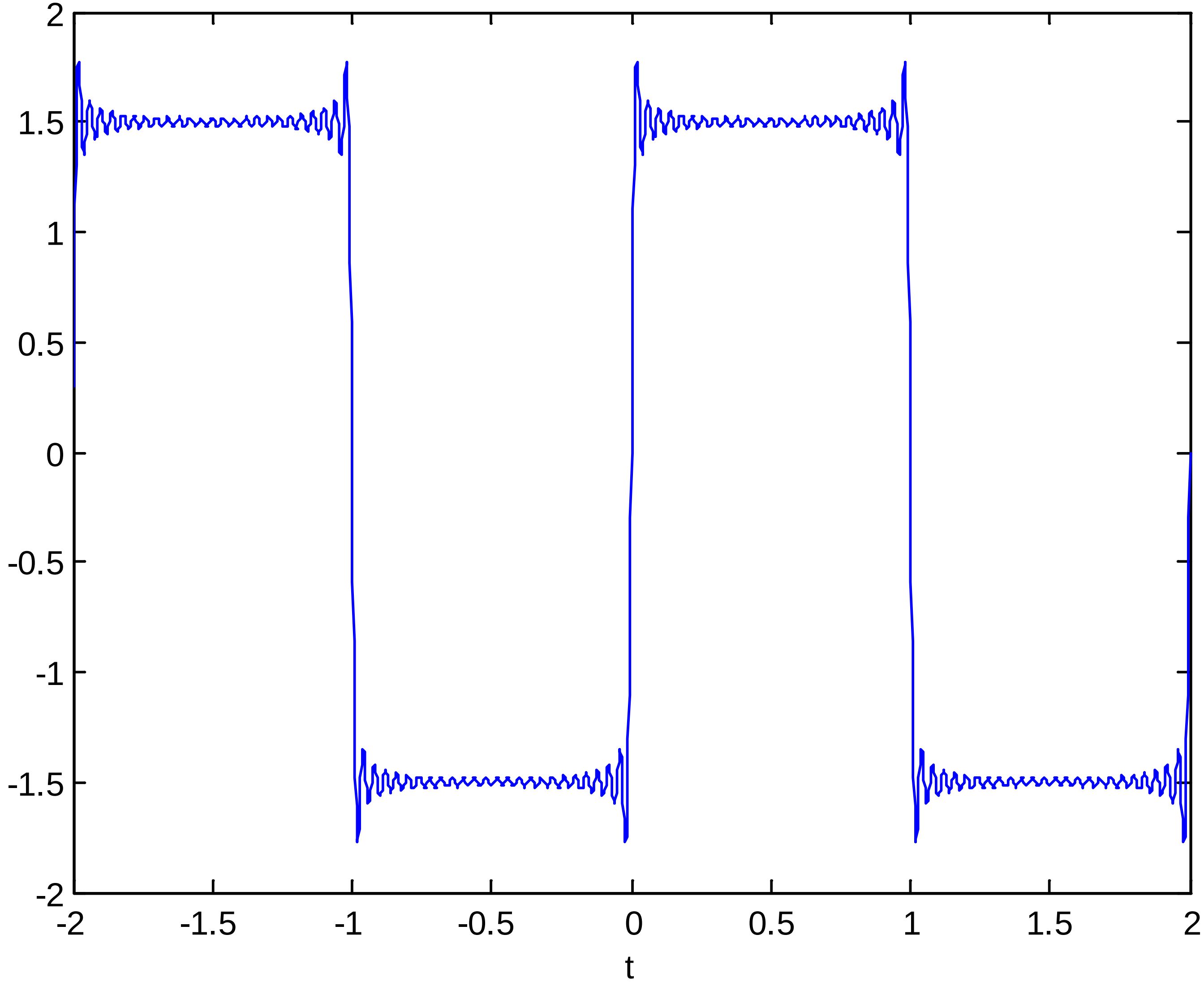


Setting $K = 10$, we may use the following code:

```
K=10;
a_p = 3./(j.*2.*[1:K].*pi).*(1-cos([1:K].*pi)); % +ve a_k
a_n = 3./(j.*2.*[-K:-1].*pi).*(1-cos([-K:-1].*pi)); %-ve a_k
a = [a_n 0 a_p]; %construct vector of a_k
for n=1:2000
 t=(n-1000)/500; %time interval of (-2,2);
      %small sampling interval of 1/500 to approximate x(t);
 e = (exp(j.*[-K:K].*pi.*t)).'; %construct exponential vector
 x(n) = a*e;
end
x=real(x); %remove imaginary parts due to precision error
n=1:2000;
t=(n-1000)./500;
plot(t,x)
xlabel('t')
```


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
t

For $K = 50$:



In summary, if $x(t)$ is periodic, it can be represented as a linear combination of complex harmonics with amplitudes $\{a_k\}$.

That is, $\{a_k\}$ correspond to the frequency domain representation of $x(t)$ and we may write:

$$x(t) \leftrightarrow X(j\Omega) \quad \text{or} \quad x(t) \leftrightarrow a_k \tag{4.7}$$

where $X(j\Omega)$, a function of frequency Ω, is characterized by $\{a_k\}$.

Both $x(t)$ and $X(j\Omega)$ represent the same signal: we observe the former in time domain while the latter in frequency domain.

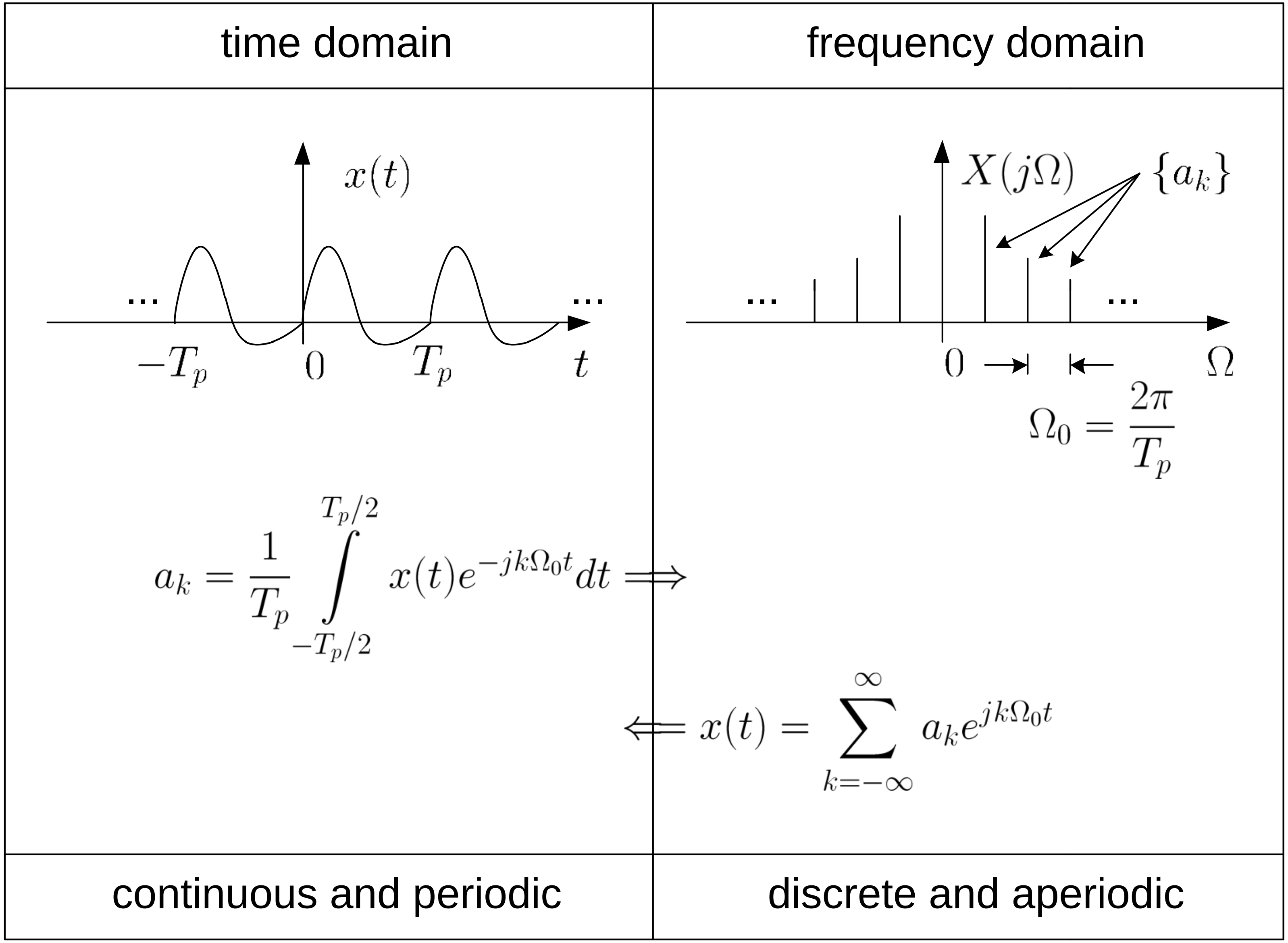


Fig.4.1: Illustration of Fourier series

Properties of Fourier Series

Linearity

Let $x(t) \leftrightarrow a_k$ and $y(t) \leftrightarrow b_k$ be two Fourier series pairs with the same period of T_p. We have:

$$Ax(t) + By(t) \leftrightarrow Aa_k + Bb_k \tag{4.8}$$

This can be proved as follows. As $x(t)$ and $y(t)$ have the same fundamental period of T_p or fundamental frequency Ω_0, we can write:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \qquad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\Omega_0 t}$$

Multiplying $x(t)$ and $y(t)$ by A and B, respectively, yields:

$$Ax(t) = A \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \qquad By(t) = B \sum_{k=-\infty}^{\infty} b_k e^{jk\Omega_0 t}$$

Summing $Ax(t)$ and $By(t)$, we get:

$$Ax(t) + By(t) = \sum_{k=-\infty}^{\infty} (Aa_k + Bb_k) e^{jk\Omega_0 t} \leftrightarrow Aa_k + Bb_k$$

Time Shifting

A shift of t_0 in $x(t)$ causes a multiplication of $e^{-jk\Omega_0 t_0}$ in a_k:

$$x(t) \leftrightarrow a_k \Rightarrow x(t - t_0) \leftrightarrow e^{-jk\Omega_0 t_0} a_k = e^{-jk(2\pi)/T_p t_0} a_k \qquad (4.9)$$

Time Reversal

$$x(t) \leftrightarrow a_k \Rightarrow x(-t) \leftrightarrow a_{-k} \qquad (4.10)$$

(4.9) and (4.10) are proved as follows.

Recall (4.3):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

Substituting t by $t - t_0$, we obtain:

$$x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 (t-t_0)} = \sum_{k=-\infty}^{\infty} \left(e^{-jk\Omega_0 t_0} a_k\right) e^{jk\Omega_0 t} \leftrightarrow e^{-jk\Omega_0 t_0} a_k$$

Substituting t by $-t$ yields:

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 (-t)} = \sum_{l=-\infty}^{\infty} a_{-l} e^{jl\Omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\Omega_0 t} \leftrightarrow a_{-k}$$

Time Scaling

For a time-scaled version of $x(t)$, $x(\alpha t)$ where $\alpha \neq 0$ is a real number, we have:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \Rightarrow x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\Omega_0)t} \qquad (4.11)$$

Multiplication

Let $x(t) \leftrightarrow a_k$ and $y(t) \leftrightarrow b_k$ be two Fourier series pairs with the same period of T_p. We have:

$$x(t)y(t) \leftrightarrow \sum_{l=-\infty}^{\infty} a_l b_{k-l} \qquad (4.12)$$

(4.12) is proved as follows.

Applying (4.3) again, the product of $x(t)$ and $y(t)$ is:

$$
\begin{aligned}
x(t)y(t) &= \sum_{l=-\infty}^{\infty} a_l e^{jl\Omega_0 t} \sum_{n=-\infty}^{\infty} b_n e^{jn\Omega_0 t} \\
&= \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_l b_n e^{j(l+n)\Omega_0 t} \\
&= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_l b_{l-k} e^{jk\Omega_0 t}, \quad k = l + n \\
&= \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} a_l b_{l-k} \right) e^{jk\Omega_0 t} \leftrightarrow \sum_{l=-\infty}^{\infty} a_l b_{k-l}
\end{aligned}
$$

Conjugation

$$x(t) \leftrightarrow a_k \Rightarrow x^*(t) \leftrightarrow a^*_{-k} \tag{4.13}$$

Parseval's Relation

The Parseval's relation addresses the power of $x(t)$:

$$\frac{1}{T_p}\int_{-T_p/2}^{T_p/2} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \tag{4.14}$$

That is, we can compute the power in either the time domain or frequency domain.

Example 4.5

Prove the Parseval's relation.

Using (4.3), we have:

$$\begin{aligned}
\frac{1}{T_p}\int_{-T_p/2}^{T_p/2}|x(t)|^2\,dt &= \frac{1}{T_p}\int_{-T_p/2}^{T_p/2}\left(\sum_{m=-\infty}^{\infty}a_m e^{jm\Omega_0 t}\right)\left(\sum_{n=-\infty}^{\infty}a_n e^{jn\Omega_0 t}\right)^* dt \\
&= \frac{1}{T_p}\int_{-T_p/2}^{T_p/2}\left(\sum_{m=-\infty}^{\infty}a_m e^{jm\Omega_0 t}\right)\left(\sum_{n=-\infty}^{\infty}a_n^* e^{-jn\Omega_0 t}\right) dt \\
&= \frac{1}{T_p}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}a_m a_n^* \int_{-T_p/2}^{T_p/2} e^{j(m-n)\Omega_0 t}dt \\
&= \frac{1}{T_p}\sum_{m=-\infty}^{\infty}|a_m|^2\int_{-T_p/2}^{T_p/2}dt = \sum_{m=-\infty}^{\infty}|a_m|^2
\end{aligned}$$

Linear Time-Invariant System with Periodic Input

Recall in a linear time-invariant (LTI) system, the input-output relationship is characterized by convolution in (3.17):

$$\begin{aligned} y(t) &= x(t) \otimes h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \end{aligned} \qquad (4.15)$$

If the input to the system with impulse response $h(t)$ is $x(t) = e^{j\Omega_0 t}$, then the output is:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{j\Omega_0(t-\tau)}d\tau \\ &= e^{j\Omega_0 t}\int_{-\infty}^{\infty} h(\tau)e^{-j\Omega_0 \tau}d\tau \end{aligned} \qquad (4.16)$$

Note that $\int_{-\infty}^{\infty} h(\tau)e^{-j\Omega_0\tau}d\tau$ is independent of t but a function of Ω_0 and we may denote it as $H(\Omega_0)$:

$$y(t) = e^{j\Omega_0 t}H(\Omega_0) = H(\Omega_0)x(t) \tag{4.17}$$

If we input a sinusoid through a LTI system, there is no change in frequency in the output but amplitude and phase are modified.

Generalizing the result to any periodic signal in (4.3) yields:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(k\Omega_0) e^{jk\Omega_0 t} \tag{4.18}$$

where only the Fourier series coefficients are modified.

Note that discrete Fourier series is used to represent discrete periodic signal in (2.7) but it will not be discussed.

Fourier Transform

Chapter Intended Learning Outcomes:

(i) Represent continuous-time aperiodic signals using Fourier transform

(ii) Understand the properties of Fourier transform

(iii) Understand the relationship between Fourier transform and linear time-invariant system

Aperiodic Signal Representation in Frequency Domain

For a periodic continuous-time signal, it can be represented in frequency domain using Fourier series.

But in general, signals are not periodic. To address this, we use Fourier transform:

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \qquad (5.1)$$

where $X(j\Omega)$ is a function of frequency Ω, also known as spectrum, and we can study the signal frequency components from it.

Unlike Fourier series, $X(j\Omega)$ is continuous in frequency, i.e., defined on a continuous range of Ω.

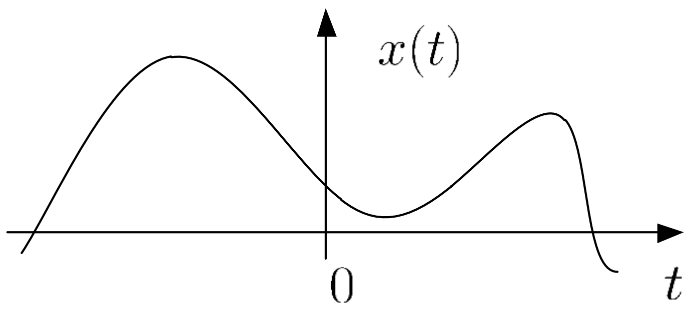
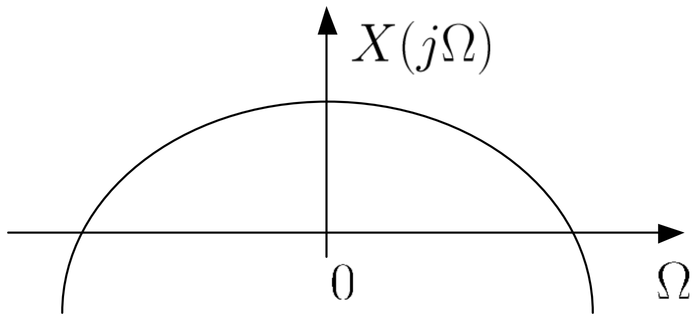
The inverse transform is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega \qquad (5.2)$$

As in (4.7), we may write:

$$x(t) \leftrightarrow X(j\Omega) \qquad (5.3)$$

That is, both $x(t)$ and $X(j\Omega)$ represent the same signal: $x(t)$ is the time domain representation while $X(j\Omega)$ is the frequency domain representation.

time domain	frequency domain
	
continuous and aperiodic	continuous and aperiodic

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \Longrightarrow$$

$$\Longleftarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

Fig.5.1: Illustration of Fourier transform

Derivation of Fourier Transform

Fourier transform can be derived from Fourier series as follows.

We start with an aperiodic $x(t)$ and then construct its periodic version $\tilde{x}(t)$ with period T.

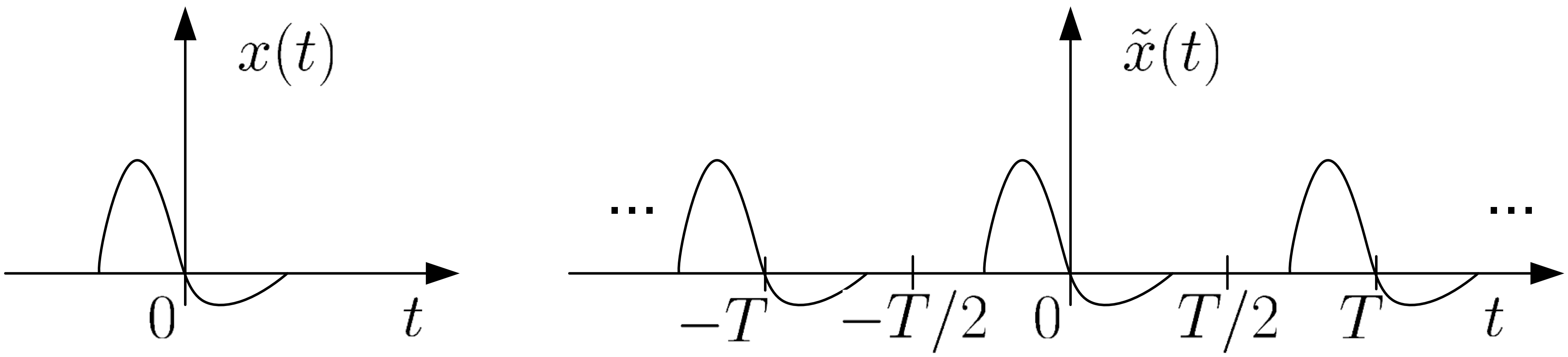


Fig.5.2: Constructing $\tilde{x}(t)$ from $x(t)$

According to (4.4), the Fourier series coefficients of $\tilde{x}(t)$ are:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\Omega_0 t} dt \qquad (5.4)$$

where $\Omega_0 = 2\pi/T$.

Noting that $x(t) = \tilde{x}(t)$ for $|t| < T/2$ and $x(t) = 0$ for $|t| > T/2$, (5.4) can be expressed as:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\Omega_0 t} dt \qquad (5.5)$$

As $X(j\Omega)$ is function of Ω, substituting $\Omega = k\Omega_0$ in (5.1) gives

$$X(jk\Omega_0) = \int_{-\infty}^{\infty} x(t) e^{-jk\Omega_0 t} dt \qquad (5.6)$$

We can express a_k as:

$$a_k = \frac{1}{T} X(jk\Omega_0) \quad (5.7)$$

According to (4.3) and using (5.7), we get the Fourier series expansion for $\tilde{x}(t)$:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\Omega_0) e^{jk\Omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t} \quad (5.8)$$

As $T \to \infty$ or $\Omega_0 \to 0$, $\tilde{x}(t) \to x(t)$.

Considering $\Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t}$ as the area of a rectangle whose height is $X(jk\Omega_0) e^{jk\Omega_0 t}$ and width corresponds to the interval of $[k\Omega_0, (k+1)\Omega_0]$, we obtain:

$$x(t) = \lim_{\Omega_0 \to 0} \tilde{x}(t) = \lim_{\Omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \qquad (5.9)$$

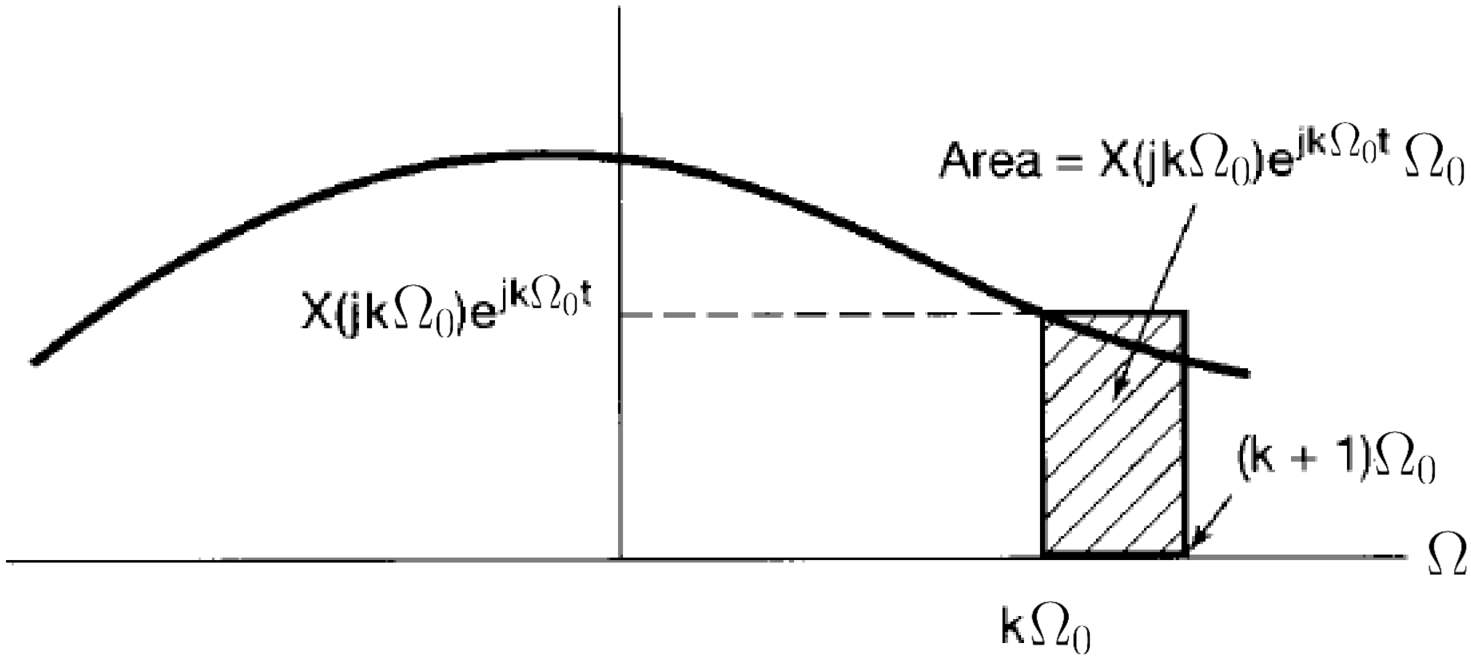


Fig. 5.3: Fourier transform from Fourier series

Example 5.1

Find the Fourier transform of $x(t)$ which is a rectangular pulse of the form:

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & \text{otherwise} \end{cases}$$

Note that the signal is of finite length and corresponds to one period of the periodic function in Example 4.3. Applying (5.1) on $x(t)$ yields:

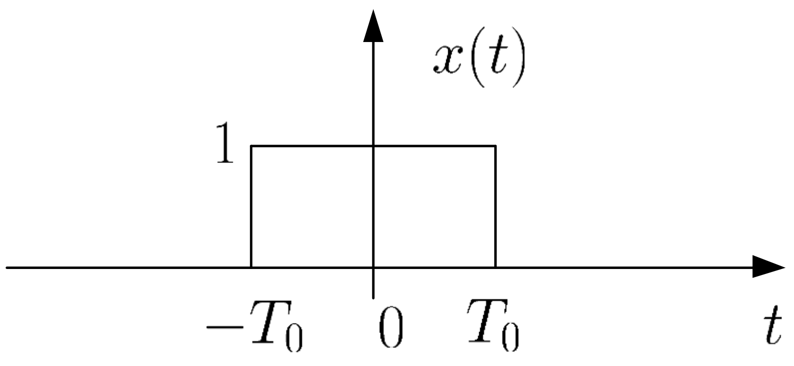
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \int_{-T_0}^{T_0} e^{-j\Omega t} dt = \frac{2\sin(\Omega T_0)}{\Omega}$$

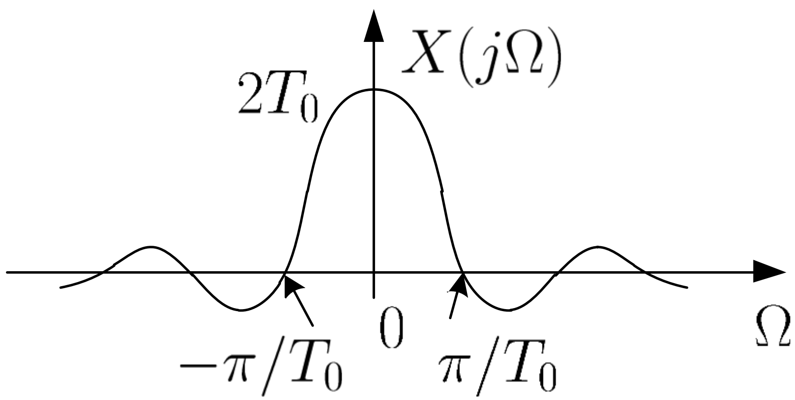
Define the sinc function:

$$\mathrm{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$

It is seen that $X(j\Omega)$ is a scaled sinc function:

$$X(j\Omega) = \frac{2\sin(\Omega T_0)}{\Omega} = 2T_0 \mathrm{sinc}\left(\frac{\Omega T_0}{\pi}\right)$$





We can see that $X(j\Omega)$ is continuous in frequency. When the pulse width decreases, it covers more frequencies and vice versa.

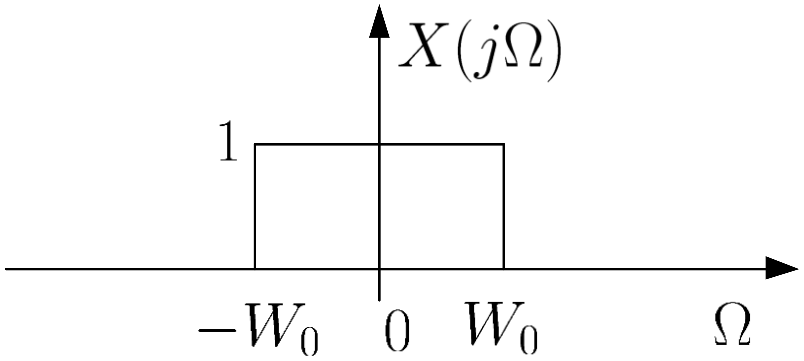
Example 5.2

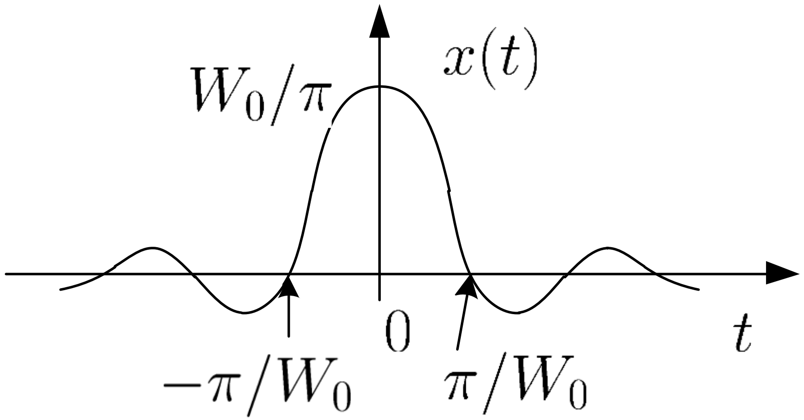
Find the inverse Fourier transform of $X(j\Omega)$ which is a rectangular pulse of the form:

$$X(j\Omega) = \begin{cases} 1, & -W_0 < \Omega < W_0 \\ 0, & \text{otherwise} \end{cases}$$

Using (5.2), we get:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega = \frac{1}{2\pi} \int_{-W_0}^{W_0} e^{j\Omega t} d\Omega = \frac{\sin(W_0 t)}{\pi t} = \frac{W_0}{\pi} \mathrm{sinc}\left(\frac{W_0 t}{\pi}\right)$$





Example 5.3

Find the Fourier transform of $x(t) = e^{-at}u(t)$ with $a > 0$.

Employing the property of $u(t)$ in (2.22) and (5.1), we get:

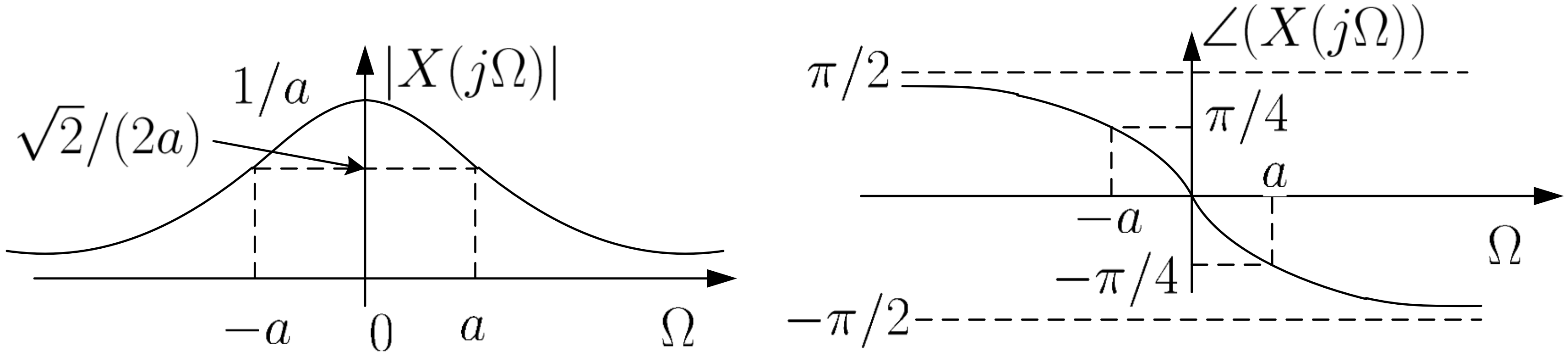
$$X(j\Omega) = \int_0^\infty e^{-at} e^{-j\Omega t} dt = -\frac{1}{a+j\Omega} e^{-(a+j\Omega)t} \Big|_0^\infty = \frac{1}{a+j\Omega} = \frac{a-j\Omega}{a^2+\Omega^2}$$

Note that when $t \to \infty$, $e^{-at} \to 0$.

$$|X(j\Omega)| = \frac{1}{\sqrt{a^2+\Omega^2}}$$

and

$$\angle(X(j\Omega)) = -\tan^{-1}\left(\frac{\Omega}{a}\right)$$



Example 5.4

Find the Fourier transform of the impulse $x(t) = \delta(t)$.

Using (2.19) and (2.20) with $x(t) = e^{-j\Omega t}$ and $t_0 = 0$, we get:

$$X(j\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega \cdot 0} dt = e^{-j\Omega \cdot 0} \int_{-\infty}^{\infty} \delta(t) dt = e^{-j\Omega \cdot 0} = 1$$

Spectrum of $\delta(t)$ has unit amplitude at all frequencies. This aligns with Example 5.1 when $T_0 \to 0$. On the other hand, at $T_0 \to \infty$, $x(t)$ will be a DC and only contains frequency 0.

Periodic Signal Representation using Fourier Transform

Fourier transform can be used to represent continuous-time periodic signals with the use of $\delta(t)$.

Instead of time domain, we consider an impulse in the frequency domain:

$$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0) \quad (5.10)$$

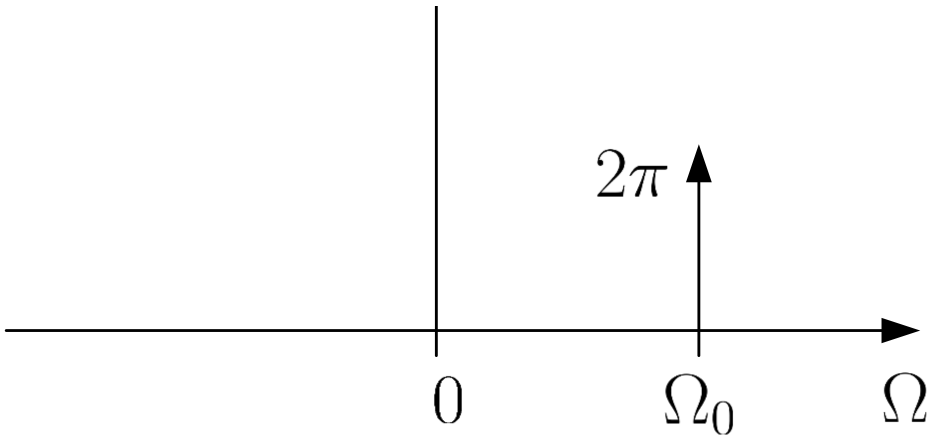


Fig.5.4: Impulse in frequency domain

Taking the inverse Fourier transform of $X(j\Omega)$ and employing the result in Example 5.4, $x(t)$ is computed as:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0) e^{j\Omega t} d\Omega = e^{j\Omega_0 t} \tag{5.11}$$

As a result, the Fourier transform pair is:

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0) \tag{5.12}$$

From (4.3) and (5.12), the Fourier transform pair for a continuous-time periodic signal is:

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0) \tag{5.13}$$

Example 5.5

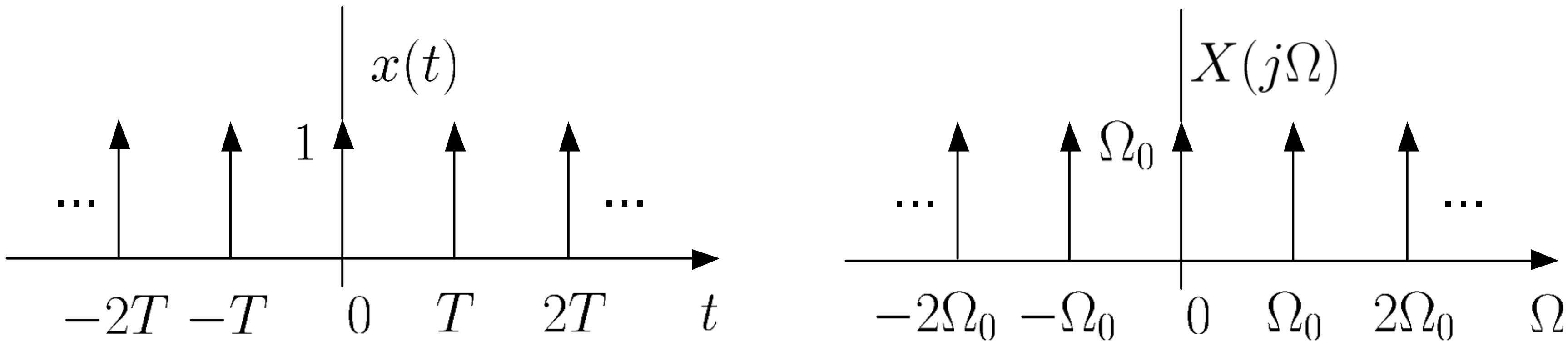
Find the Fourier transform of $x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)$ which is called an impulse train.

Clearly, $x(t)$ is a periodic signal with a period of T. Using (4.4) and Example 5.4, the Fourier series coefficients are:

$$a_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\Omega_0 t}dt=\frac{1}{T}$$

with $\Omega_0=2\pi/T$. According to (5.13), the Fourier transform is:

$$X(j\Omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\Omega-\frac{2\pi k}{T}\right)=\Omega_0\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_0)$$



Properties of Fourier Transform

Linearity

Let $x(t) \leftrightarrow X(j\Omega)$ and $y(t) \leftrightarrow Y(j\Omega)$ be two Fourier transform pairs. We have:

$$ax(t) + by(t) \leftrightarrow aX(j\Omega) + bY(j\Omega) \tag{5.14}$$

Time Shifting

A shift of t_0 in $x(t)$ causes a multiplication of $e^{-j\Omega t_0}$ in $X(j\Omega)$:

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x(t - t_0) \leftrightarrow e^{-j\Omega t_0} X(j\Omega) \quad (5.15)$$

Time Reversal

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x(-t) \leftrightarrow X(-j\Omega) \quad (5.16)$$

Time Scaling

For a time-scaled version of $x(t)$, $x(\alpha t)$ where $\alpha \neq 0$ is a real number, we have:

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x(\alpha t) \leftrightarrow \frac{1}{|\alpha|} X\left(\frac{j\Omega}{\alpha}\right) \quad (5.17)$$

Multiplication

Let $x(t) \leftrightarrow X(j\Omega)$ and $y(t) \leftrightarrow Y(j\Omega)$ be two Fourier transform pairs. We have:

$$x(t) \cdot y(t) \leftrightarrow \frac{1}{2\pi} X(j\Omega) \otimes Y(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\tau) Y(j(\Omega - \tau)) d\tau \quad (5.18)$$

Conjugation

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x^*(t) \leftrightarrow X^*(-j\Omega) \quad (5.19)$$

Parseval's Relation

The Parseval's relation addresses the energy of $x(t)$:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 dt \quad (5.20)$$

Convolution

Let $x(t) \leftrightarrow X(j\Omega)$ and $y(t) \leftrightarrow Y(j\Omega)$ be two Fourier transform pairs. We have:

$$x(t) \otimes y(t) \leftrightarrow X(j\Omega)Y(j\Omega) \quad (5.21)$$

which can be derived as:

$$\begin{aligned}
&\int_{-\infty}^{\infty} x(t) \otimes y(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)y(t-\tau)e^{-j\Omega t} d\tau dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)y(u)e^{-j\Omega\tau}e^{-j\Omega u} d\tau du, \quad u = t-\tau \\
&= \left[\int_{-\infty}^{\infty} x(\tau)e^{-j\Omega\tau} d\tau\right] \cdot \left[\int_{-\infty}^{\infty} y(u)e^{-j\Omega u} du\right] \\
&= X(j\Omega) \cdot Y(j\Omega) \quad (5.22)
\end{aligned}$$

Differentiation

Differentiating $x(t)$ with respect to t corresponds to multiplying $X(j\Omega)$ by $j\Omega$ in the frequency domain:

$$\frac{dx(t)}{dt} \leftrightarrow j\Omega X(j\Omega) \Rightarrow \frac{d^k x(t)}{dt^k} \leftrightarrow (j\Omega)^k X(j\Omega) \qquad (5.23)$$

Integration

On the other hand, if we perform integration on $x(t)$, then the frequency domain representation becomes:

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\Omega}X(j\Omega) + \pi X(0)\delta(\Omega) \qquad (5.24)$$

Fourier Transform and Linear Time-Invariant System

Recall in a linear time-invariant (LTI) system, the input-output relationship is characterized by convolution in (3.17):

$$\begin{aligned} y(t) &= x(t) \otimes h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \end{aligned} \quad (5.25)$$

Using (5.21), we can consider (5.25) in frequency domain:

$$y(t) = x(t) \otimes h(t) \leftrightarrow Y(j\Omega) = X(j\Omega)H(j\Omega) \quad (5.26)$$

This suggests apart from computing the output using time-domain approach via convolution, we can convert the input and impulse response to frequency domain, then $y(t)$ is computed from inverse Fourier transform of $X(j\Omega)H(j\Omega)$.

In fact, $H(j\Omega)$ represents the LTI system in the frequency domain, is called the system frequency response.

Recall (3.25) that the input and output of a LTI system satisfy the differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (5.27)$$

Taking the Fourier transform and using the linearity and differentiation properties, we get:

$$Y(j\Omega)\left[\sum_{k=0}^{N} a_k (j\Omega)^k\right] = X(j\Omega)\left[\sum_{k=0}^{M} b_k (j\Omega)^k\right] \qquad (5.28)$$

The system frequency response can also be computed as:

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{\sum_{k=0}^{M} b_k (j\Omega)^k}{\sum_{k=0}^{M} a_k (j\Omega)^k} \quad (5.29)$$

<u>Example 5.6</u>

Determine the system frequency response for a LTI system described by the following differential equation:

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

Applying (5.29), we easily obtain:

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{1}{j\Omega + a}$$

Discrete-Time Fourier Transform

Chapter Intended Learning Outcomes:

(i) Represent discrete-time signals using discrete-time Fourier transform

(ii) Understand the properties of discrete-time Fourier transform

(iii) Understand the relationship between discrete-time Fourier transform and linear time-invariant system

Discrete-Time Signals in Frequency Domain

For continuous-time signals, we can use Fourier series and Fourier transform to study them in frequency domain.

With the use of sampled version of a continuous-time signal $x(t)$, we can obtain the discrete-time Fourier transform (DTFT) or Fourier transform of discrete-time signals as follows.

We start with studying the sampled signal $x_s(t)$ produced by multiplying $x(t)$ by the impulse train $i(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$:

$$x_s(t) = x(t) \cdot i(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t - kT) \quad (6.1)$$

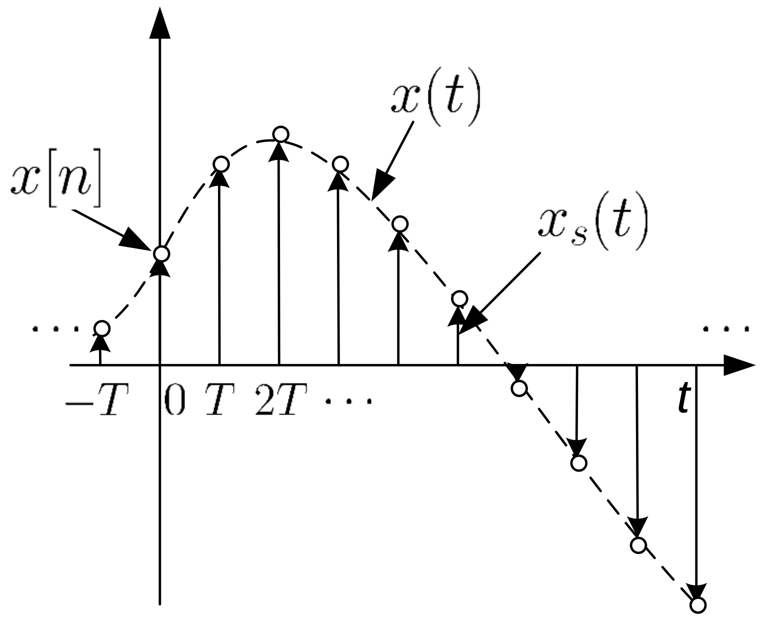


Fig. 6.1: Continuous-time signal multiplied by impulse train

Using (2.20) and assigning $x[k] = x(kT)$, (6.1) becomes:

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT) = \sum_{k=-\infty}^{\infty} x[k]\delta(t - kT) \tag{6.2}$$

where $x_s(t)$ is still a continuous-time signal, although $x[n]$ is discrete-time.

Taking Fourier transform of $x_s(t)$ with the use of the properties of $\delta(t)$, we obtain:

$$X_s(j\Omega) = \int_{-\infty}^{\infty} x_s(t)e^{-j\Omega t}dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)e^{-j\Omega t}dt$$
$$= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t-nT)e^{-j\Omega t}dt = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nT} \quad (6.3)$$

Defining $\omega = \Omega T$ as the discrete-time frequency parameter and writing $X_s(j\Omega)$ as $X(e^{j\omega})$, (6.3) becomes

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (6.4)$$

which is the DTFT of the discrete-time signal $x[n]$.

As in Fourier transform, $X(e^{j\omega})$ is also called spectrum and is a continuous function of the frequency parameter ω.

Nevertheless, $X(e^{j\omega})$ is periodic with period 2π:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n} = X(e^{j(\omega+2k\pi)}) \quad (6.5)$$

for any integer k.

To convert $X(e^{j\omega})$ to $x[n]$, we use inverse DTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \quad (6.6)$$

which is obtained by putting (6.4) into (6.6):

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right]e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} x[m]\int_{-\pi}^{\pi} e^{-j\omega m}e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} x[m]\frac{2\sin((n-m)\pi)}{n-m} \\
&= x[n] \qquad (6.7)
\end{aligned}
$$

Note that $\sin((n-m)\pi)/(n-m)=0$ if $m \neq n$ while when $m=n$, we have $\int_{-\pi}^{\pi} e^{-j\omega m}e^{j\omega n}d\omega = \int_{-\pi}^{\pi} d\omega = 2\pi$.

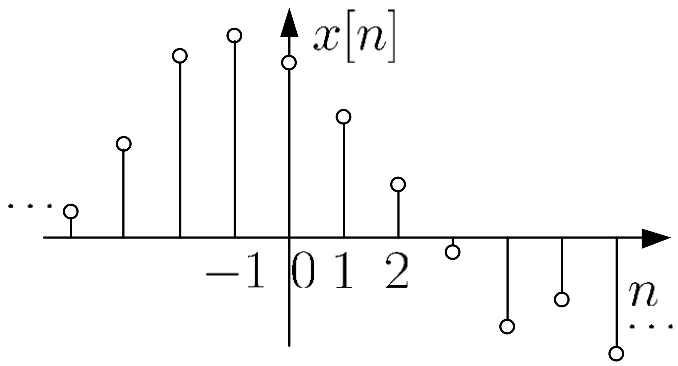
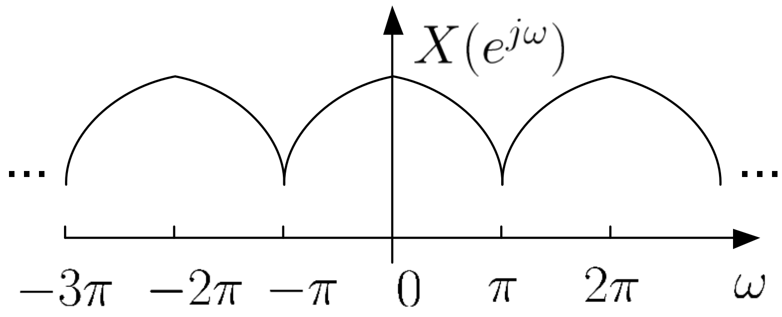
time domain	frequency domain
	
$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \Longrightarrow$	
	$\Longleftarrow x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
discrete and aperiodic	continuous and periodic

Fig.6.1: Illustration of DTFT

As $X(e^{j\omega})$ is generally complex, we can illustrate $X(e^{j\omega})$ using the magnitude and phase spectra, i.e., $|X(e^{j\omega})|$ and $\angle(X(e^{j\omega}))$:

$$|X(e^{j\omega})| = \sqrt{(\Re\{X(e^{j\omega})\})^2 + (\Im\{X(e^{j\omega})\})^2} \quad (6.8)$$

and

$$\angle(X(e^{j\omega})) = \tan^{-1}\left(\frac{\Im\{X(e^{j\omega})\}}{\Re\{X(e^{j\omega})\}}\right) \quad (6.9)$$

where both are continuous in frequency and periodic with period 2π.

Example 6.1

Find the DTFT of $x[n]$ which has the form of:

$$x[n] = a^n u[n], \quad |a| < 1$$

Using (6.4), the DTFT is:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

As $|ae^{-j\omega}| = |a| < 1 \Rightarrow |ae^{-j\omega}|^{\infty} = 0$ and applying the geometric sum formula, we have

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

where we see that $X(e^{j\omega})$ is complex.

Example 6.2

Find the DTFT of $x[n] = \delta[n]$.

Using (6.4), we have

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = e^{-j\omega \cdot 0} = 1$$

Analogous to Example 5.4 that the spectrum of the continuous-time $\delta(t)$ has unit amplitude at all frequencies, the spectrum of $\delta[n]$ also has unit amplitude at all frequencies in $(-\pi, \pi)$.

Example 6.3

Find the DTFT of $x[n] = u[n] - u[n-N]$. Plot the magnitude and phase spectra for $N = 10$.

Using (6.4), we have

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{e^{-j\omega \cdot 0}\left(1 - e^{-j\omega N}\right)}{1 - e^{-j\omega}} = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

We can also further express $X(e^{j\omega})$ as:

$$X(e^{j\omega}) = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = \frac{e^{-j\omega N/2}}{e^{-j\omega/2}} \cdot \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}} = e^{-j\omega(N-1)/2} \cdot \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

In doing so, $|X(e^{j\omega})|$ and $\angle(X(e^{j\omega}))$ can be written in closed-forms as:

$$|X(e^{j\omega})| = \left|\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right|$$

and

$$\angle(X(e^{j\omega})) = -\frac{\omega(N-1)}{2} + \angle\left(\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right)$$

Although $(\sin(\omega N/2)/\sin(\omega/2))$ is real, its phase is π if it is negative while the phase is 0 if it is positive.

Note that we generally employ (6.8) and (6.9) for magnitude and phase computation.

In using MATLAB to plot $|X(e^{j\omega})|$ and $\angle(X(e^{j\omega}))$, we utilize the command `sinc` so that there is no need to separately handle the “0/0” cases due to the sine functions. Recall:

$$\text{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$

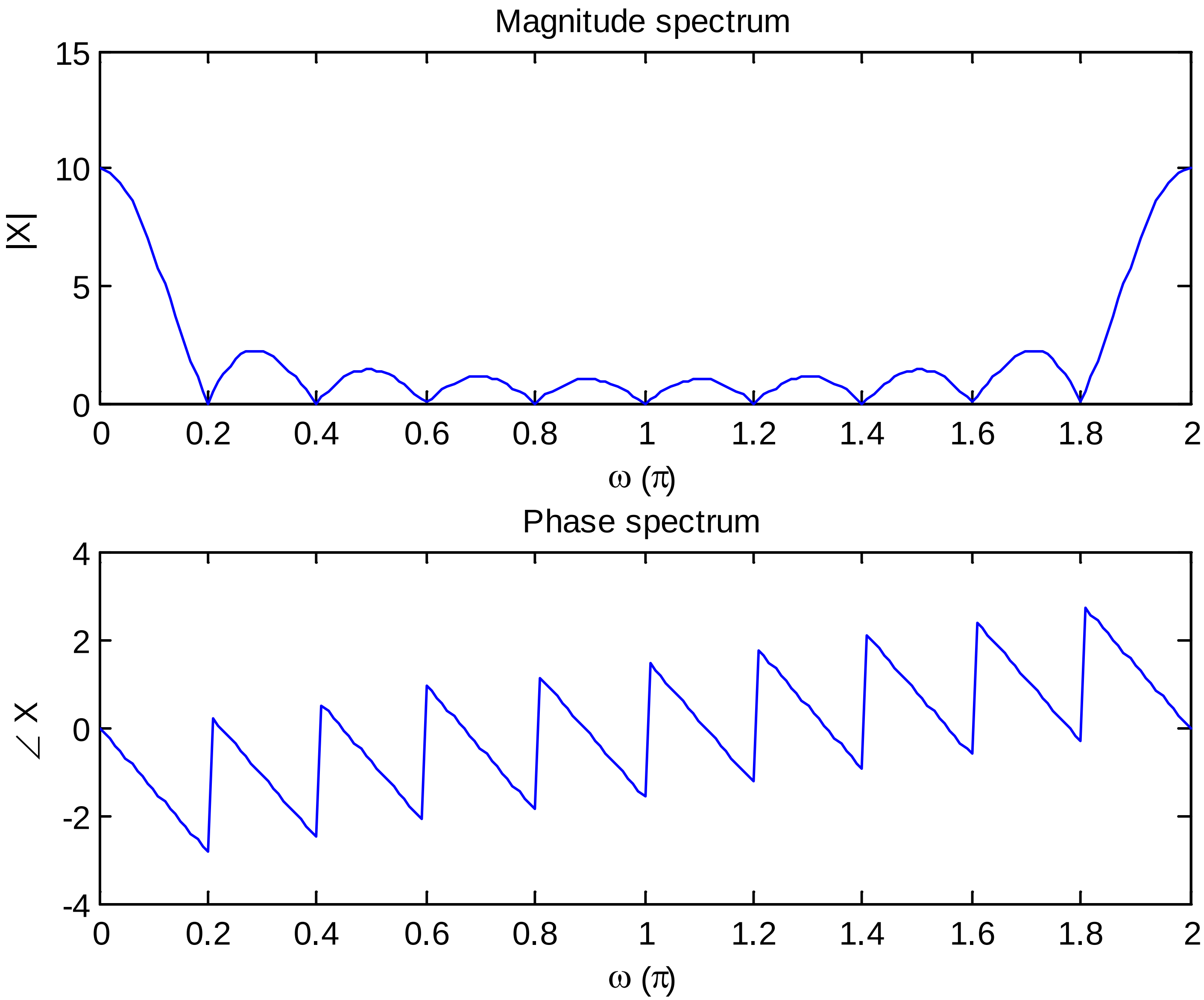
As a result, we have:

$$\begin{aligned}\frac{\sin(\omega N/2)}{\sin(\omega/2)} &= \frac{\sin(\omega \cdot N\pi/(2\pi))}{\omega \cdot N\pi/(2\pi)} \cdot \frac{\omega \cdot N\pi}{2\pi} \cdot \frac{\omega\pi/(2\pi)}{\sin(\omega\pi/(2\pi))} \cdot \frac{2\pi}{\omega\pi} \\ &= N \cdot \frac{\text{sinc}(\omega N/(2\pi))}{\text{sinc}(\omega/(2\pi))}\end{aligned}$$

The key MATLAB code for $N=10$ is

```
N=10;                   %N=10
w=0:0.01*pi:2*pi;       %successive frequency point
                        %separation is 0.01pi
dtft=N.*sinc(w.*N./2./pi)./(sinc(w./2./pi)).*exp(-
j.*w.*(N-1)./2);        %define DTFT function
subplot(2,1,1)
Mag=abs(dtft);          %compute magnitude
plot(w./pi,Mag);        %plot magnitude
subplot(2,1,2)
Pha=angle(dtft);        %compute phase
plot(w./pi,Pha);        %plot phase
```

There are 201 uniformly-spaced points for plotting the continuous functions $|X(e^{j\omega})|$ and $\angle(X(e^{j\omega}))$.


Magnitude spectrum
|X|
ω (π)
Phase spectrum
∠ X
ω (π)

Example 6.4

Find the inverse DTFT of $X(e^{j\omega})$ which is a rectangular pulse. Within the period of $[-\pi, \pi]$, $X(e^{j\omega})$ has the form of:

$$X(e^{j\omega}) = \begin{cases} 1, & -w_0 < \omega < w_0 \\ 0, & \text{otherwise} \end{cases}$$

where $0 < w_0 < \pi$.

Using (6.6), we get:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-w_0}^{w_0} e^{j\omega n}d\omega = \frac{\sin(w_0 n)}{\pi n} = \frac{w_0}{\pi}\text{sinc}\left(\frac{w_0 n}{\pi}\right)$$

That is, $x[n]$ is an infinite-duration sequence whose values are drawn from a scaled sinc function.

Note also that $x[n]$ corresponds to the discrete-time version in Example 5.2.

Example 6.5

Given a discrete-time finite-duration sinusoid:

$$x[n] = 2\cos(0.5\pi n + 1), \quad n = 0, 1, \cdots, 20$$

Find the tone frequency using DTFT.

Consider the continuous-time case first. According to (5.10), the Fourier transform pair for a complex continuous-time tone of frequency Ω_0 is:

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

That is, Ω_0 can be found by locating the peak of the Fourier transform. Moreover, a real-valued tone $\cos(\Omega_0 t)$ is:

$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

This means that Ω_0 and $-\Omega_0$ can be found from the two impulses of the Fourier transform of $\cos(\Omega_0 t)$.

Analogously, we expect that there are two peaks which correspond to frequencies 0.5π and -0.5π in the DTFT for $x[n]$.

The MATLAB code is

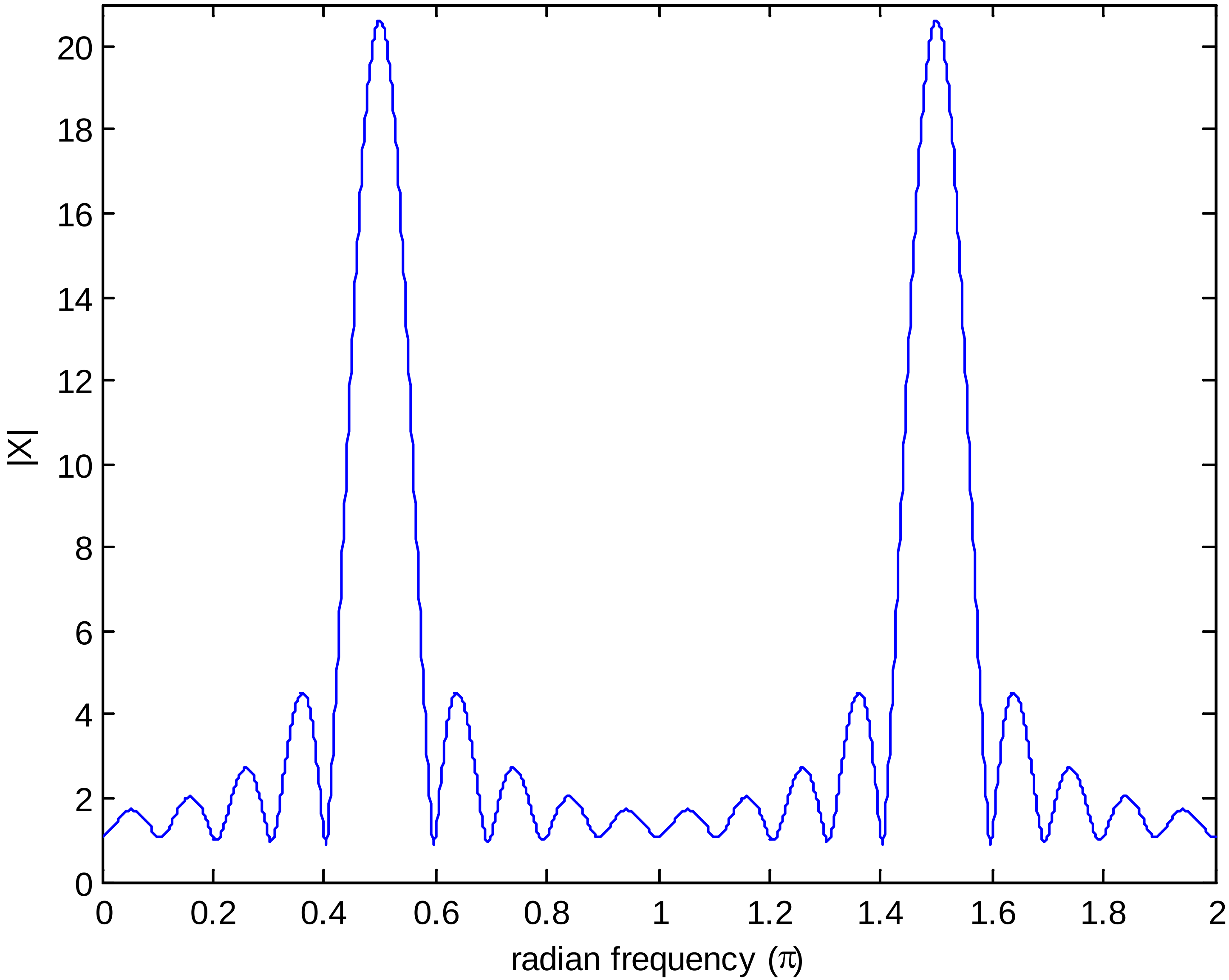
```
N=21;                %number of samples is 21
A=2;                 %tone amplitude is 2
w=0.5*pi;            %frequency is 0.5*pi
p=1;                 %phase is 1
n=0:N-1;             %define a vector of size N
x=A*cos(w*n+p);      %generate tone
for k=1:2001         %frequency index k
w=(k-1)*pi/1000; %frequency interval of [0,2pi];
%compute DTFT at frequency points w only
e=(exp(j.*w.*n)).'; %construct exponential vector
X(k) = x*e;
end
```

```
X=abs(X); %compute magnitude
k=1:2001;
f=(k-1)./1000;
plot(f,X)
```

Note that $X(e^{j\omega})$ is continuous in ω and we cannot compute all points. Instead, here we only compute $X(e^{j\omega})$ at $\omega_k = 2\pi k/1000$ for $k = 0, 1, \cdots, 2000$. That is, k corresponds to frequency `w=(k-1)*pi/1000`.

With the use of `max(abs(X))`, we find that the peak magnitude corresponds to the index k=501, then the signal frequency is correctly determined as:

$$\frac{501-1}{1000}\pi = 0.5\pi$$



|X|
radian frequency (π)
0
2
4
6
8
10
12
14
16
18
20
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Properties of DTFT

Linearity

If $x_1[n] \leftrightarrow X_1(e^{j\omega})$ and $x_2[n] \leftrightarrow X_2(e^{j\omega})$ are two DTFT pairs, then:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega}) \quad (6.10)$$

Time Shifting

A shift of n_0 in $x[n]$ causes a multiplication of $e^{-j\omega n_0}$ in $X(e^{j\omega})$:

$$x[n] \leftrightarrow X(e^{j\omega}) \Rightarrow x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(e^{j\omega}) \quad (6.11)$$

Time Reversal

The DTFT pair for $x[-n]$ is given as:

$$x[n] \leftrightarrow X(e^{j\omega}) \Rightarrow x[-n] \leftrightarrow X(e^{-j\omega}) \quad (6.12)$$

Multiplication

Multiplication in the time domain corresponds to convolution in the frequency domain:

$$x_1[n] \cdot x_2[n] \leftrightarrow X_1(e^{j\omega}) \tilde{\otimes} X_2(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\tau}) X_2(e^{j(\omega-\tau)}) d\tau \quad (6.13)$$

where $\tilde{\otimes}$ denotes convolution within one period.

Conjugation

The DTFT pair for $x^*[n]$ is given as:

$$x[n] \leftrightarrow X(e^{j\omega}) \Rightarrow x^*[n] \leftrightarrow X^*(e^{-j\omega}) \quad (6.14)$$

Multiplication by an Exponential Sequence

Multiplying $x[n]$ by $e^{j\omega_0 n}$ in time domain corresponds to a shift of ω_0 in the frequency domain:

$$x[n] \leftrightarrow X(e^{j\omega}) \Rightarrow e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)}) \quad (6.15)$$

Differentiation

Differentiating $X(e^{j\omega})$ with respect to ω corresponds to multiplying $x[n]$ by n:

$$x[n] \leftrightarrow X(e^{j\omega}) \Rightarrow nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega} \quad (6.16)$$

Parseval's Relation

The Parseval's relation addresses the energy of $x[n]$:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega \quad (6.17)$$

With the use of (6.6), (6.17) is proved as:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |x[n]|^2 &= \sum_{n=-\infty}^{\infty} x[n]x^*[n] \\
&= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \right)^* \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega}) \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right) d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X(e^{j\omega}) \right|^2 d\omega
\end{aligned} \tag{6.18}$$

Convolution

If $x_1[n] \leftrightarrow X_1(e^{j\omega})$ and $x_2[n] \leftrightarrow X_2(e^{j\omega})$ are two DTFT pairs, then:

$$x_1[n] \otimes x_2[n] \leftrightarrow X_1(e^{j\omega})X_2(e^{j\omega}) \tag{6.19}$$

which can be derived as:

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} x_1[n] \otimes x_2[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1[m] x_2[n-m] e^{-j\omega n} \\
&= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1[m] x_2[k] e^{-j\omega(k+m)}, \quad k = n - m \\
&= \left[\sum_{m=-\infty}^{\infty} x_1[m] e^{-j\omega m}\right] \cdot \left[\sum_{k=-\infty}^{\infty} x_2[k] e^{-j\omega k}\right] \\
&= X_1(e^{j\omega}) \cdot X_2(e^{j\omega})
\end{aligned} \tag{6.20}$$

DTFT and Linear Time-Invariant System

Recall in a discrete-time LTI system, the input-output relationship is characterized by convolution in (3.11):

$$\begin{aligned} y[n] &= x[n] \otimes h[n] \\ &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \end{aligned} \tag{6.21}$$

Using (6.19), we can consider (6.21) in frequency domain:

$$y[n] = x[n] \otimes h[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \tag{6.22}$$

This suggests apart from computing the output using time-domain approach via convolution, we can convert the input and impulse response to frequency domain, then $y[n]$ is computed from inverse DTFT of $X(e^{j\omega})H(e^{j\omega})$.

In fact, $H(e^{j\omega})$ represents the LTI system in the frequency domain, is called the system frequency response.

Recall (3.22) that the input and output of a discrete-time LTI system satisfy the difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{6.23}$$

Taking the DTFT and using the linearity and time shifting properties, we get:

$$Y(e^{j\omega}) \sum_{k=0}^{N} a_k e^{j\omega k} = X(e^{j\omega}) \sum_{k=0}^{M} b_k e^{j\omega k} \tag{6.24}$$

The system frequency response can also be computed as:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{j\omega k}}{\sum_{k=0}^{M} a_k e^{j\omega k}} \quad (6.25)$$

<u>Example 6.6</u>

Determine the system frequency response for a causal LTI system described by the following difference equation:

$$y[n] = 0.1y[n-1] + x[n] + x[n-1]$$

Applying (6.25), we easily obtain:

$$Y(e^{j\omega})\left(1 - 0.1e^{-j\omega}\right) = X(e^{j\omega})\left(1 + e^{-j\omega}\right) \Rightarrow H(e^{j\omega}) = \frac{1 + e^{-j\omega}}{1 - 0.1e^{-j\omega}}$$

Example 6.7

The moving average (MA) is in fact a LTI system. Consider the close price of Dow Jones Industrial Average (DJIA) index as input $x[n]$ and the output $y[n]$ is the 20-day MA. Establish the input-output relationship using a difference equation. Then compute the system impulse response and frequency response. Plot the system magnitude frequency response.

In stock market (or other applications), future data are unavailable. The best we can do is to use the today value and close prices of previous 19 trading days in MA calculation, that is:

$$y[n] = \frac{1}{20}\sum_{k=0}^{19} x[n-k]$$

Following Example 3.18, we can easily deduce the impulse response as:

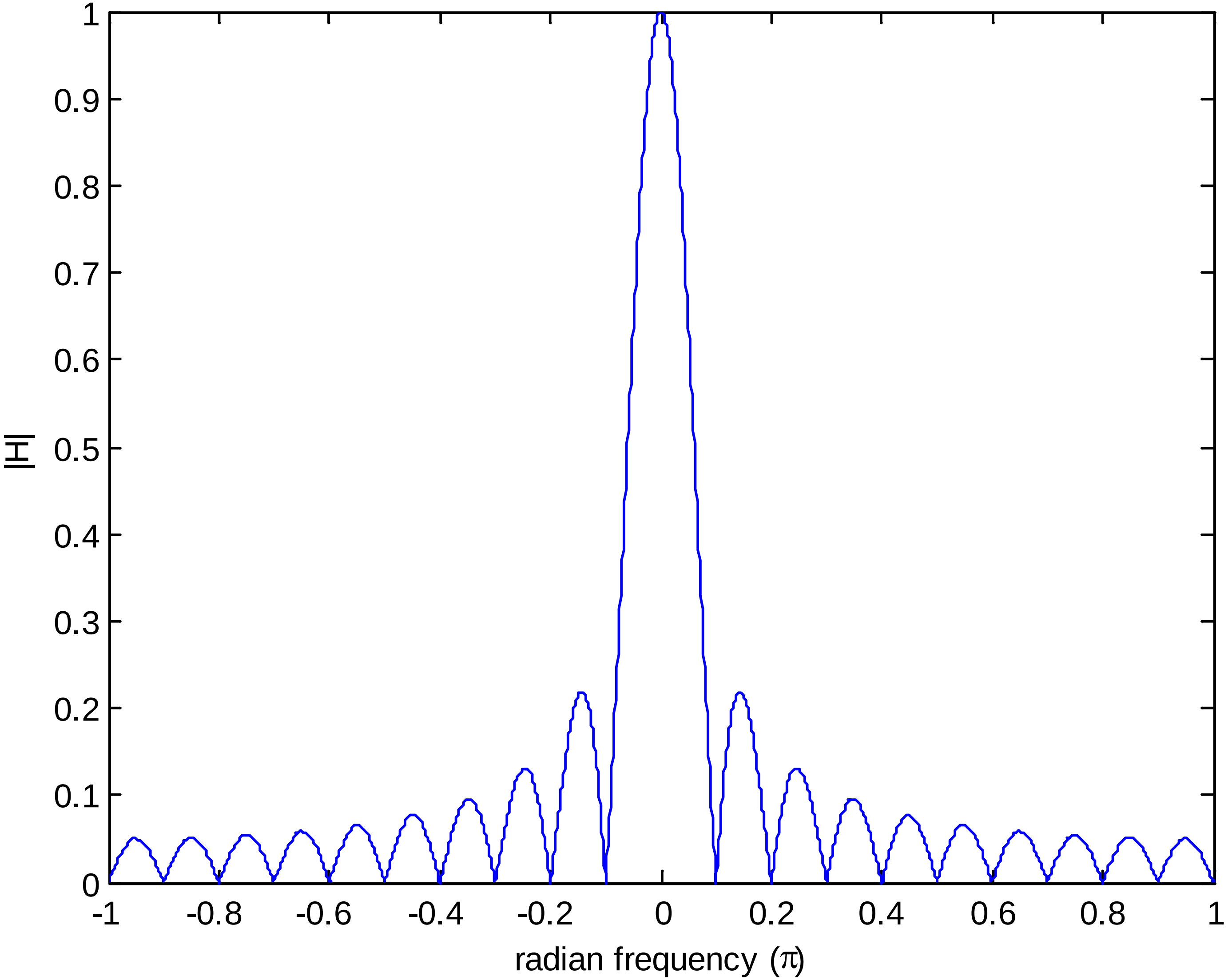
$$h[n] = \frac{1}{20}(\delta[n] + \delta[n-1] + \cdots + \delta[n-19]) = \frac{1}{20}\sum_{k=0}^{19}\delta[n-k]$$

Applying (6.25), the system frequency response is:

$$H(e^{j\omega}) = \frac{1}{20}\sum_{k=0}^{19} e^{-j\omega k}$$

From the magnitude plot, the frequency is concentrated around the DC. It is called a lowpass filter (also for Example 6.3).

From Fig. 1.11, we see that low-frequency components (smooth part) are kept while high-frequency components (fluctuations) are suppressed in the system output.



The MATLAB code for the plot is provided as `ex6_7.m`.

Sampling and Reconstruction

Chapter Intended Learning Outcomes:

(i) Convert a continuous-time signal to a discrete-time signal via sampling

(ii) Construct a continuous-time signal from a discrete-time signal

(iii) Understand the conditions when a sampled signal can uniquely represent its continuous-time counterpart

Sampling

Process of converting a continuous-time signal $x(t)$ into a discrete-time signal $x[n]$.

$x[n]$ is obtained by extracting $x(t)$ every T s where T is known as the sampling period or interval.

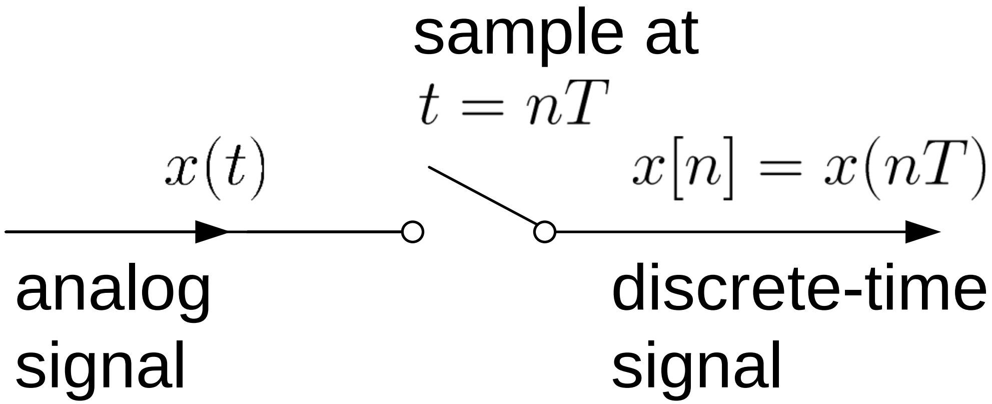


Fig.7.1: Conversion of analog signal to discrete-time signal

Relationship between $x(t)$ and $x[n]$ is:

$$x[n] = x(t)|_{t=nT} = x(nT), \quad n = \cdots - 1, 0, 1, 2, \cdots \qquad (7.1)$$

Conceptually, conversion of $x(t)$ to $x[n]$ is achieved by a continuous-time to discrete-time (CD) converter:

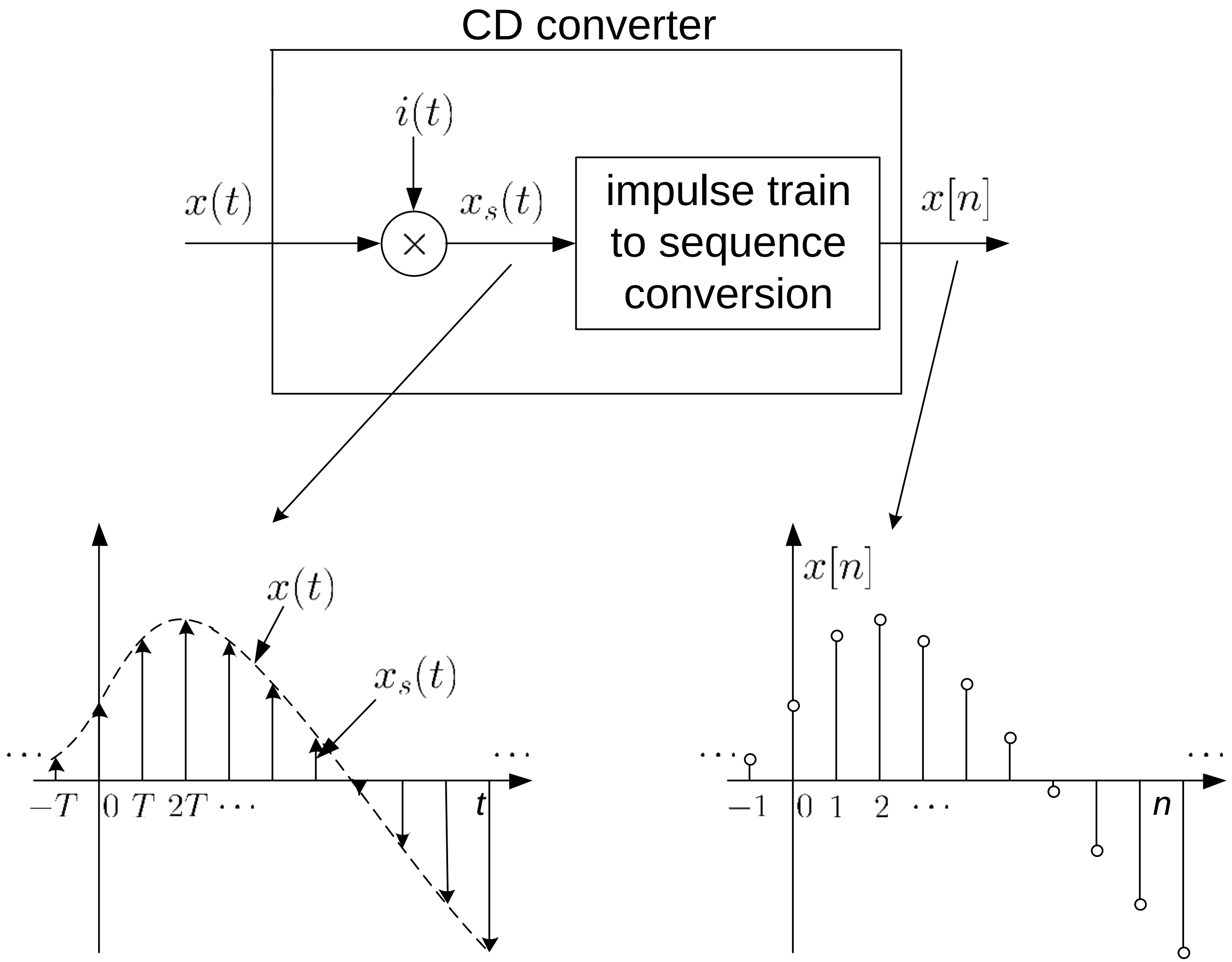


Fig.7.2: Block diagram of CD converter

A fundamental question is whether $x[n]$ can uniquely represent $x(t)$ or if we can use $x[n]$ to reconstruct $x(t)$.

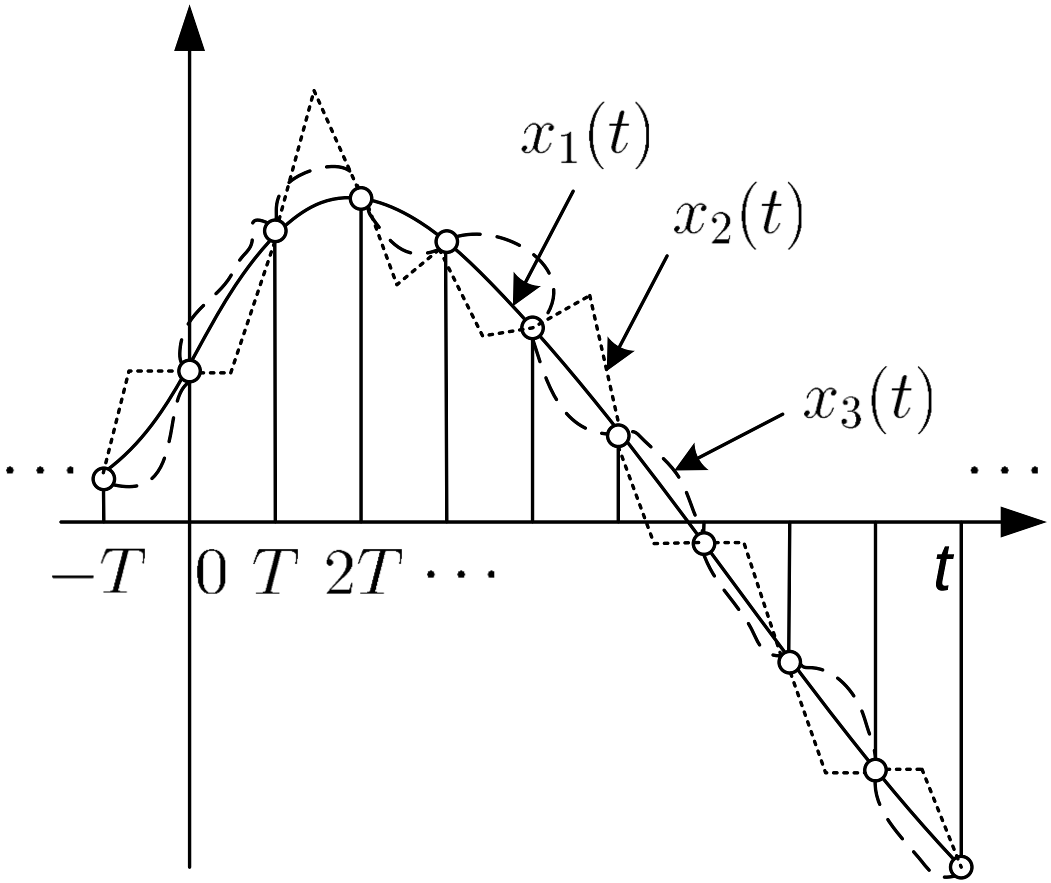


Fig.7.3: Different analog signals map to same sequence

But, the answer is yes when:

(1) $x(t)$ is bandlimited such that its Fourier transform $X(j\Omega)=0$ for $|\Omega| \geq \Omega_b$ where Ω_b is called the bandwidth.

(2) Sampling period T is sufficiently small.

Example 7.1

The continuous-time signal $x(t)=\cos(200\pi t)$ is used as the input for a CD converter with the sampling period $1/300$ s. Determine the resultant discrete-time signal $x[n]$.

According to (7.1), $x[n]$ is

$$x[n] = x(nT) = \cos(200n\pi T) = \cos\left(\frac{2\pi n}{3}\right), \quad n = \cdots -1, 0, 1, 2, \cdots$$

The frequency in $x(t)$ is $200\pi \text{ rads}^{-1}$ while that of $x[n]$ is $2\pi/3$.

Frequency Domain Representation of Sampled Signal

In the time domain, $x_s(t)$ is obtained by multiplying $x(t)$ by the impulse train $i(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)$. From (6.2), we have:

$$x_s(t) = x(t)\sum_{k=-\infty}^{\infty}\delta(t-kT) = \sum_{k=-\infty}^{\infty} x[k]\delta(t-kT) \qquad (7.2)$$

Let the sampling frequency in radian be $\Omega_s = 2\pi/T$ (or $F_s = 1/T = \Omega_s/(2\pi)$ in Hz). From Example 5.5, we have:

$$I(j\Omega) = \Omega_s \sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_s) \qquad (7.3)$$

Using multiplication property of Fourier transform in (5.18):

$$x_1(t) \cdot x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(j\Omega) \otimes X_2(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\tau) X_2(j(\Omega - \tau)) d\tau \quad (7.4)$$

where the convolution operation corresponds to continuous-time signals.

Using (7.2)-(7.4) and the properties of $\delta(t)$, $X_s(j\Omega)$ is determined as follows:

$$
\begin{aligned}
X_s(j\Omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} I(j\tau)X(j(\Omega-\tau))d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\Omega_s \sum_{k=-\infty}^{\infty} \delta(\tau-k\Omega_s)\right) X(j(\Omega-\tau))d\tau \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(j(\Omega-\tau))\delta(\tau-k\Omega_s)d\tau\right) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\Omega-k\Omega_s)) \left(\int_{-\infty}^{\infty} \delta(\tau-k\Omega_s)d\tau\right) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\Omega-k\Omega_s)) \qquad (7.5)
\end{aligned}
$$

which is the sum of infinite copies of $X(j\Omega)$ scaled by $1/T$.

When Ω_s is chosen sufficiently large such that all copies of $X(j\Omega)/T$ do not overlap, that is, $\Omega_s - \Omega_b > \Omega_b$ or $\Omega_s > 2\Omega_b$, we can get $X(j\Omega)$ from $X_s(j\Omega)$.

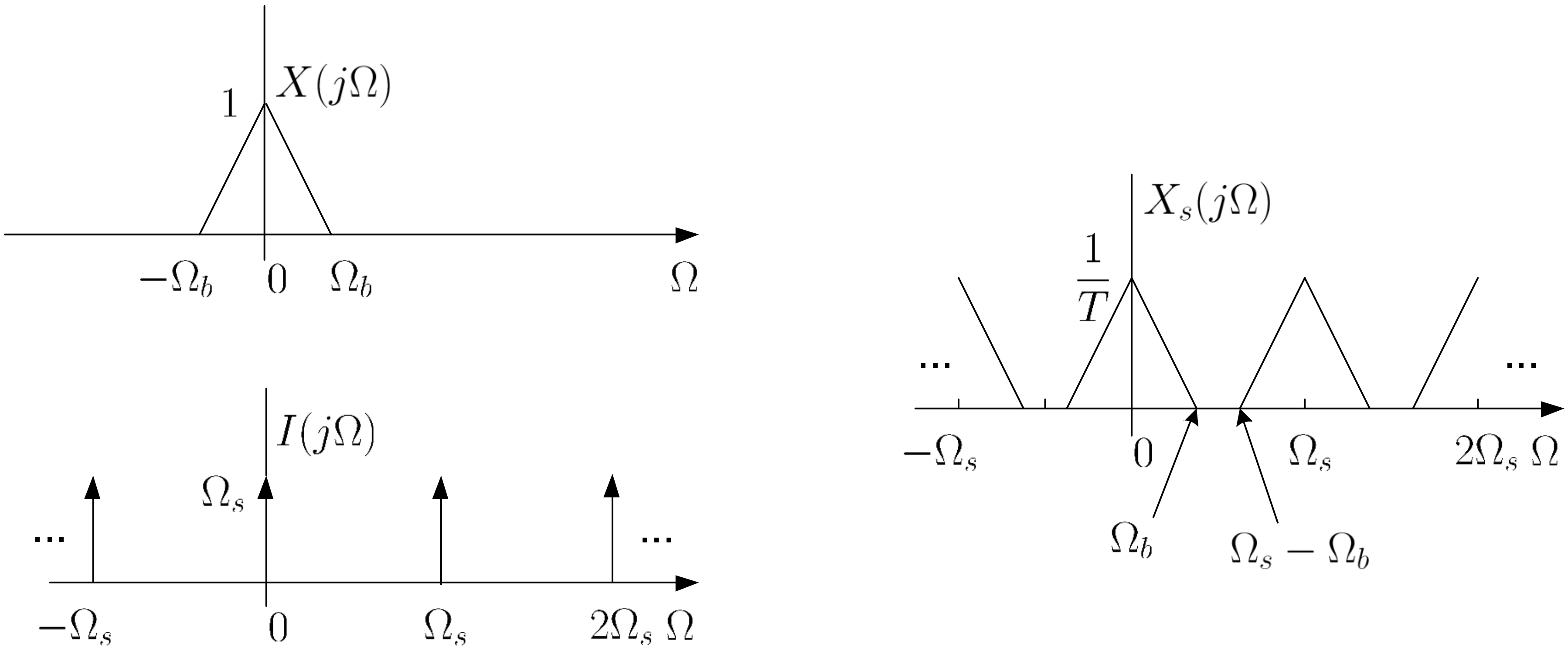


Fig.7.4: $X_s(j\Omega) = X(j\Omega) \otimes I(j\Omega)$ for sufficiently large Ω_s

When Ω_s is not chosen sufficiently large such that $\Omega_s < 2\Omega_b$, copies of $X(j\Omega)/T$ overlap, we cannot get $X(j\Omega)$ from $X_s(j\Omega)$, which is referred to aliasing.

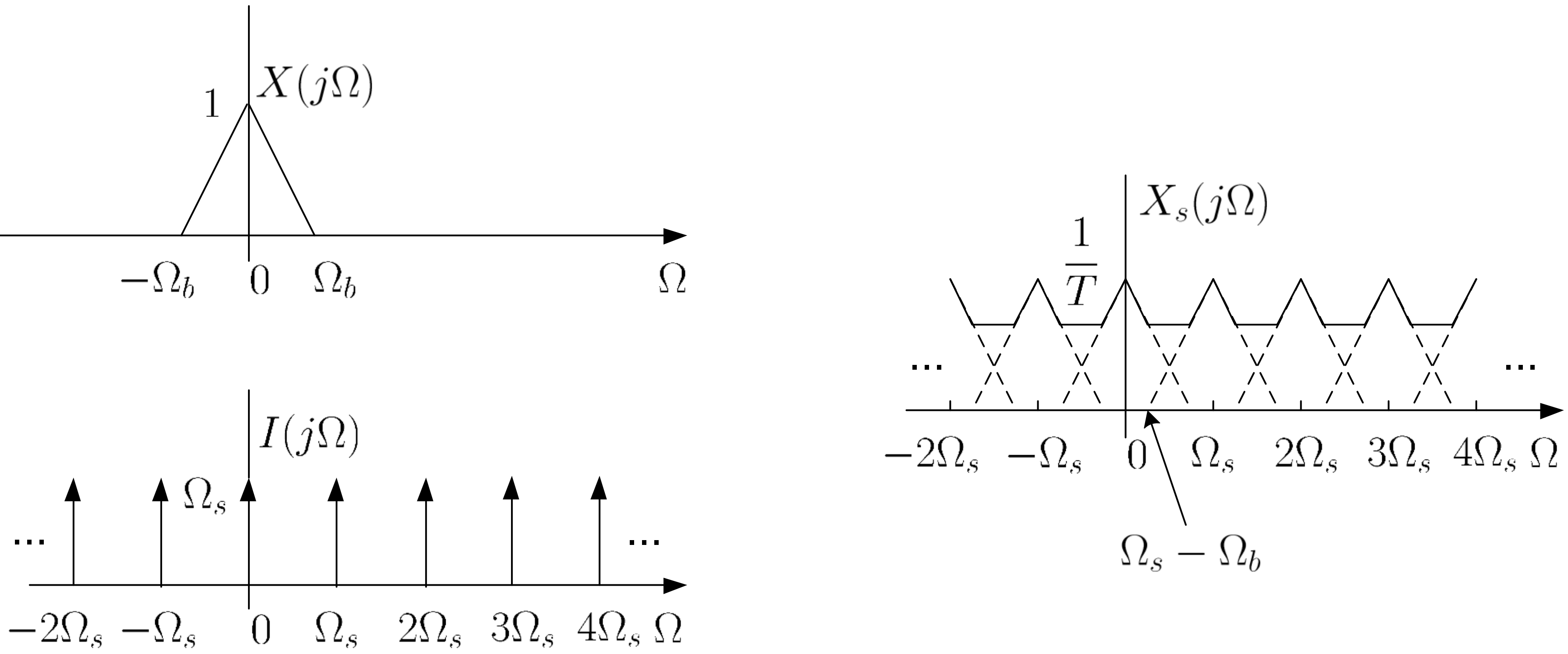


Fig.7.5: $X_s(j\Omega) = X(j\Omega) \otimes I(j\Omega)$ when Ω_s is not large enough

These findings can be summarized as sampling theorem:

Let $x(t)$ be a bandlimited continuous-time signal with

$$X(j\Omega) = 0, \quad |\Omega| \geq \Omega_b \tag{7.6}$$

Then $x(t)$ is uniquely determined by its samples $x[n] = x(nT)$, $n = \cdots - 1, 0, 1, 2, \cdots$, if

$$\Omega_s = \frac{2\pi}{T} > 2\Omega_b \tag{7.7}$$

In order to avoid aliasing, the sampling frequency must exceed $2\Omega_b$.

Application	$f_b = \Omega_b/(2\pi)$	$f_s = \Omega_s/(2\pi)$
Biomedical	< 500 Hz	1 kHz
Telephone speech	< 4 kHz	8 kHz
Music	< 20 kHz	44.1 kHz
Ultrasonic	< 100 kHz	250 kHz
Radar	< 100 MHz	200 MHz

Table 7.1: Typical bandwidths and sampling frequencies in signal processing applications

Example 7.2

Consider the continuous-time signal $x(t)$:

$$x(t) = 1 + \sin(2000\pi t) + \cos(4000\pi t)$$

Determine minimum sampling frequency to avoid aliasing.

The frequencies are 0, 2000π and 4000π . The minimum sampling frequency must exceed $8000\pi \text{ rads}^{-1}$.

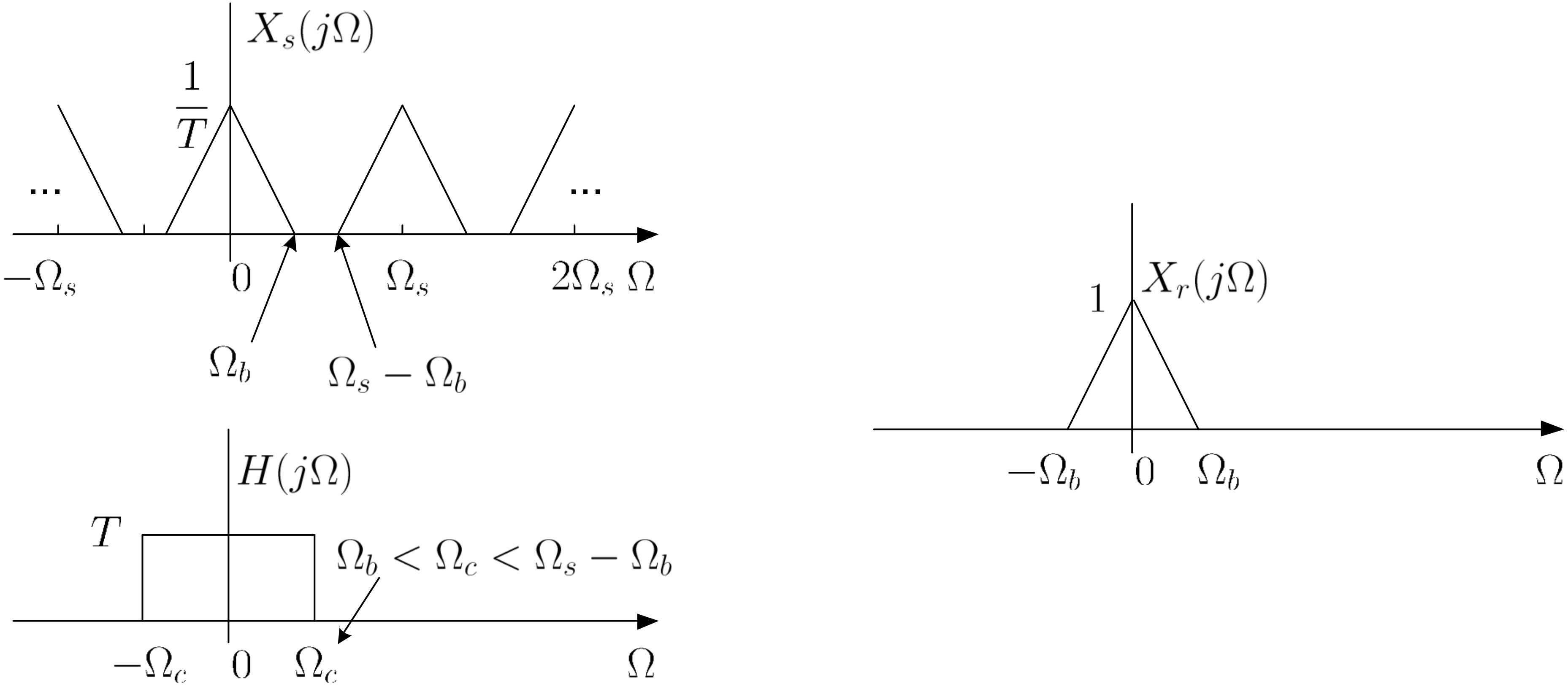


Fig.7.6: Multiplying $X_s(j\Omega)$ by $H(j\Omega)$ to recover $X(j\Omega)$

In frequency domain, we multiply $X_s(j\Omega)$ by $H(j\Omega)$ with amplitude T and bandwidth Ω_c with $\Omega_b < \Omega_c < \Omega_s - \Omega_b$, to obtain $X_r(j\Omega)$, and it corresponds to $x_r(t) = x_s(t) \otimes h(t)$, according to (5.26).

Reconstruction

Process of transforming $x[n]$ back to $x(t)$ via a discrete-time to continuous-time (DC) converter.

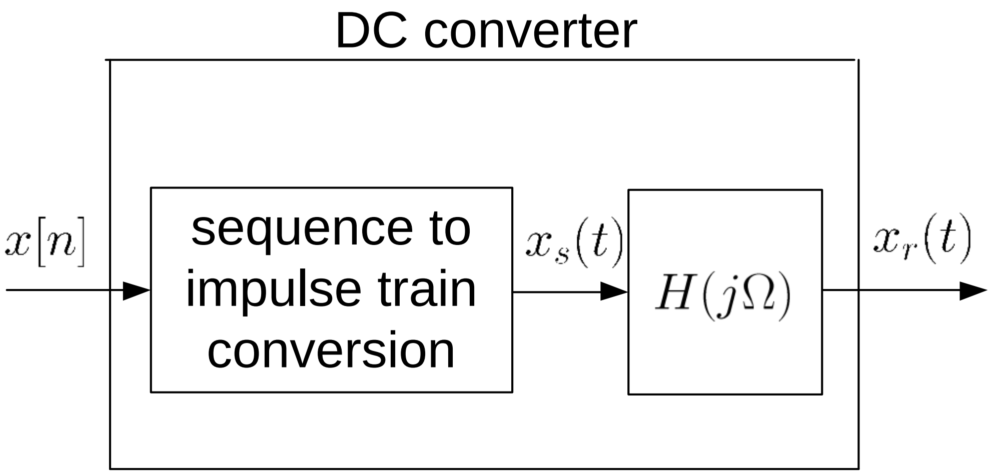


Fig.7.7: Block diagram of DC converter

From Fig.7.6, the requirements of $H(j\Omega)$ are:

$$H(j\Omega) = \begin{cases} T, & -\Omega_c < \Omega < \Omega_c \\ 0, & \text{otherwise} \end{cases} \quad (7.8)$$

where $\Omega_b < \Omega_c < \Omega_s - \Omega_b$, which is a lowpass filter.

For simplicity, we set Ω_c as the average of Ω_b and $(\Omega_s - \Omega_b)$:

$$\Omega_c = \frac{\Omega_s}{2} = \frac{\pi}{T} \tag{7.9}$$

To get the impulse response $h(t)$, we take inverse Fourier transform of $H(j\Omega)$ or use Example 5.2:

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\Omega)e^{j\Omega t}d\Omega = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} Te^{j\Omega t}d\Omega = \frac{T\sin(\pi t/T)}{\pi t} \\ &= \mathrm{sinc}\left(\frac{t}{T}\right) \end{aligned} \tag{7.10}$$

where $\mathrm{sinc}(u) = \sin(\pi u)/(\pi u)$.

Using (7.2) and the properties of $\delta(t)$, $x_r(t)$ is:

$$\begin{aligned} x_r(t) &= x_s(t) \otimes h(t) \\ &= \left(\sum_{k=-\infty}^{\infty} x[k]\delta(t-kT) \right) \otimes h(t) \\ &= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]\delta(\tau-kT)h(t-\tau)d\tau \\ &= \sum_{k=-\infty}^{\infty} x[k]h(t-kT) \\ &= \sum_{k=-\infty}^{\infty} x[k]\mathrm{sinc}\left(\frac{t-kT}{T}\right) \end{aligned} \tag{7.11}$$

which holds for all real values of t.

The interpolation formula can be verified at $t = nT$:

$$x_r(nT) = \sum_{k=-\infty}^{\infty} x[k]\operatorname{sinc}(n-k) \tag{7.12}$$

It is easy to see that

$$\operatorname{sinc}(n-k) = \frac{\sin((n-k)\pi)}{(n-k)\pi} = 0, \quad n \neq k \tag{7.13}$$

For $n = k$, we use $\mathrm{L'H\hat{o}pital}$'s rule to obtain:

$$\operatorname{sinc}(0) = \lim_{m \to 0} \frac{\sin(m\pi)}{m\pi} = \lim_{m \to 0} \frac{\dfrac{d\sin(m\pi)}{dm}}{\dfrac{dm\pi}{dm}} = \lim_{m \to 0} \frac{\pi\cos(m\pi)}{\pi} = 1 \tag{7.14}$$

Substituting (7.13)-(7.14) into (7.12) yields:

$$x_r(nT) = x[n] = x(nT) \tag{7.15}$$

which aligns with $x_r(t) = x(t)$.

<u>Example 7.3</u>

Suppose a continuous-time signal $x(t)=\cos(\Omega_0 t)$ is sampled at a sampling frequency of 1000Hz to produce $x[n]$:

$$x[n]=\cos\left(\frac{\pi n}{4}\right)$$

Determine 2 possible positive values of Ω_0, say, Ω_1 and Ω_2. Discuss if $\cos(\Omega_1 t)$ or $\cos(\Omega_2 t)$ will be obtained when passing $x[n]$ through the DC converter.

According to (7.1) with $T=1/1000$ s:

$$\cos\left(\frac{\pi n}{4}\right)=x[n]=x(nT)=\cos\left(\frac{\Omega_0 n}{1000}\right)$$

Ω_1 is easily computed as:

$$\frac{\pi n}{4}=\frac{\Omega_1 n}{1000}\Rightarrow \Omega_1=\frac{1000\pi}{4}=250\pi$$

Ω_2 can be obtained by noting the periodicity of a sinusoid:

$$\cos\left(\frac{\pi n}{4}\right) = \cos\left(\frac{\pi n}{4} + 2n\pi\right) = \cos\left(\frac{9\pi n}{4}\right) = \cos\left(\frac{\Omega_2 n}{1000}\right)$$

As a result, we have:

$$\frac{9\pi n}{4} = \frac{\Omega_2 n}{1000} \Rightarrow \Omega_2 = \frac{9000\pi}{4} = 2250\pi$$

This is illustrated using the MATLAB code:

```
O1=250*pi;                    %first frequency
O2=2250*pi;                   %second frequency
Ts=1/100000;%successive sample separation is 0.01T
t=0:Ts:0.02;%observation interval
x1=cos(O1.*t);                %tone from first frequency
x2=cos(O2.*t);                 %tone from second frequency
```

There are 2001 samples in 0.02s and interpolating the successive points based on `plot` yields good approximation.

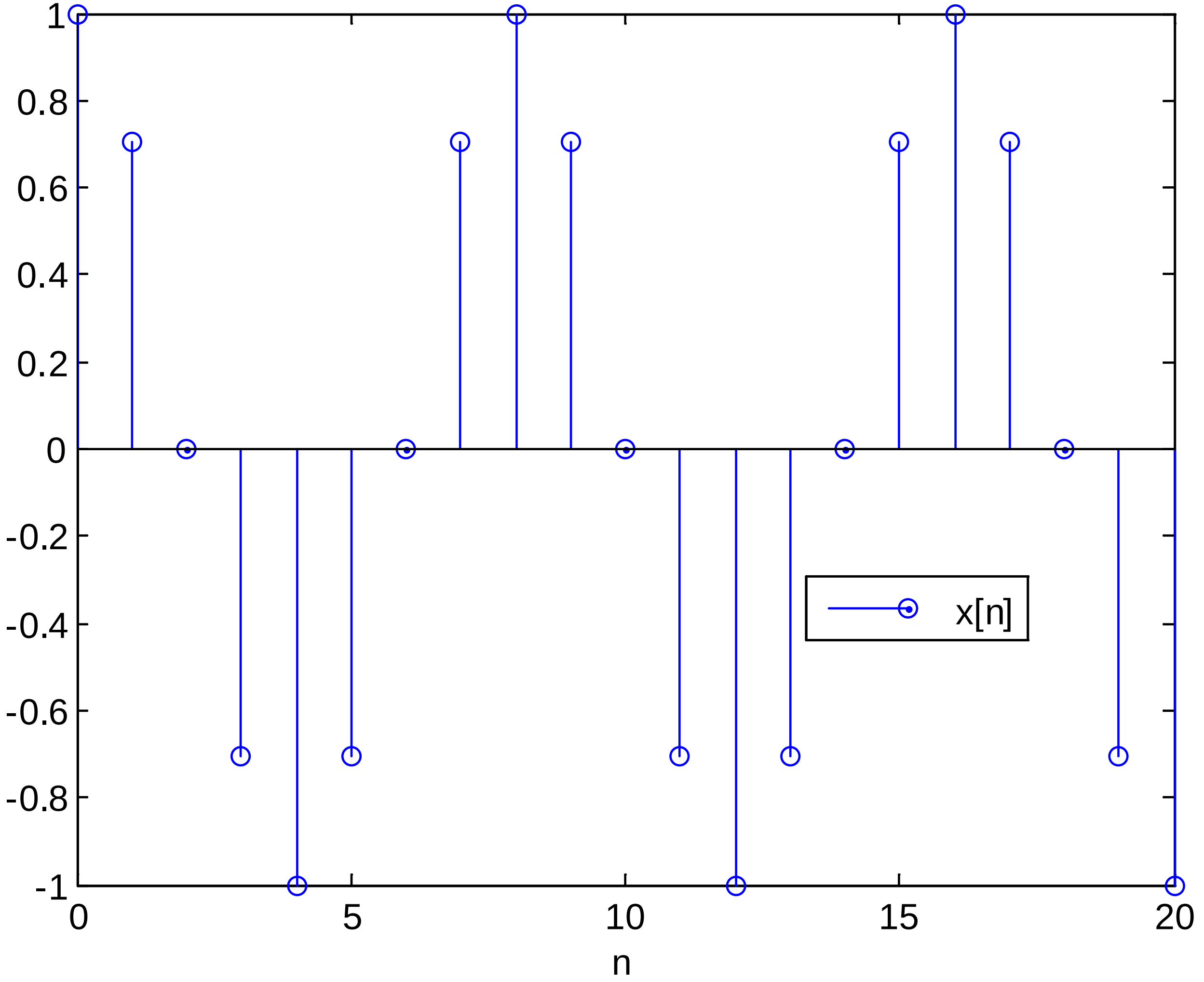


Fig.7.8: Discrete-time sinusoid

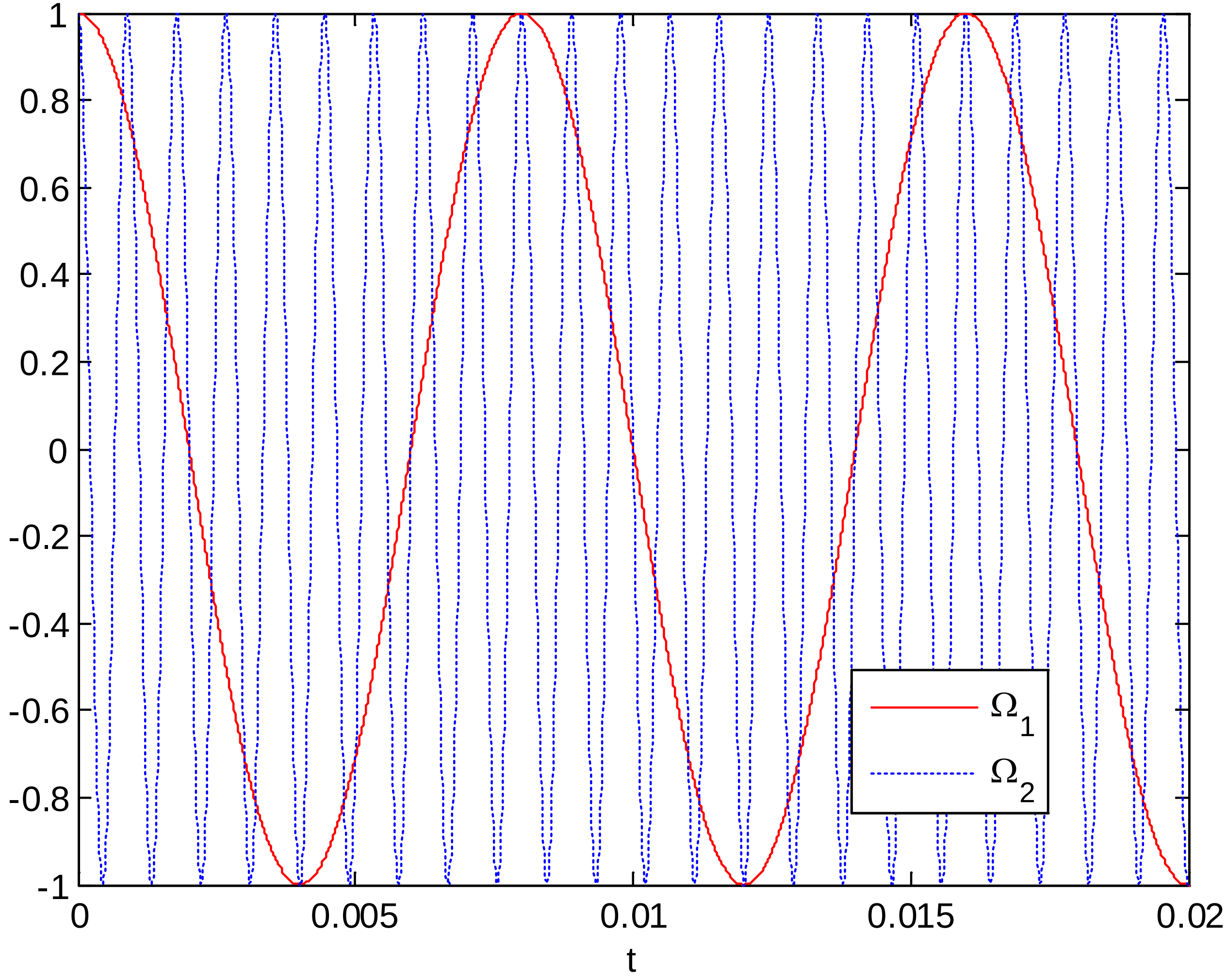


Fig.7.9: Continuous-time sinusoids

Passing $x[n]$ through the DC converter only produces $\cos(\Omega_1 t)$ but not $\cos(\Omega_2 t)$.

The signal frequency of $\cos(\Omega_2 t)$ is $2250\pi \text{ rads}^{-1}$ and hence the sampling frequency without aliasing is $\Omega_s > 4500\pi$.

Given $F_s = 1000$ Hz or $\Omega_s = 2000\pi \quad \text{rads}^{-1}$, $\cos(\Omega_2 t)$ does not correspond to $x[n]$.

We can recover $x_r(t) = \cos(\Omega_1 t)$ because the signal frequency of $\cos(\Omega_1 t)$ is $250\pi \text{ rads}^{-1}$, and $\Omega_s = 2000\pi > 2 \cdot 250\pi$.

Based on (7.11), $x_r(t) = \cos(\Omega_1 t)$ is:

$$x_r(t) = \sum_{k=-\infty}^{\infty} x[k]\text{sinc}\left(\frac{t-kT}{T}\right) \approx \sum_{k=-10}^{30} x[k]\text{sinc}\left(\frac{t-kT}{T}\right)$$

with $T = 1/1000$ s.

The MATLAB code for reconstructing $\cos(\Omega_1 t)$ is:

```
n=-10:30;              %add 20 past and future samples
x=cos(pi.*n./4);
T=1/1000;              %sampling interval is 1/1000
for l=1:2000           %observed interval is [0,0.02]
t=(l-1)*T/100;%successive sample separation is 0.01T
h=sinc((t-n.*T)./T);
xr(l)=x*h.'; %approximate interpolation of (7.11)
end
```

We compute 2000 samples of $x_r(t)$ in $t \in [0, 0.02]$s.

The value of each $x_r(t)$ at time `t` is approximated as `x*h.'` where the sinc vector is updated for each computation.

The MATLAB program is provided as `ex7_3.m`.

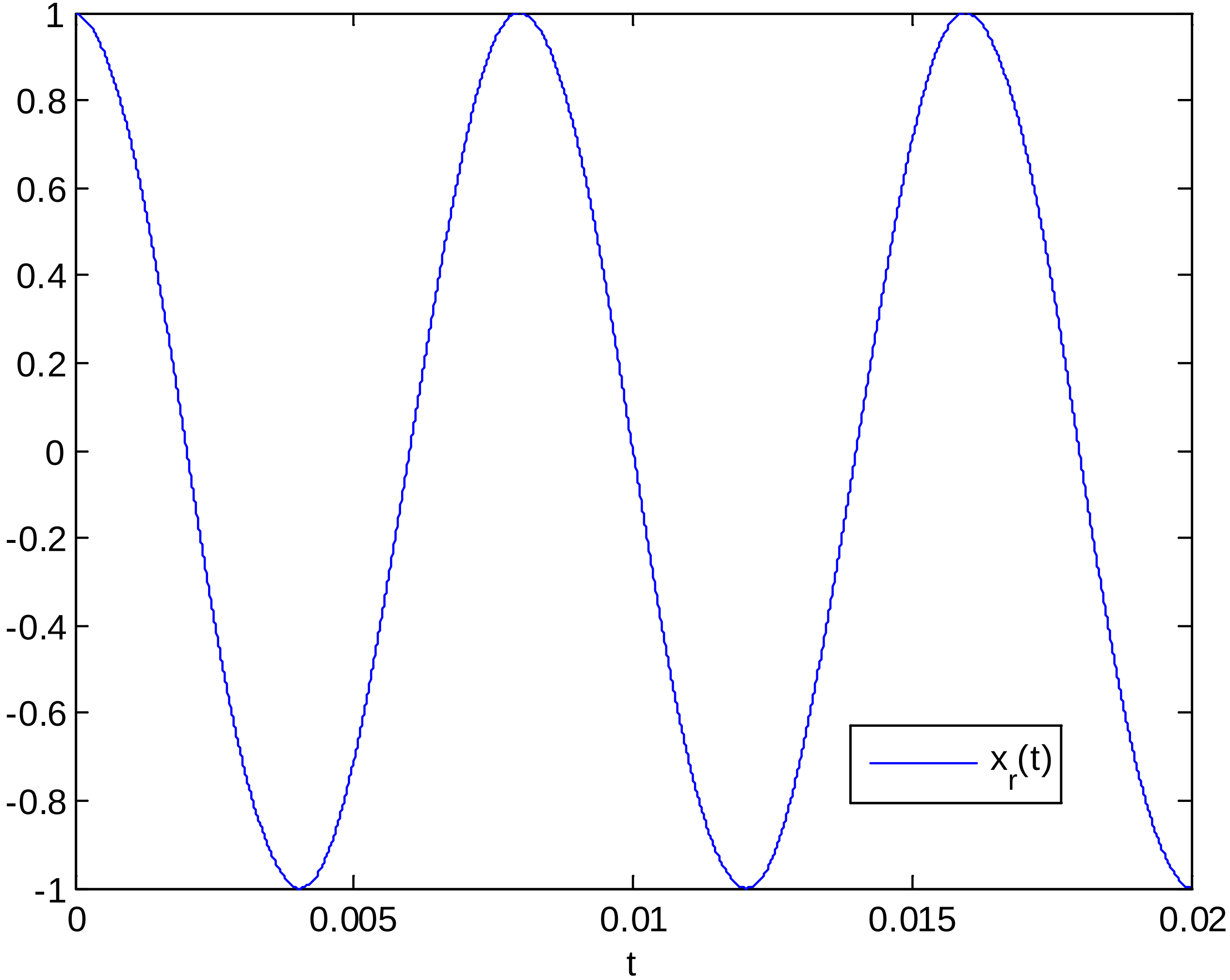


Fig.7.10: Reconstructed continuous-time sinusoid

z Transform

Chapter Intended Learning Outcomes:

(i) Represent discrete-time signals using z transform

(ii) Understand the relationship between z transform and discrete-time Fourier transform

(iii) Understand the properties of z transform

(iv) Perform operations on z transform and inverse z transform

(v) Apply z transform for analyzing linear time-invariant systems

Discrete-Time Signal Representation with z Transform

Apart from discrete-time Fourier transform (DTFT), we can also use z transform to represent discrete-time signals.

The z transform of $x[n]$, denoted by $X(z)$, is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad (8.1)$$

where z is a continuous complex variable.

We can also express z as:

$$z = re^{j\omega} \quad (8.2)$$

where $r = |z| > 0$ is magnitude and $\omega = \angle(z)$ is angle of z.

Employing (8.2), the z transform can be written as:

$$X(z)\big|_{z=re^{j\omega}} = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right) e^{-j\omega n} \quad (8.3)$$

Comparing (8.3) and the DTFT formula in (6.4):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (8.4)$$

That is, z transform of $x[n]$ is equal to the DTFT of $x[n]r^{-n}$.

When $r = 1$ or $z = e^{j\omega}$, (8.3) and (8.4) are identical:

$$X(z)\big|_{z=e^{j\omega}} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (8.5)$$

That is, z transform generalizes the DTFT.

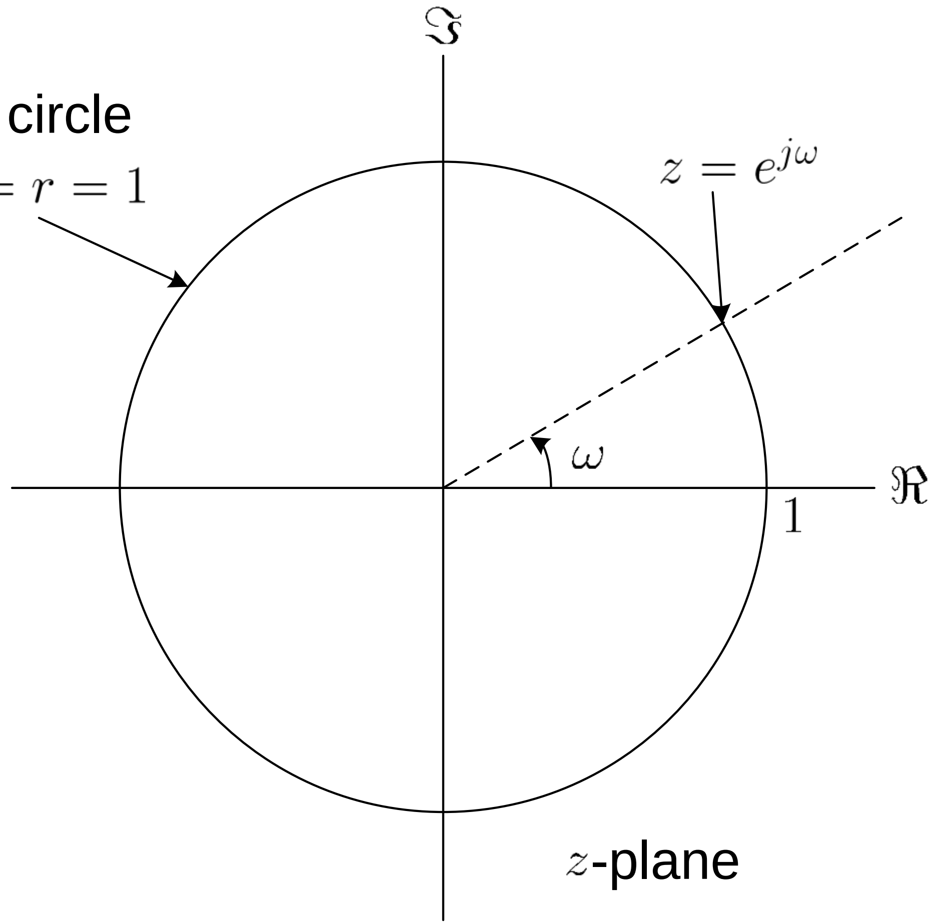


Fig.8.1: Relationship between $X(z)$ and $X(e^{j\omega})$ on the z-plane

Region of Convergence (ROC)

ROC indicates when z transform of a sequence converges.

Generally there exists some z such that

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \rightarrow \infty \quad (8.6)$$

where the z transform does not converge.

The set of values of z for which $X(z)$ converges or

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} \left| x[n] z^{-n} \right| < \infty \quad (8.7)$$

is called the ROC, which must be specified along with $X(z)$ in order for the z transform to be complete.

Note also that if

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \rightarrow \infty \quad (8.8)$$

then the DTFT does not exist. While the DTFT converges if

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} \left|x[n]e^{-j\omega n}\right| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad (8.9)$$

That is, it is possible that the DTFT of $x[n]$ does not exist.

Also, the z transform does not exist if there is no value of z satisfies (8.7).

Assuming that $x[n]$ is of infinite length, we decompose $X(z)$:

$$X(z) = X_-(z) + X_+(z) \tag{8.10}$$

where

$$X_-(z) = \sum_{n=-\infty}^{-1} x[n]z^{-n} = \sum_{m=1}^{\infty} x[-m]z^{m} \tag{8.11}$$

and

$$X_+(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \tag{8.12}$$

Let $f_n(z) = x[n]z^{-n}$, $X_+(z)$ is expanded as:

$$\begin{aligned} X_+(z) &= x[0]z^{-0} + x[1]z^{-1} + \cdots + x[n]z^{-n} + \cdots \\ &= f_0(z) + f_1(z) + \cdots + f_n(z) + \cdots \end{aligned} \tag{8.13}$$

According to the ratio test, convergence of $X_+(z)$ requires

$$\lim_{n\to\infty}\left|\frac{f_{n+1}(z)}{f_n(z)}\right| < 1 \tag{8.14}$$

Let $\lim_{n\to\infty}|x[n+1]/x[n]| = R_+ > 0$. $X_+(z)$ converges if

$$\lim_{n\to\infty}\left|\frac{x[n+1]z^{-n-1}}{x[n]z^{-n}}\right| = \lim_{n\to\infty}\left|\frac{x[n+1]}{x[n]}\right||z^{-1}| < 1$$
$$\Rightarrow |z| > \lim_{n\to\infty}\left|\frac{x[n+1]}{x[n]}\right| = R_+ \tag{8.15}$$

That is, the ROC for $X_+(z)$ is $|z| > R_+$.

Let $\lim_{m\to\infty} |x[-m]/x[-m-1]| = R_- > 0$. $X_-(z)$ converges if

$$\lim_{m\to\infty}\left|\frac{x[-m-1]z^{m+1}}{x[-m]z^m}\right| = \lim_{m\to\infty}\left|\frac{x[-m-1]}{x[-m]}\right||z| < 1$$

$$\Rightarrow |z| < \lim_{m\to\infty}\left|\frac{x[-m]}{x[-m-1]}\right| = R_- \quad (8.16)$$

As a result, the ROC for $X_-(z)$ is $|z| < R_-$.

Combining the results, the ROC for $X(z)$ is $R_+ < |z| < R_-$:

- ROC is a ring when $R_+ < R_-$
- No ROC if $R_- < R_+$ and $X(z)$ does not exist

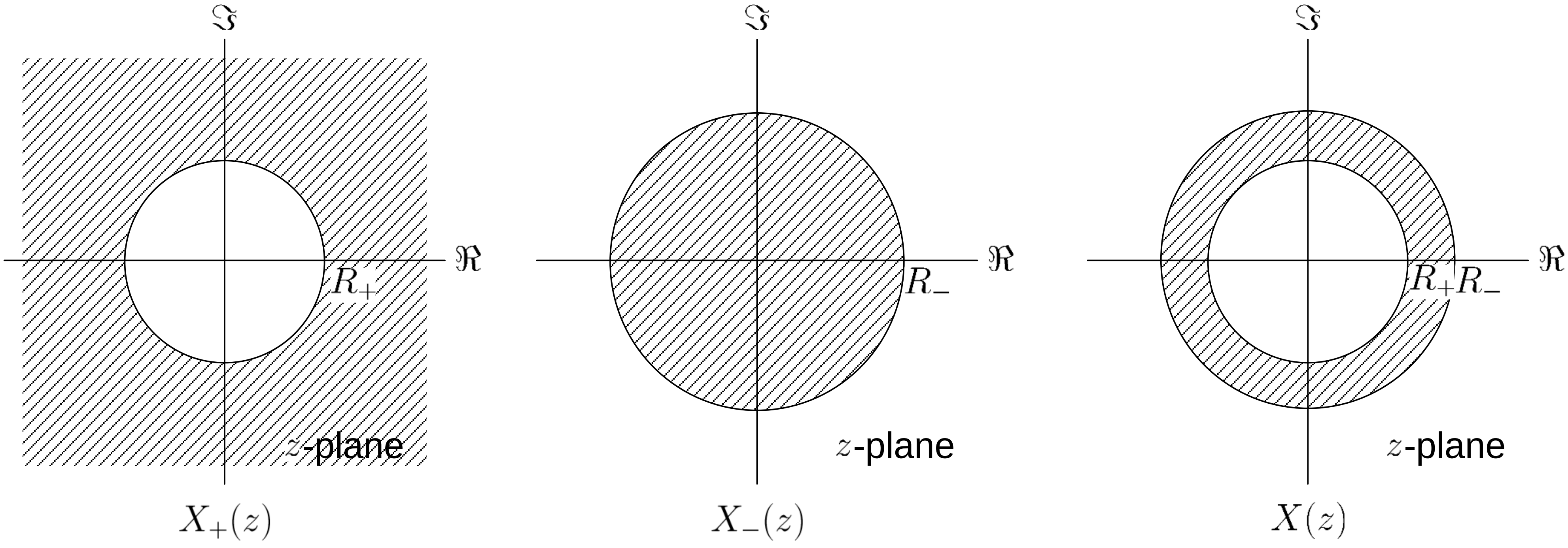


Fig.8.2: ROCs for $X_+(z)$, $X_-(z)$ and $X(z)$

Poles and Zeros

Values of z for which $X(z) = 0$ are the zeros of $X(z)$.

Values of z for which $X(z) = \infty$ are the poles of $X(z)$.

Example 8.1

In many real-world applications, $X(z)$ is represented as a rational function in z^{-1}:

$$X(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Discuss the poles and zeros of $X(z)$.

Multiplying both $P(z)$ and $Q(z)$ by z^{M+N} and then perform factorization yields:

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}} = \frac{z^N b_0 (z-d_1)(z-d_2)\cdots(z-d_M)}{z^M a_0 (z-c_1)(z-c_2)\cdots(z-c_N)}$$

We see that there are M nonzero zeros, namely, $d_1, d_2, \cdots, d_M$, and N nonzero poles, namely, $c_1, c_2, \cdots, c_N$.

If $M > N$, there are $(M-N)$ poles at zero location.

On the other hand, if $M < N$, there are $(N-M)$ zeros at zero location.

Note that we use a "$\circ$" to represent a zero and a "$\times$" to represent a pole on the z-plane.

Example 8.2

Determine the z transform of $x[n] = a^n u[n]$ where $u[n]$ is the unit step function. Then determine the condition when the DTFT of $x[n]$ exists.

Using (8.1) and (2.34), we have

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

According to (8.7), $X(z)$ converges if

$$\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty$$

Applying the ratio test, the convergence condition is

$$\left|az^{-1}\right| < 1 \Leftrightarrow |z| > |a|$$

which aligns with the ROC for $X_+(z)$ in (8.15).

Note that we cannot write $|z| > a$ because a may be complex.

For $|z| > |a|$, $X(z)$ is computed as

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1 - \left(az^{-1}\right)^{\infty}}{1 - az^{-1}} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

Together with the ROC, the z transform of $x[n] = a^n u[n]$ is:

$$X(z) = \frac{z}{z - a}, \quad |z| > |a|$$

It is clear that $X(z)$ has a zero at $z = 0$ and a pole at $z = a$. Using (8.5), we substitute $z = e^{j\omega}$ to obtain

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}, \quad |e^{j\omega}| = 1 > |a|$$

As a result, the existence condition for DTFT of $x[n]$ is $|a| < 1$.

Otherwise, its DTFT does not exist. In general, the DTFT $X(e^{j\omega})$ exists if its ROC includes the unit circle. If $|z|>|a|$ includes $|z|=1$, $|a|<1$ is required.

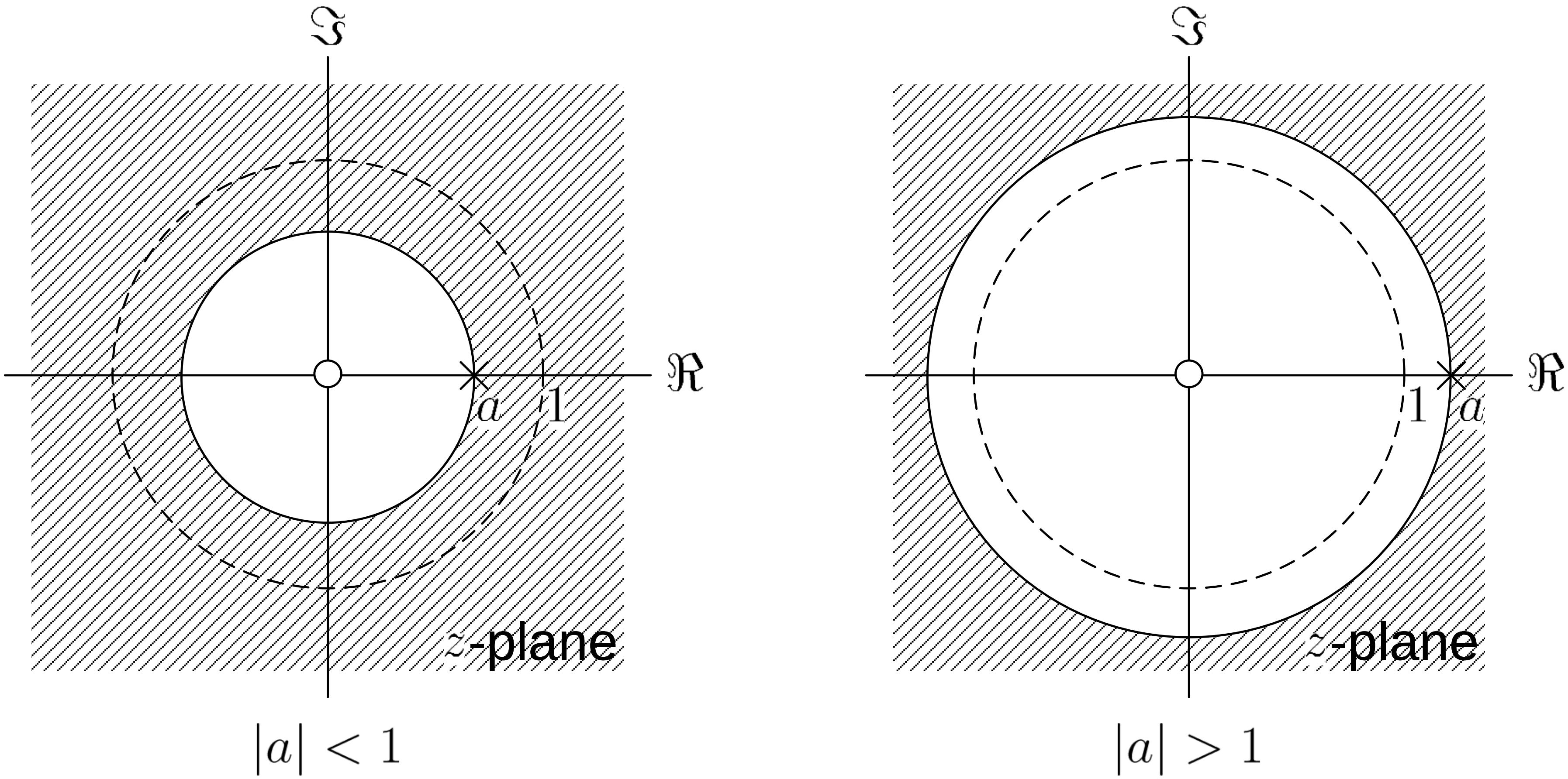


Fig.8.3: ROCs for $|a|<1$ and $|a|>1$ when $x[n]=a^n u[n]$

Example 8.3

Determine the z transform of $x[n]=-a^n u[-n-1]$. Then determine the condition when the DTFT of $x[n]$ exists.

Using (8.1) and (2.34), we have

$$X(z)=\sum_{n=-\infty}^{-1} -a^n z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^m = -\sum_{m=1}^{\infty} \left(a^{-1}z\right)^m$$

Similar to Example 8.2, $X(z)$ converges if $\left|a^{-1}z\right|<1$ or $|z|<|a|$, which aligns with the ROC for $X_-(z)$ in (8.16). This gives

$$X(z)=-\sum_{m=1}^{\infty}\left(a^{-1}z\right)^m = -\frac{a^{-1}z\left(1-\left(a^{-1}z\right)^{\infty}\right)}{1-a^{-1}z} = -\frac{a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a}$$

Together with ROC, the z transform of $x[n]=-a^n u[-n-1]$ is:

$$X(z)=\frac{z}{z-a}, \quad |z|<|a|$$

Using (8.5), we substitute $z = e^{j\omega}$ to obtain

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}, \quad |e^{j\omega}| = 1 < |a|$$

As a result, the existence condition for DTFT of $x[n]$ is $|a| > 1$.

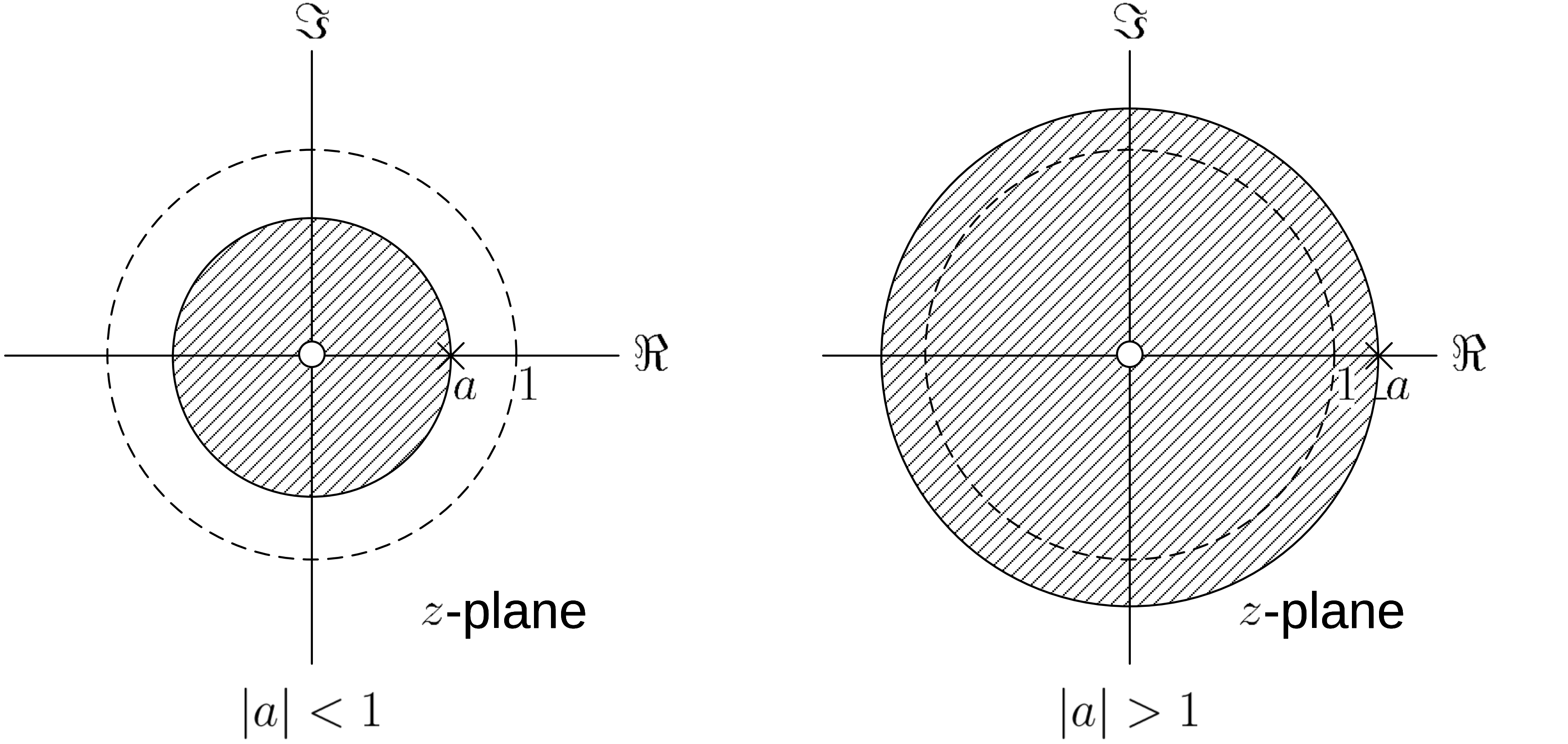


Fig.8.4: ROCs for $|a| < 1$ and $|a| > 1$ when $x[n] = -a^n u[-n-1]$

Example 8.4

Determine the z transform of $x[n] = a^n u[n] + b^n u[-n-1]$ where $|a| < |b|$.

Employing the results in Examples 8.2 and 8.3, we have

$$\begin{aligned} X(z) &= \frac{1}{1-az^{-1}} + \left(-\frac{1}{1-bz^{-1}}\right), \quad |z| > |a| \quad \text{and} \quad |z| < |b| \\ &= \frac{(a-b)z^{-1}}{(1-az^{-1})(1-bz^{-1})} \\ &= \frac{(a-b)z}{(z-a)(z-b)}, \quad |a| < |z| < |b| \end{aligned}$$

Note that its ROC agrees with Fig. 8.2.

What are the pole(s) and zero(s) of *X*(*z*)?

Example 8.5

Determine the z transform of $x[n] = \delta[n+1]$.

Using (8.1) and (2.33), we have

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n} = z$$

Example 8.6

Determine the z transform of $x[n]$ which has the form of:

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Using (8.1), we have

$$X(z) = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

What are the ROCs in Examples 8.5 and 8.6?

Finite-Duration and Infinite-Duration Sequences

Finite-duration sequence: values of $x[n]$ are nonzero only for a finite time interval.

Otherwise, $x[n]$ is called an infinite-duration sequence:

- Right-sided: if $x[n]=0$ for $n<N_+<\infty$ where N_+ is an integer (e.g., $x[n]=a^n u[n]$ with $N_+=0$; $x[n]=a^n u[n-10]$ with $N_+=10$; $x[n]=a^n u[n+10]$ with $N_+=-10$).
- Left-sided: if $x[n]=0$ for $n>N_->-\infty$ where N_- is an integer (e.g., $x[n]=-a^n u[-n-1]$ with $N_-=-1$).
- Two-sided: neither right-sided nor left-sided (e.g., Example 8.4).

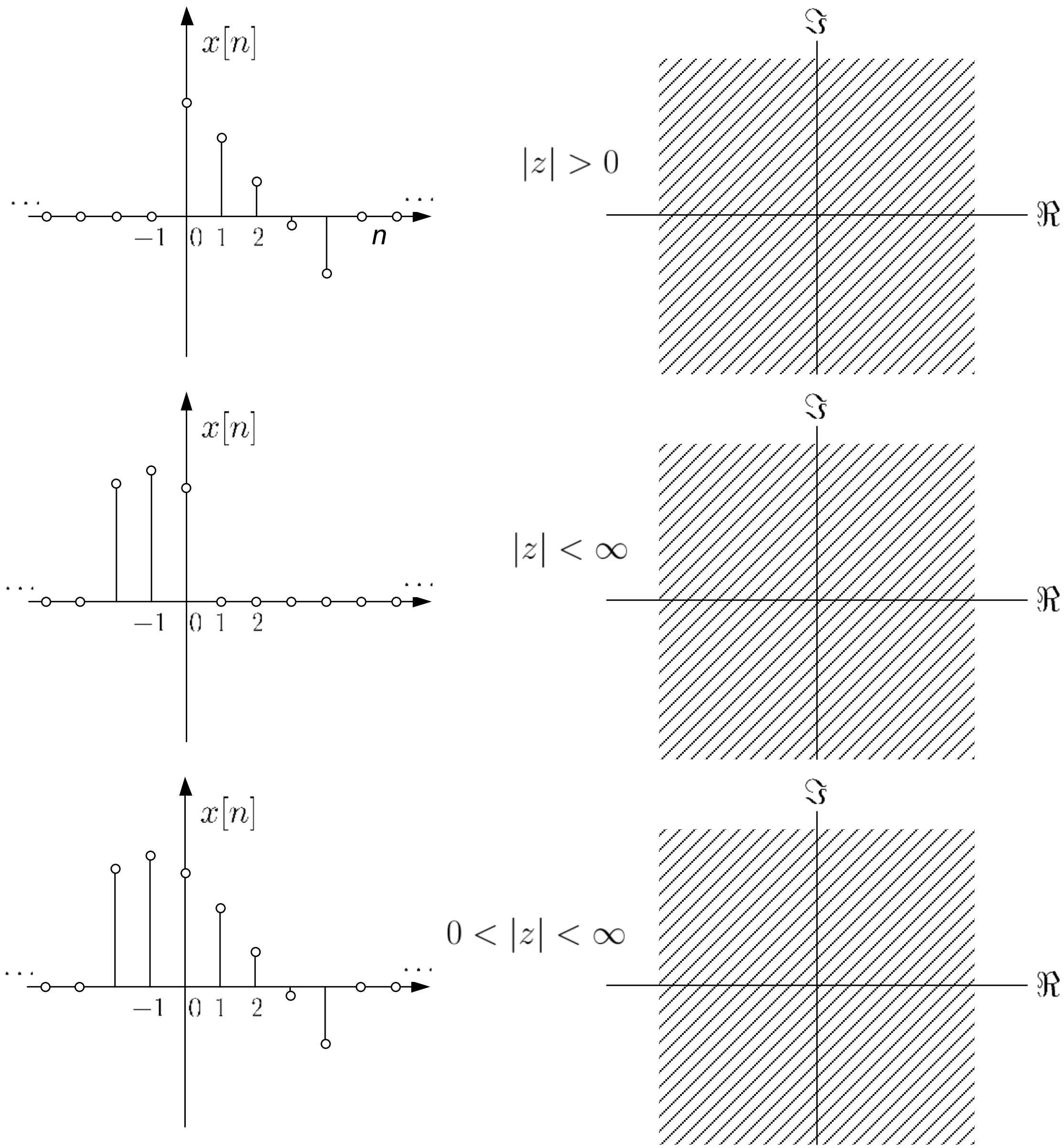


Fig.8.5: Finite-duration sequences

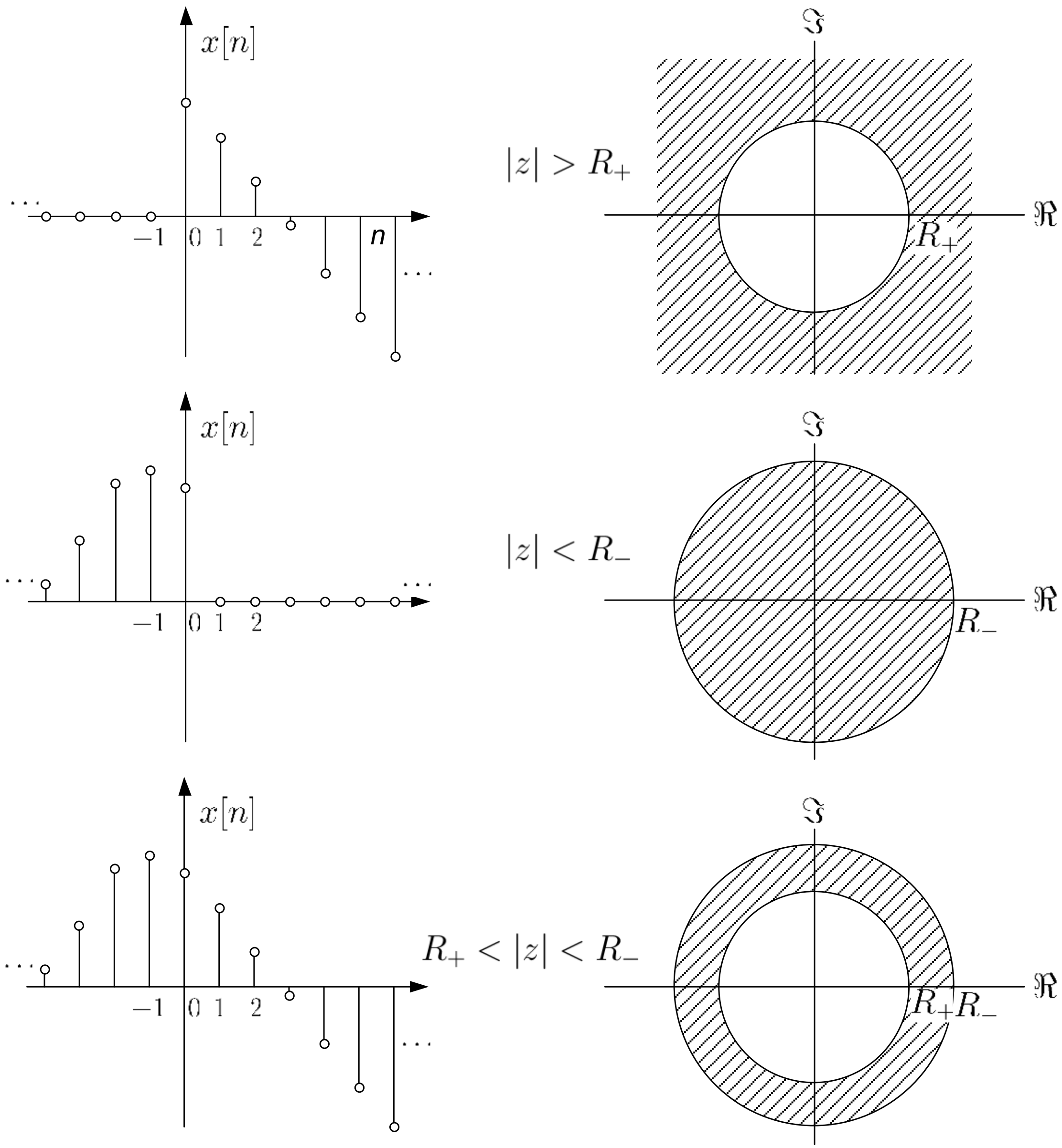


Fig.8.6: Infinite-duration sequences

Sequence	Transform	ROC
$\delta[n]$	1	All z
$\delta[n-m]$	z^{-m}	$\lvert z\rvert > 0$, $m>0$; $\lvert z\rvert < \infty$, $m<0$
$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$a^n \cos(bn) u[n]$	$\dfrac{1-a\cos(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$a^n \sin(bn) u[n]$	$\dfrac{a\sin(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Table 8.1: z transforms for common sequences

Summary of ROC Properties

P1. There are four possible shapes for ROC, namely, the entire region except possibly $z=0$ and/or $z=\infty$, a ring, or inside or outside a circle in the z-plane centered at the origin (e.g., Figures 8.6 and 8.7).

P2. The DTFT of a sequence $x[n]$ exists if and only if the ROC of the z transform of $x[n]$ includes the unit circle (e.g., Examples 8.2 and 8.3).

P3: The ROC cannot contain any poles (e.g., Examples 8.2 to 8.4).

P4: When $x[n]$ is a finite-duration sequence, the ROC is the entire z-plane except possibly $z=0$ and/or $z=\infty$ (e.g., Examples 8.5 and 8.6).

P5: When $x[n]$ is a right-sided sequence, the ROC is of the form $|z| > |p_{\max}|$ where $p_{\max}$ is the pole with the largest magnitude in $X(z)$ (e.g., Example 8.2).

P6: When $x[n]$ is a left-sided sequence, the ROC is of the form $|z| < |p_{\min}|$ where $p_{\min}$ is the pole with the smallest magnitude in $X(z)$ (e.g., Example 8.3).

P7: When $x[n]$ is a two-sided sequence, the ROC is of the form $|p_a| < |z| < |p_b|$ where p_a and p_b are two poles with the successive magnitudes in $X(z)$ such that $|p_a| < |p_b|$ (e.g., Example 8.4).

P8: The ROC must be a connected region.

<u>Example 8.7</u>

A z transform $X(z)$ contains three poles, namely, a, b and c with $|a| < |b| < |c|$. Determine all possible ROCs.

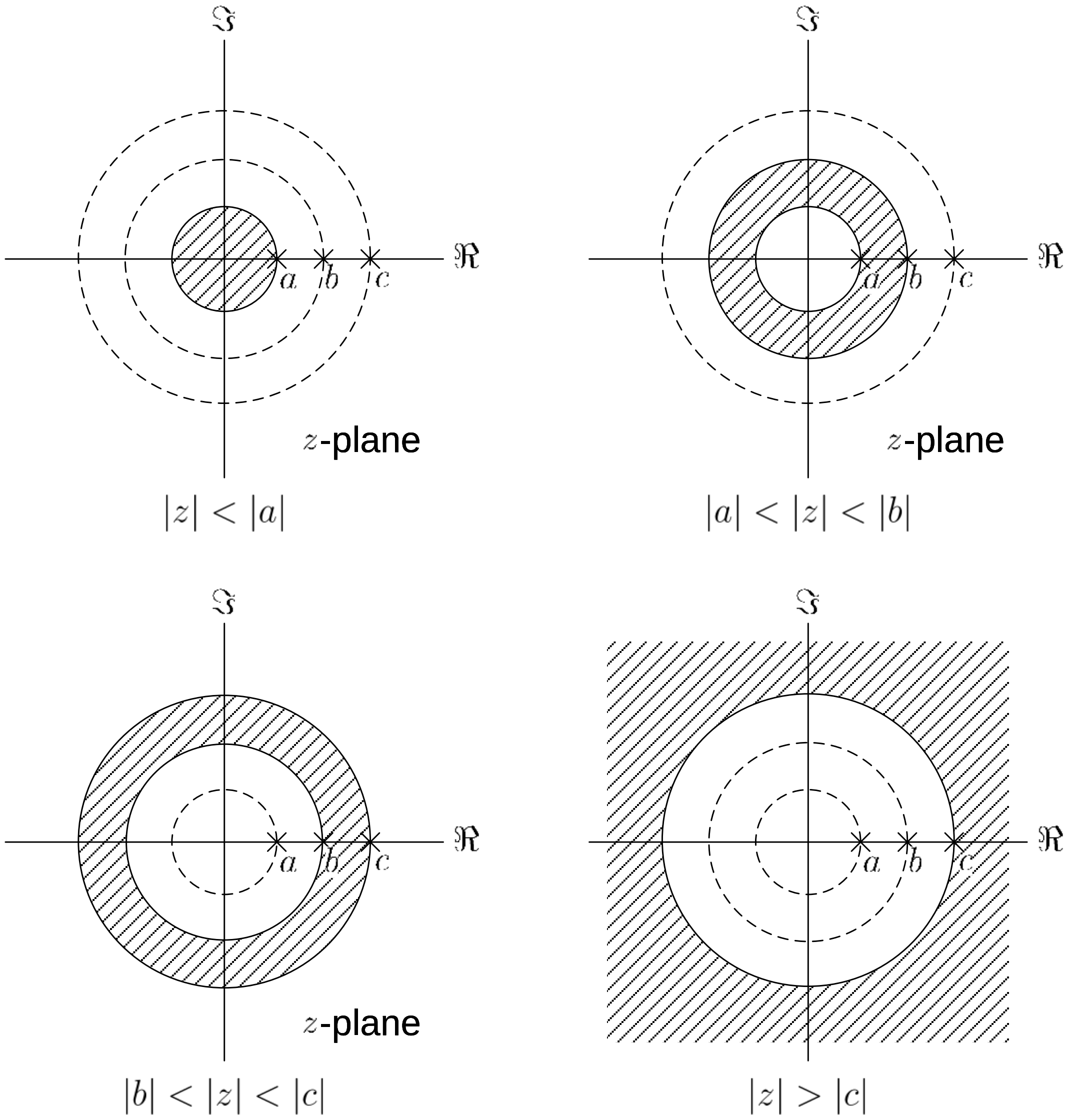


Fig.8.7: ROC possibilities for three poles

What are other possible ROCs?

Properties of z Transform

Linearity

Let $x_1[n] \leftrightarrow X_1(z)$ and $x_2[n] \leftrightarrow X_2(z)$ be two z transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$, respectively, we have

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z) \quad (8.17)$$

Its ROC is denoted by $\mathcal{R}$, which includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$ where $\cap$ is the intersection operator. That is, $\mathcal{R}$ contains at least the intersection of $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$.

Example 8.8

Determine the z transform of $y[n]$ which is expressed as:

$$y[n] = x_1[n] + x_2[n]$$

where $x_1[n] = (0.2)^n u[n]$ and $x_2[n] = (-0.3)^n u[n]$.

From Table 8.1, the z transforms of $x_1[n]$ and $x_2[n]$ are:

$$x_1[n] = (0.2)^n u[n] \leftrightarrow \frac{1}{1-0.2z^{-1}}, \quad |z| > 0.2$$

and

$$x_2[n] = (-0.3)^n u[n] \leftrightarrow \frac{1}{1+0.3z^{-1}}, \quad |z| > 0.3$$

According to the linearity property, the z transform of $y[n]$ is

$$Y(z) = \frac{1}{1-0.2z^{-1}} + \frac{1}{1+0.3z^{-1}}, \quad |z| > 0.3$$

Why the ROC is |z|>0.3 instead of |z|>0.2?

Example 8.9

Determine the ROC of the z transform of $x[n]$ which is expressed as:

$$x[n] = a^n u[n] - a^n u[n-1]$$

Noting that $a^n u[n] - a^n u[n-1] = \delta[n]$, we know that the ROC of $x[n]$ is the entire z-plane.

On the other hand, both ROCs of $a^n u[n]$ and $a^n u[n-1]$ are $|z| > |a|$. We see that the ROC of $x[n]$ contains the intersections of $a^n u[n]$ and $a^n u[n-1]$, which is $|z| > |a|$.

Time Shifting

A time-shift of n_0 in $x[n]$ causes a multiplication of z^{-n_0} in $X(z)$

$$x[n-n_0] \leftrightarrow z^{-n_0} X(z) \quad (8.18)$$

The ROC for $x[n-n_0]$ is basically identical to that of $X(z)$ except for the possible addition or deletion of $z = 0$ or $z = \infty$.

Example 8.10

Find the z transform of $x[n]$ which has the form of:

$$x[n] = a^{n-1}u[n-1]$$

Employing the time shifting property with $n_0 = 1$ and:

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

we easily obtain

$$a^{n-1}u[n-1] \leftrightarrow z^{-1} \cdot \frac{1}{1-az^{-1}} = \frac{z^{-1}}{1-az^{-1}}, \quad |z| > |a|$$

Note that using (8.1) with $|z| > |a|$ also produces the same result but this approach is less efficient:

$$X(z) = \sum_{n=1}^{\infty} a^{n-1} z^{-n} = a^{-1} \sum_{n=1}^{\infty} \left(az^{-1}\right)^n = a^{-1} \frac{az^{-1}\left[1-\left(az^{-1}\right)^{\infty}\right]}{1-az^{-1}} = \frac{z^{-1}}{1-az^{-1}}$$

Multiplication by an Exponential Sequence

If we multiply $x[n]$ by z_0^n in the time domain, the variable z will be changed to z/z_0 in the z transform domain. That is:

$$z_0^n x[n] \leftrightarrow X(z/z_0) \tag{8.19}$$

If the ROC for $x[n]$ is $R_+ < |z| < R_-$, then the ROC for $z_0^n x[n]$ is $|z_0|R_+ < |z| < |z_0|R_-$.

Example 8.11

With the use of the following z transform pair:

$$u[n] \leftrightarrow \frac{1}{1-z^{-1}}, \quad |z| > 1$$

Find the z transform of $x[n]$ which has the form of:

$$x[n] = a^n \cos(bn) u[n]$$

Noting that $\cos(bn) = (e^{jbn} + e^{-jbn})/2$, $x[n]$ can be written as:

$$x[n] = \frac{1}{2}\left(ae^{jb}\right)^n u[n] + \frac{1}{2}\left(ae^{-jb}\right)^n u[n]$$

By means of the property of (8.19) with the substitution of $z_0 = ae^{jb}$ and $z_0 = ae^{-jb}$, we obtain:

$$\frac{1}{2}\left(ae^{jb}\right)^n u[n] \leftrightarrow \frac{1}{2}\frac{1}{1-(z/(ae^{jb}))^{-1}} = \frac{1}{2}\frac{1}{1-ae^{jb}z^{-1}}, \quad |z| > |a|$$

and

$$\frac{1}{2}\left(ae^{-jb}\right)^n u[n] \leftrightarrow \frac{1}{2}\frac{1}{1-(z/(ae^{-jb}))^{-1}} = \frac{1}{2}\frac{1}{1-ae^{-jb}z^{-1}}, \quad |z| > |a|$$

By means of the linearity property, it follows that

$$X(z) = \frac{1}{2}\frac{1}{1-ae^{jb}z^{-1}} + \frac{1}{2}\frac{1}{1-ae^{-jb}z^{-1}} = \frac{1-a\cos(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}, |z| > |a|$$

which agrees with Table 8.1.

Differentiation

Differentiating $X(z)$ with respect to z corresponds to multiplying $x[n]$ by n in the time domain:

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz} \tag{8.20}$$

The ROC for $nx[n]$ is basically identical to that of $X(z)$ except for the possible addition or deletion of $z = 0$ or $z = \infty$.

Example 8.12

Determine the z transform of $x[n] = na^n u[n]$.

We have:

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

and

$$\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right)=\frac{d\left(1-az^{-1}\right)^{-1}}{d\left(1-az^{-1}\right)}\cdot\frac{d\left(1-az^{-1}\right)}{dz}=-\frac{az^{-2}}{\left(1-az^{-1}\right)^2}$$

By means of the differentiation property, we obtain:

$$na^n u[n] \leftrightarrow -z\cdot -\frac{az^{-2}}{\left(1-az^{-1}\right)^2}=\frac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad |z|>|a|$$

which agrees with Table 8.1.

Conjugation

The z transform pair for $x^*[n]$ is:

$$x^*[n] \leftrightarrow X^*(z^*) \tag{8.21}$$

The ROC for $x^*[n]$ is identical to that of $x[n]$.

Time Reversal

The z transform pair for $x[-n]$ is:

$$x[-n] \leftrightarrow X(z^{-1}) \tag{8.22}$$

If the ROC for $x[n]$ is $R_+ < |z| < R_-$, the ROC for $x[-n]$ is $1/R_- < |z| < 1/R_+$.

Example 8.13

Determine the z transform of $x[n] = -na^{-n}u[-n]$.

Using Example 8.12:

$$na^n u[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$$

and from the time reversal property:

$$X(z) = \frac{az}{(1-az)^2} = \frac{a^{-1}z^{-1}}{(1-a^{-1}z^{-1})^2}, \quad |z| < |a^{-1}|$$

Convolution

Let $x_1[n] \leftrightarrow X_1(z)$ and $x_2[n] \leftrightarrow X_2(z)$ be two z transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$, respectively. Then we have:

$$x_1[n] \otimes x_2[n] \leftrightarrow X_1(z)X_2(z) \tag{8.23}$$

and its ROC includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$.

The proof is given as follows.

Let

$$y[n] = x_1[n] \otimes x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \tag{8.24}$$

With the use of the time shifting property, $Y(z)$ is:

$$
\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \right] z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x_1[k] \left[\sum_{n=-\infty}^{\infty} x_2[n-k] z^{-n} \right] \\
&= \sum_{k=-\infty}^{\infty} x_1[k] X_2(z) z^{-k} \\
&= X_1(z) X_2(z)
\end{aligned}
\qquad (8.25)
$$

Causality and Stability Investigation with ROC

Suppose $h[n]$ is the impulse response of a discrete-time linear time-invariant (LTI) system. Recall (3.19), which is the causality condition:

$$h[n] = 0, \quad n < 0 \tag{8.26}$$

If the system is causal and $h[n]$ is of finite duration, the ROC should include ∞ (See Example 8.5 and Figure 8.5).

If the system is causal and $h[n]$ is of infinite duration, the ROC is of the form $|z| > |p_{\max}|$ and should include ∞ (See Example 8.2 and Figure 8.6). According to P5, $h[n]$ must be a right-sided sequence.

Example 8.14

Consider a LTI system with impulse response $h[n]$:

$$h[n] = a^{n+10}u[n+10]$$

Discuss the causality of the system.

According to (8.26), the system is not causal. Although it is a right-sided sequence, the ROC of $H(z)$ does not include ∞:

$$H(z) = \sum_{n=-\infty}^{\infty} a^{n+10}u[n+10]z^{-n} = a^{10}\left(\left(\frac{a}{z}\right)^{-10} + \left(\frac{a}{z}\right)^{-9} + \cdots\right)$$

where z cannot be equal to ∞ for convergence.

Applying the time shifting property, we get:

$$a^{n+10}u[n+10] \leftrightarrow z^{10} \cdot \frac{1}{1-az^{-1}} = \frac{z^{10}}{1-az^{-1}} = \frac{z^{11}}{z-a}, \quad |z| > |a|$$

The numerator has degree 11 while the denominator has degree 1, making the ROC cannot include ∞.

Generalizing the results, for a rational $H(z)$, it will be a causal system if its ROC has the form of $|z| > |p_{\max}|$ and the order of the numerator is not greater than that of the denominator.

Recall the stability condition in (3.21):

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \qquad (8.27)$$

Based on (8.9), this also means that the DTFT of $h[n]$ exists.

According to P2, (8.27) indicates that the ROC of $H(z)$ should include the unit circle.

<u>Example 8.15</u>

Consider a LTI system with impulse response $h[n]$:

$$h[n] = a^{n+10}u[n+10]$$

Discuss the stability of the system.

Using the result in Example 8.14, we have:

$$H(z) = \frac{z^{10}}{1 - az^{-1}}, \quad |z| > |a|$$

That is, if $|a| < 1$, then the system is stable. Otherwise, the system is not stable.

Inverse z Transform

Inverse z transform corresponds to finding $x[n]$ given $X(z)$ and its ROC.

The z transform and inverse z transform are one-to-one mapping provided that the ROC is given:

$$x[n] \leftrightarrow X(z) \tag{8.28}$$

There are 4 commonly used techniques to evaluate the inverse z transform. They are

1. Inspection
2. Partial Fraction Expansion
3. Power Series Expansion
4. Cauchy Integral Theorem

Inspection

When we are familiar with certain transform pairs, we can do the inverse z transform by inspection.

Example 8.16

Determine the inverse z transform of $X(z)$ which is expressed as:

$$X(z) = \frac{z}{2z-1}, \quad |z| > 0.5$$

We first rewrite $X(z)$ as:

$$X(z) = \frac{0.5}{1-0.5z^{-1}}$$

Making use of the following transform pair in Table 8.1:

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

and putting $a = 0.5$, we have:

$$\frac{0.5}{1-0.5z^{-1}} \leftrightarrow 0.5(0.5)^n u[n]$$

By inspection, the inverse z transform is:

$$x[n] = (0.5)^{n+1} u[n]$$

Partial Fraction Expansion

We consider that $X(z)$ is a rational function in z^{-1}:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad (8.29)$$

To obtain the partial fraction expansion from (8.29), the first step is to determine the N nonzero poles, $c_1, c_2, \cdots, c_N$.

There are 4 cases to be considered:

Case 1: $M < N$ and all poles are of first order

For first-order poles, all $\{c_k\}$ are distinct. $X(z)$ is:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - c_k z^{-1}} \tag{8.30}$$

For each first-order term of $A_k / \left(1 - c_k z^{-1}\right)$, its inverse z transform can be easily obtained by inspection.

Multiplying both sides by $\left(1 - c_k z^{-1}\right)$ and evaluating for $z = c_k$

$$A_k = \left(1 - c_k z^{-1}\right) X(z)\Big|_{z=c_k} \tag{8.31}$$

An illustration for computing A_1 with $N = 2 > M$ is:

$$X(z) = \frac{A_1}{1 - c_1 z^{-1}} + \frac{A_2}{1 - c_2 z^{-1}}$$
$$\Rightarrow \left(1 - c_1 z^{-1}\right) X(z) = A_1 + \frac{A_2 \left(1 - c_1 z^{-1}\right)}{1 - c_2 z^{-1}} \tag{8.32}$$

Substituting $z = c_1$, we get A_1.

In summary, three steps are:

- Find poles.
- Find $\{A_k\}$.
- Perform inverse z transform for the fractions by inspection.

Example 8.17

Find the pole and zero locations of $H(z)$:

$$H(z) = -\frac{1 + 0.1z^{-1}}{1 - 2.05z^{-1} + z^{-2}}$$

Then determine the inverse z transform of $H(z)$.

We first multiply z^2 to both numerator and denominator polynomials to obtain:

$$H(z) = -\frac{z\left(z + 0.1\right)}{z^2 - 2.05z + 1}$$

Apparently, there are two zeros at $z = 0$ and $z = -0.1$. On the other hand, by solving the quadratic equation at the denominator polynomial, the poles are determined as $z = 0.8$ and $z = 1.25$.

According to (8.30), we have:

$$H(z) = \frac{A_1}{1-0.8z^{-1}} + \frac{A_2}{1-1.25z^{-1}}$$

Employing (8.31), A_1 is calculated as:

$$A_1 = \left(1-0.8z^{-1}\right) H(z)\Big|_{z=0.8} = -\left.\frac{1+0.1z^{-1}}{1-1.25z^{-1}}\right|_{z=0.8} = 2$$

Similarly, A_2 is found to be -3. As a result, the partial fraction expansion for $H(z)$ is

$$H(z) = \frac{2}{1-0.8z^{-1}} - \frac{3}{1-1.25z^{-1}}$$

As the ROC is not specified, we investigate all possible scenarios, namely, $|z| > 1.25$, $0.8 < |z| < 1.25$, and $|z| < 0.8$.

For $|z| > 1.25$, we notice that

$$(0.8)^n u[n] \leftrightarrow \frac{1}{1-0.8z^{-1}}, \quad |z| > 0.8$$

and

$$(1.25)^n u[n] \leftrightarrow \frac{1}{1-1.25z^{-1}}, \quad |z| > 1.25$$

where both ROCs agree with $|z| > 1.25$. Combining the results, the inverse z transform $h[n]$ is:

$$h[n] = \left(2(0.8)^n - 3(1.25)^n\right) u[n]$$

which is a right-sided sequence and aligns with P5.

For $0.8 < |z| < 1.25$, we make use of

$$(0.8)^n u[n] \leftrightarrow \frac{1}{1-0.8z^{-1}}, \quad |z| > 0.8$$

and

$$-(1.25)^n u[-n-1] \leftrightarrow \frac{1}{1-1.25z^{-1}}, \quad |z| < 1.25$$

where both ROCs agree with $0.8 < |z| < 1.25$. This implies:

$$h[n] = 2(0.8)^n u[n] + 3(1.25)^n u[-n-1]$$

which is a two-sided sequence and aligns with P7.

Finally, for $|z| < 0.8$:

$$-(0.8)^n u[-n-1] \leftrightarrow \frac{1}{1-0.8z^{-1}}, \quad |z| < 0.8$$

and

$$-(1.25)^n u[-n-1] \leftrightarrow \frac{1}{1-1.25z^{-1}}, \quad |z| < 1.25$$

where both ROCs agree with $|z| < 0.8$. As a result, we have:

$$h[n] = \left(-2(0.8)^n + 3(1.25)^n\right) u[-n-1]$$

which is a left-sided sequence and aligns with P6.

Suppose $h[n]$ is the impulse response of a discrete-time LTI system.

In terms of causality and stability, there are three possible cases:

- $h[n] = (2(0.8)^n - (1.25)^n)\, u[n]$ is the impulse response of a causal but unstable system (ROC: $|z| > 1.25$).

- $h[n] = 2(0.8)^n u[n] + (1.25)^n u[-n-1]$ corresponds to a non-causal but stable system (ROC: $0.8 < |z| < 1.25$).

- $h[n] = (-2(0.8)^n + (1.25)^n)\, u[-n-1]$ is non-causal and unstable (ROC: $|z| < 0.8$).

Case 2: $M \geq N$ and all poles are of first order

In this case, $X(z)$ can be expressed as:

$$X(z) = \sum_{l=0}^{M-N} B_l z^{-l} + \sum_{k=1}^{N} \frac{A_k}{1 - c_k z^{-1}} \qquad (8.33)$$

- B_l are obtained by long division of the numerator by the denominator, with the division process terminating when the remainder is of lower degree than the denominator.
- A_k can be obtained using (8.31).

Example 8.18

Determine $x[n]$ which has z transform of the form:

$$X(z) = \frac{4 - 2z^{-1} + z^{-2}}{1 - 1.5z^{-1} + 0.5z^{-2}}, \quad |z| > 1$$

The poles are easily determined as $z = 0.5$ and $z = 1$

According to (8.33) with $M = N = 2$:

$$X(z) = B_0 + \frac{A_1}{1 - 0.5z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

The value of B_0 is found by dividing the numerator polynomial by the denominator polynomial as follows:

$$\begin{array}{r}
2 \phantom{-2z^{-1}+4} \\
0.5z^{-2} - 1.5z^{-1} + 1 \overline{\big) z^{-2} - 2z^{-1} + 4} \\
\underline{z^{-2} - 3z^{-1} + 2} \\
z^{-1} + 2
\end{array}$$

That is, $B_0 = 2$. Thus $X(z)$ is expressed as

$$X(z) = 2 + \frac{2 + z^{-1}}{(1 - 0.5z^{-1})(1 - z^{-1})} = 2 + \frac{A_1}{1 - 0.5z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

According to (8.31), A_1 and A_2 are calculated as

$$A_1 = \left. \frac{4 - 2z^{-1} + z^{-2}}{1 - z^{-1}} \right|_{z=0.5} = -4$$

and

$$A_2 = \left. \frac{4 - 2z^{-1} + z^{-2}}{1 - 0.5z^{-1}} \right|_{z=1} = 6$$

With $|z| > 1$:

$$\delta[n] \leftrightarrow 1$$

$$(0.5)^n u[n] \leftrightarrow \frac{1}{1 - 0.5z^{-1}}, \quad |z| > 0.5$$

and

$$u[n] \leftrightarrow \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

the inverse z transform $x[n]$ is:

$$x[n] = 2\delta[n] - 4(0.5)^n u[n] + 6u[n]$$

Case 3: $M < N$ with multiple-order pole(s)

If $X(z)$ has a s-order pole at $z = c_i$ with $s \geq 2$, this means that there are s repeated poles with the same value of c_i. $X(z)$ is:

$$X(z) = \sum_{k=1, k \neq i}^{N} \frac{A_k}{1 - c_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1 - c_i z^{-1}\right)^m} \qquad (8.34)$$

- When there are two or more multiple-order poles, we include a component like the second term for each corresponding pole
- A_k can be computed according to (8.31)
- C_m can be calculated from:

$$C_m = \frac{1}{(s-m)!(-c_i)^{s-m}} \cdot \frac{d^{s-m}}{dw^{s-m}} \left[(1 - c_i w)^s X(w^{-1}) \right] \Big|_{w = c_i^{-1}} \qquad (8.35)$$

Example 8.19

Determine the partial fraction expansion for $X(z)$:

$$X(z)=\frac{4}{\left(1+z^{-1}\right)\left(1-z^{-1}\right)^2}$$

It is clear that $X(z)$ corresponds to Case 3 with $N=3>M$ and one second-order pole at $z=1$. Hence $X(z)$ is:

$$X(z)=\frac{A_1}{1+z^{-1}}+\frac{C_1}{1-z^{-1}}+\frac{C_2}{\left(1-z^{-1}\right)^2}$$

Employing (8.31), A_1 is:

$$A_1=\left.\frac{4}{\left(1-z^{-1}\right)^2}\right|_{z=-1}=1$$

Applying (8.35), C_1 is:

$$\begin{aligned} C_1 &= \frac{1}{(2-1)!(-1)^{2-1}} \cdot \frac{d}{dw}\left[(1-1\cdot w)^2 \frac{4}{(1+w)(1-w)^2}\right]\Bigg|_{w=1} \\ &= -\frac{d}{dw}\frac{4}{1+w}\bigg|_{w=1} \\ &= \frac{4}{(1+w)^2}\bigg|_{w=1} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} C_2 &= \frac{1}{(2-2)!(-1)^{2-2}} \cdot \left[(1-1\cdot w)^2 \frac{4}{(1+w)(1-w)^2}\right]\Bigg|_{w=1} \\ &= \frac{4}{1+w}\bigg|_{w=1} \\ &= 2 \end{aligned}$$

Therefore, the partial fraction expansion for $X(z)$ is

$$X(z) = \frac{1}{1+z^{-1}} + \frac{1}{1-z^{-1}} + \frac{2}{\left(1-z^{-1}\right)^2}$$

Case 4: $M \geq N$ with multiple-order pole(s)

This is the most general case and the partial fraction expansion of $X(z)$ is

$$X(z) = \sum_{l=0}^{M-N} B_l z^{-l} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1-c_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1-c_i z^{-1}\right)^m} \quad (8.36)$$

assuming that there is only one multiple-order pole of order $s \geq 2$ at $z = c_i$. It is easily extended to the scenarios when there are two or more multiple-order poles as in Case 3. The A_k , B_l and C_m can be calculated as in Cases 1, 2 and 3.

Power Series Expansion

When $X(z)$ is expanded as power series according to (5.1):

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \cdots + x[-1]z^{1} + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \quad (8.37)$$

any particular value of $x[n]$ can be determined by finding the coefficient of the appropriate power of z^{-1}

Example 8.20

Determine $x[n]$ which has z transform of the form:

$$X(z) = 2z^2\left(1 - 0.5z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right), \quad 0 < |z| < \infty$$

Expanding $X(z)$ yields

$$X(z) = 2z^2 - z - 2 + z^{-1}$$

From (8.37), $x[n]$ is deduced as:

$$x[n] = 2\delta[n+2] - \delta[n+1] - 2\delta[n] + \delta[n-1]$$

Example 8.21

Determine $x[n]$ whose z transform has the form of:

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

With the use of

$$\frac{1}{1-\lambda} = 1 + \lambda + \lambda^2 + \cdots, \quad |\lambda| < 1$$

Carrying out long division in $X(z)$ with $|az^{-1}| < 1$:

$$X(z) = 1 + az^{-1} + \left(az^{-1}\right)^2 + \cdots$$

From (8.37), $x[n]$ is deduced as:

$$x[n] = a^n u[n]$$

which agrees with Example 8.2 and Table 8.1.

Example 8.22

Determine $x[n]$ whose z transform has the form of:

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

We first express $X(z)$ as:

$$X(z) = \frac{-a^{-1}z}{-a^{-1}z} \cdot \frac{1}{1 - az^{-1}} = \frac{-a^{-1}z}{1 - a^{-1}z}$$

Carrying out long division in $X(z)$ with $|a^{-1}z| < 1$:

$$X(z) = -a^{-1}z\left(1 + a^{-1}z + \left(a^{-1}z\right)^2 + \cdots\right)$$

From (8.37), $x[n]$ is deduced as:

$$x[n] = -a^n u[-n-1]$$

which agrees with Example 8.3 and Table 8.1.

Transfer Function of Linear Time-Invariant System

A LTI system can be characterized by the transfer function, which is a z transform expression

Starting with:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad (8.38)$$

Applying z transform on (8.38) with the use of the linearity and time shifting properties, we have:

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k} \quad (8.39)$$

The transfer function, denoted by $H(z)$, is defined as:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{8.40}$$

The system impulse response $h[n]$ is given by the inverse z transform of $H(z)$ with an appropriate ROC, that is, $h[n] \leftrightarrow H(z)$, such that $y[n]=x[n]\otimes h[n]$. This suggests that we can first take the z transforms for $x[n]$ and $h[n]$, then multiply $X(z)$ by $H(z)$, and finally perform the inverse z transform of $X(z)H(z)$.

Comparing with (6.25), we see that the system frequency response can be obtained as $H(z)|_{z=e^{j\omega}}=H(e^{j\omega})$ if it exists.

Example 8.23

Determine the transfer function for a LTI system whose input $x[n]$ and output $y[n]$ are related by:

$$y[n] = 0.1y[n-1] + x[n] + x[n-1]$$

Applying z transform on the difference equation with the use of the linearity and time shifting properties, $H(z)$ is:

$$Y(z)\left(1-0.1z^{-1}\right) = X(z)\left(1+z^{-1}\right) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-0.1z^{-1}}$$

Note that there are two ROC possibilities, namely, $|z| > 0.1$ and $|z| < 0.1$, and we cannot uniquely determine $h[n]$. However, if it is known that the system is causal, $h[n]$ can be uniquely found because the ROC should be $|z| > 0.1$.

Example 8.24

Find the difference equation of a LTI system whose transfer function is given by

$$H(z)=\frac{\left(1+z^{-1}\right)\left(1-2z^{-1}\right)}{\left(1-0.5z^{-1}\right)\left(1+2z^{-1}\right)}$$

Let $H(z)=Y(z)/X(z)$. Performing cross-multiplication and inverse z transform, we obtain:

$$\left(1-0.5z^{-1}\right)\left(1+2z^{-1}\right)Y(z)=\left(1+z^{-1}\right)\left(1-2z^{-1}\right)X(z)$$
$$\Rightarrow \left(1+1.5z^{-1}-z^{-2}\right)Y(z)=\left(1-z^{-1}-2z^{-2}\right)X(z)$$
$$\Rightarrow y[n]+1.5y[n-1]-y[n-2]=x[n]-x[n-1]-2x[n-2]$$

Examples 8.23 and 8.24 imply the equivalence between the difference equation and transfer function.

Example 8.25

Compute the impulse response $h[n]$ for a LTI system which is characterized by the following difference equation:

$$y[n] = x[n] - x[n-1]$$

Applying z transform on the difference equation with the use of the linearity and time shifting properties, $H(z)$ is:

$$Y(z) = X(z)\left(1 - z^{-1}\right) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1}$$

There is only one ROC possibility, namely, $|z| > 0$. Taking the inverse z transform on $H(z)$, we get:

$$h[n] = \delta[n] - \delta[n-1]$$

which agrees with Example 3.18.

Example 8.26

Determine the output $y[n]$ if the input is $x[n] = u[n]$ and the LTI system impulse response is $h[n] = \delta[n] + 0.5\delta[n-1]$

The z transforms for $x[n]$ and $h[n]$ are

$$X(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

and

$$H(z) = 1 + 0.5z^{-1} \quad |z| > 0$$

As a result, we have:

$$Y(z) = X(z)H(z) = \frac{1}{1-z^{-1}} + 0.5\frac{z^{-1}}{1-z^{-1}}, \quad |z| > 1$$

Taking the inverse z transform of $Y(z)$ with the use of the time shifting property yields:

$$y[n] = u[n] + 0.5u[n-1]$$

which agrees with Example 3.13.

Laplace Transform

Chapter Intended Learning Outcomes:

(i) Represent continuous-time signals using Laplace transform

(ii) Understand the relationship between Laplace transform and Fourier transform

(iii) Understand the properties of Laplace transform

(iv) Perform operations on Laplace transform and inverse Laplace transform

(v) Apply Laplace transform for analyzing linear time-invariant systems

Analog Signal Representation with Laplace Transform

Apart from Fourier transform, we can also use Laplace transform to represent continuous-time signals.

The Laplace transform of $x(t)$, denoted by $X(s)$, is defined as:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad (9.1)$$

where s is a continuous complex variable.

We can also express s as:

$$s = \sigma + j\Omega \qquad (9.2)$$

where σ and Ω are the real and imaginary parts of s, respectively.

Employing (9.2), the Laplace transform can be written as:

$$X(\sigma + j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\Omega)t}dt = \int_{-\infty}^{\infty} \left(x(t)e^{-\sigma t}\right) e^{-j\Omega t}dt \quad (9.3)$$

Comparing (9.3) and the Fourier transform formula in (5.1):

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \quad (9.4)$$

Laplace transform of $x(t)$ is equal to the Fourier transform of $x(t)e^{-\sigma t}$.

When $\sigma = 0$ or $s = j\Omega$, (9.3) and (9.4) are identical:

$$X(s)|_{s=j\Omega} = X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \quad (9.5)$$

That is, Laplace transform generalizes Fourier transform, as z transform generalizes the discrete-time Fourier transform.

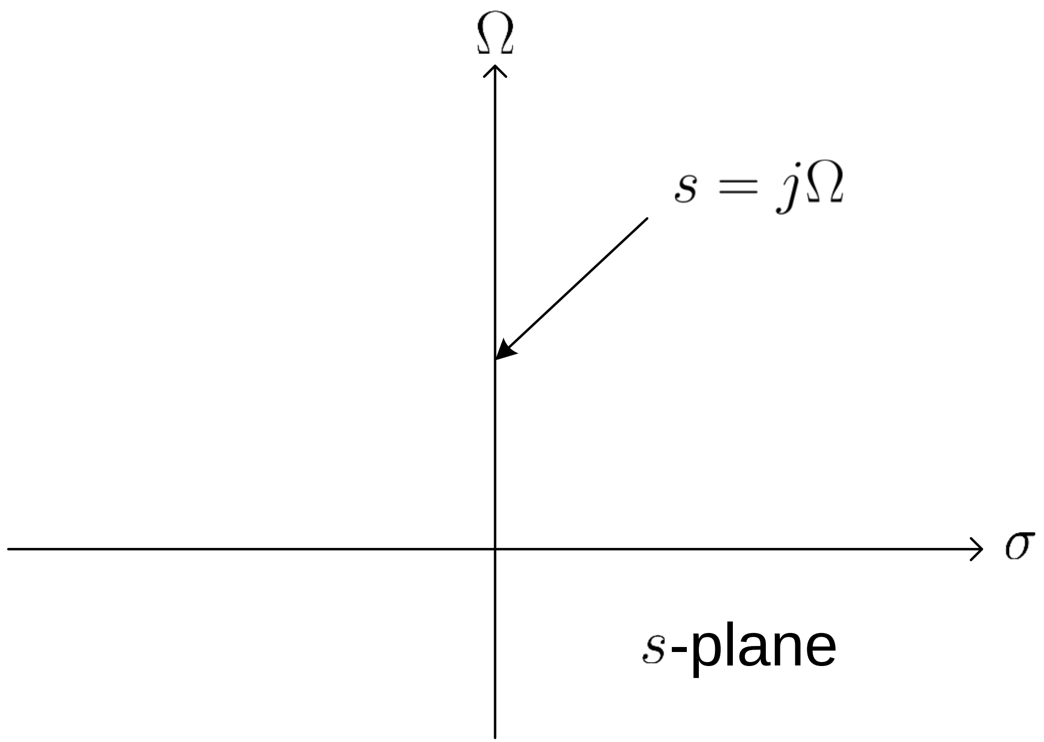


Fig.9.1: Relationship between $X(s)$ and $X(j\Omega)$ on the s-plane

Region of Convergence (ROC)

As in z transform of discrete-time signals, ROC indicates when Laplace transform of $x(t)$ converges.

That is, if

$$|X(s)| = \left| \int_{-\infty}^{\infty} x(t) e^{-st} dt \right| \rightarrow \infty \quad (9.6)$$

then the Laplace transform does not converge at point s.

Employing $s = \sigma + j\Omega$ and $|e^{j\Omega t}| = 1$, Laplace transform exists if

$$|X(\sigma + j\Omega)| \leq \int_{-\infty}^{\infty} \left| x(t) e^{-(\sigma + j\Omega)t} \right| dt = \int_{-\infty}^{\infty} \left| x(t) e^{-\sigma t} \right| dt < \infty \quad (9.7)$$

The set of values of σ which satisfies (9.7) is called the ROC, which must be specified along with $X(s)$ in order for the Laplace transform to be complete.

Note also that if

$$|X(j\Omega)| = \left| \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \right| \to \infty \quad (9.8)$$

then the Fourier transform does not exist. While it exists if

$$|X(j\Omega)| \leq \int_{-\infty}^{\infty} \left| x(t) e^{-j\Omega t} \right| dt = \int_{-\infty}^{\infty} |x(t)| \, dt < \infty \quad (9.9)$$

Hence it is possible that the Fourier transform of $x(t)$ does not exist.

Also, the Laplace transform does not exist if there is no value of σ satisfies (9.7).

Poles and Zeros

Values of s for which $X(s)=0$ are the zeros of $X(s)$.

Values of s for which $X(s)=\infty$ are the poles of $X(s)$.

Example 9.1

In many real-world applications, $X(s)$ is represented as a rational function in s:

$$X(s)=\frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Discuss the poles and zeros of $X(s)$.

Performing factorization on $X(s)$ yields:

$$X(s)=\frac{\displaystyle\sum_{k=0}^{M} b_k s^k}{\displaystyle\sum_{k=0}^{N} a_k s^k}=\frac{b_M(s-d_1)(s-d_2)\cdots(s-d_M)}{a_N(s-c_1)(s-c_2)\cdots(s-c_N)}$$

We see that there are M nonzero zeros, namely, $d_1, d_2, \cdots, d_M$, and N nonzero poles, namely, $c_1, c_2, \cdots, c_N$.

As in z transform, we use a "$\circ$" to represent a zero and a "$\times$" to represent a pole on the s-plane.

Example 9.2

Determine the Laplace transform of $x(t)=e^{-at}u(t)$ where $u(t)$ is the unit step function and a is a real number. Determine the condition when the Fourier transform of $x(t)$ exists.

Using (9.1) and (2.22), we have

$$X(s)=\int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt=\int_{0}^{\infty} e^{-(s+a)t}dt$$

Employing $s=\sigma+j\Omega$ yields

$$X(\sigma+j\Omega)=\int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\Omega t}dt=-\frac{1}{\sigma+a+j\Omega}e^{-(\sigma+a+j\Omega)t}\bigg|_{0}^{\infty}$$

It converges if $e^{-(\sigma+a)t}$ is bounded at $t\rightarrow\infty$, indicating that the ROC is

$$\sigma+a>0 \text{ or } \Re\{s\}=\sigma>-a$$

For $\sigma + a > 0$, $X(s)$ is computed as

$$X(s) = -\frac{1}{\sigma + a + j\Omega} e^{-(\sigma+a+j\Omega)t} \Big|_0^\infty = \frac{1}{(\sigma + a) + j\Omega} = \frac{1}{s + a}$$

With the ROC, the Laplace transform of $x(t) = e^{-at}u(t)$ is:

$$X(s) = \frac{1}{s + a}, \quad \Re\{s\} > -a$$

It is clear that $X(s)$ does not have zero but has a pole at $s = -a$. Using (9.5), we substitute $s = j\Omega$ to obtain

$$X(j\Omega) = \frac{1}{j\Omega + a}, \quad \Re\{s\} = 0 > -a$$

As a result, the existence condition for Fourier transform of $x(t)$ is $a > 0$. Otherwise, the Fourier transform does not exist.

In general, $X(j\Omega)$ exists if its ROC includes the imaginary axis. If $\Re\{s\} > -a$ includes $j\Omega$ axis, $a > 0$ is required.

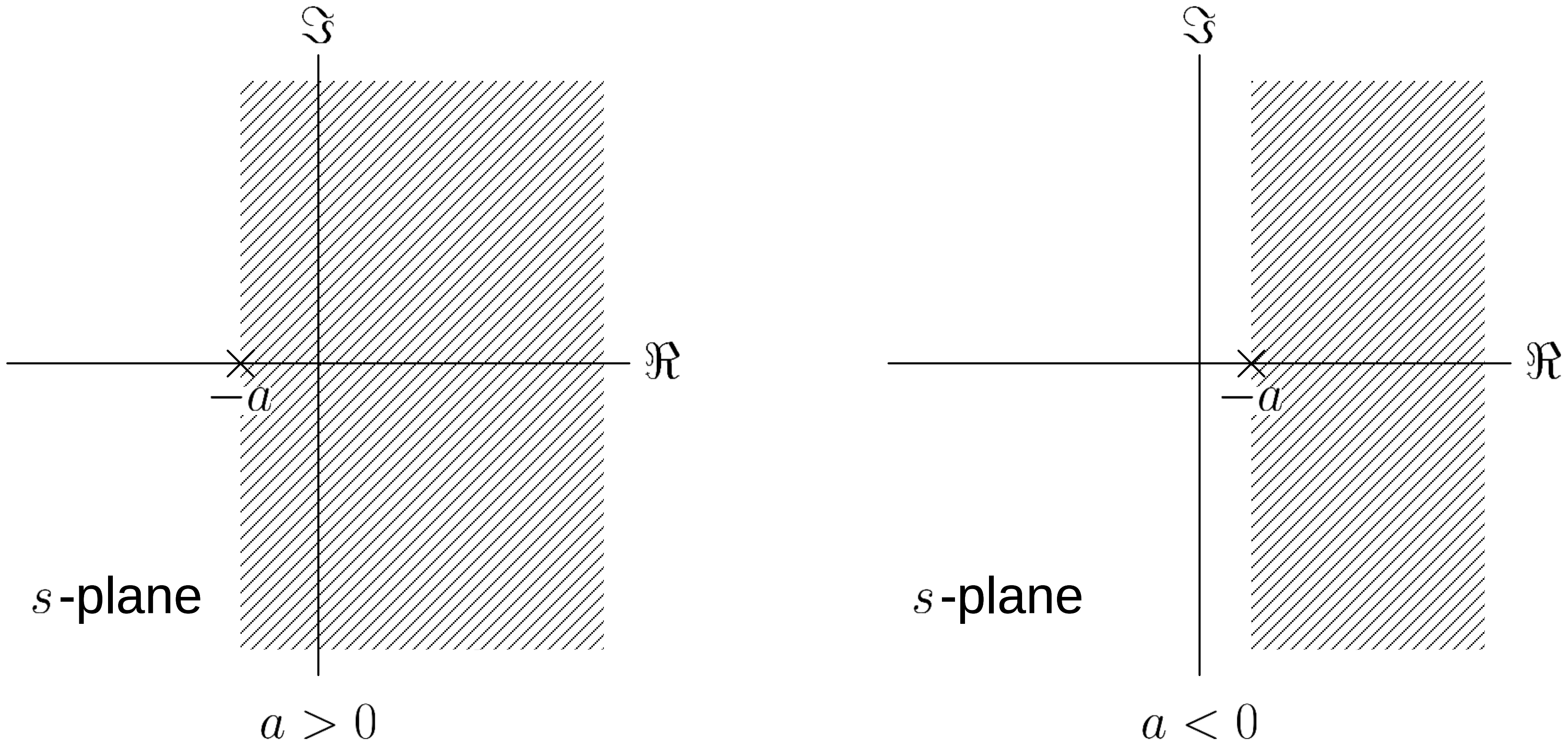


Fig.9.2: ROCs for $a > 0$ and $a < 0$ when $x(t) = e^{-at}u(t)$

Example 9.3

Determine the Laplace transform of $x(t) = -e^{-at}u(-t)$ where a is a real number. Then determine the condition when the Fourier transform of $x(t)$ exists.

Using (9.1) and (2.22), we have

$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt = -\int_{-\infty}^{0} e^{-(s+a)t}dt$$

Employing $s = \sigma + j\Omega$ yields

$$X(\sigma + j\Omega) = -\int_{-\infty}^{0} e^{-(\sigma+a)t}e^{-j\Omega t}dt = \frac{1}{\sigma + a + j\Omega} e^{-(\sigma+a+j\Omega)t}\bigg|_{-\infty}^{0}$$

It converges if $e^{-(\sigma+a)t}$ is bounded at $t \to -\infty$, indicating that:

$$\sigma + a < 0 \text{ or } \Re\{s\} = \sigma < -a$$

For $\sigma + a < 0$, $X(s)$ is computed as

$$X(s) = \frac{1}{\sigma + a + j\Omega} e^{-(\sigma + a + j\Omega)t}\Big|_{-\infty}^{0} = \frac{1}{(\sigma + a) + j\Omega} = \frac{1}{s + a}$$

With the ROC, the Laplace transform of $x(t) = -e^{-at}u(-t)$ is:

$$X(s) = \frac{1}{s + a}, \quad \Re\{s\} < -a$$

It is clear that $X(s)$ does not have zero but has a pole at $s = -a$. Using (9.5), we substitute $s = j\Omega$ to obtain

$$X(j\Omega) = \frac{1}{j\Omega + a}, \quad \Re\{s\} = 0 < -a$$

As a result, the existence condition for Fourier transform of $x(t)$ is $a < 0$. Otherwise, the Fourier transform does not exist.

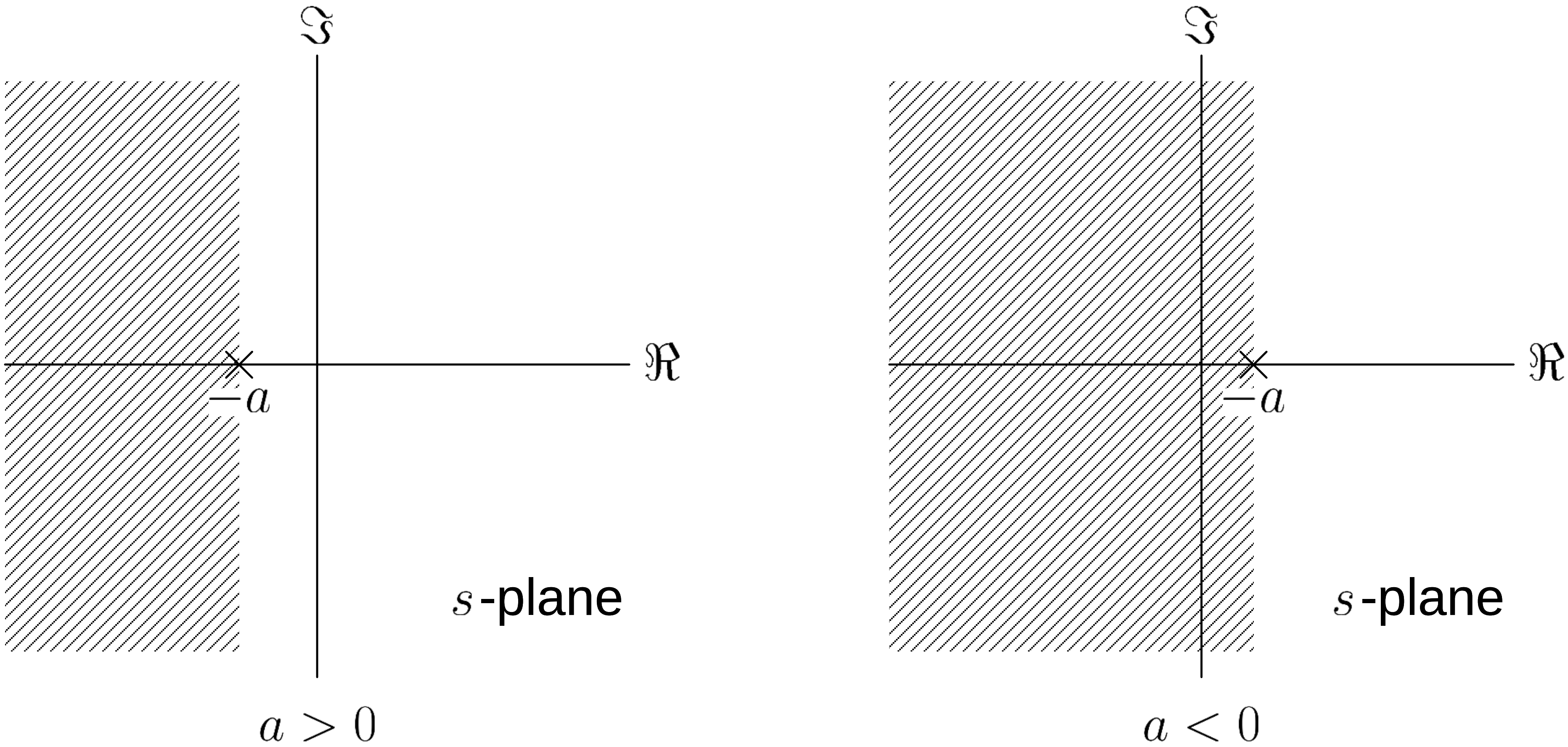


Fig.9.3: ROCs for $a > 0$ and $a < 0$ when $x(t) = -e^{-at}u(-t)$

We also see that $X(j\Omega)$ exists if its ROC includes the imaginary axis.

Example 9.4

Determine the Laplace transform of $x(t)=e^{-at}u(t)+e^{bt}u(-t)$, assuming that a and b are real such that $b>-a$.

Employing the results in Examples 9.2 and 9.3, we have

$$\begin{aligned} X(s) &= \frac{1}{s+a}-\frac{1}{s-b}, \quad \Re\{s\}>-a,\ \Re\{s\}<b \\ &= \frac{-(a+b)}{(s+a)(s-b)}, \quad b>\Re\{s\}>-a \end{aligned}$$

Note that there is no zero while there are two poles, namely, $s=-a$ and $s=b$.

If $b<-a$, then there is no intersection between $\Re\{s\}>-a$ and $\Re\{s\}<b$, and $X(s)$ does not exist for any s.

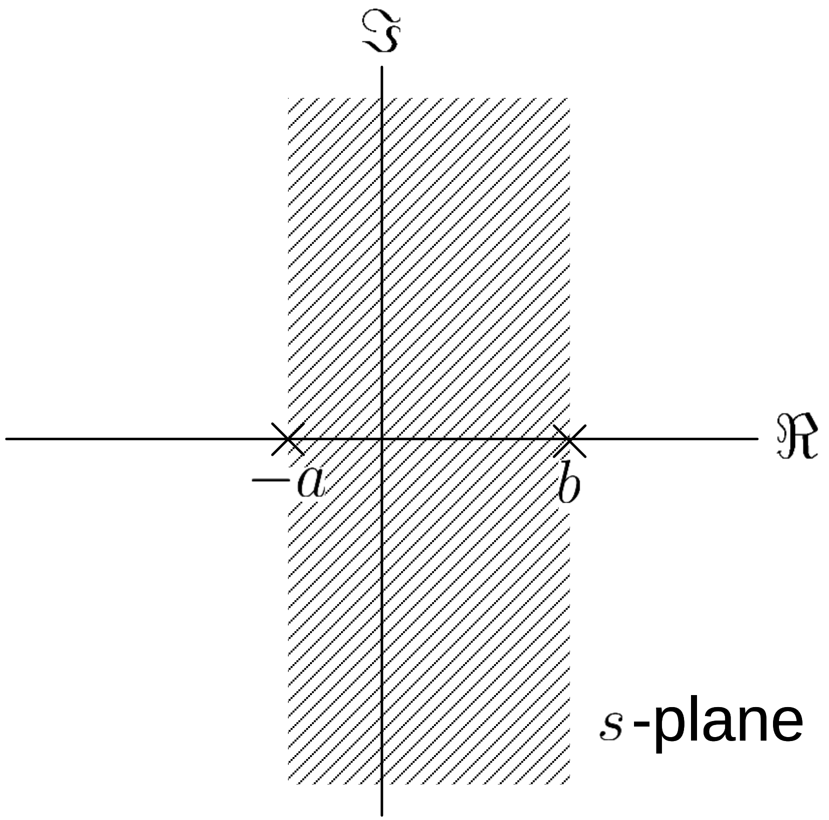


Fig.9.4: ROC for $x(t) = e^{-at}u(t) + e^{bt}u(-t)$

Does the Fourier transform of *x(t)* exist?

Example 9.5

Determine the Laplace transform of $x(t) = \delta(t)$.

Using (9.1) and (2.19), we have

$$X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = \int_{-\infty}^{\infty} \delta(t) e^{-s \cdot 0} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

Example 9.6

Determine the Laplace transform of $x(t) = \delta(t+1) + \delta(t-1)$.

Similar to Example 9.5, we have

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} [\delta(t+1) + \delta(t-1)] e^{-st} dt \\ &= \int_{-\infty}^{\infty} \delta(t+1) e^{-s \cdot -1} dt + \int_{-\infty}^{\infty} \delta(t-1) e^{-s \cdot 1} dt \\ &= e^{s} + e^{-s} \end{aligned}$$

Example 9.7

Determine the Laplace transform of $x(t) = e^{-at}[u(t) - u(t-10)]$

$$
\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} e^{-at}[u(t) - u(t-10)]e^{-st}dt \\
&= \int_{0}^{10} e^{-(s+a)t}dt \\
&= -\frac{1}{s+a}e^{-(s+a)t}\bigg|_{0}^{10} \\
&= \frac{1 - e^{-10(s+a)}}{s+a}
\end{aligned}
$$

What are the ROCs in Examples 9.5, 9.6 and 9.7?

Finite-Duration and Infinite-Duration Signals

Finite-duration signal: values of $x(t)$ are nonzero only for a finite time interval. If $x(t)$ is absolutely integrable, then the ROC of $X(s)$ is the entire s-plane.

Example 9.8

Given a finite-duration $x(t)$ such that:

$$x(t) = \begin{cases} \text{nonzero}, & T_1 < t < T_2 \\ 0, & \text{otherwise} \end{cases}$$

It is also absolutely integrable:

$$\int_{-\infty}^{\infty} |x(t)| dt = \int_{T_1}^{T_2} |x(t)| dt < \infty$$

Show that the ROC of $X(s)$ is the entire s-plane.

According to (9.7), $X(s)$ converges if

$$\int_{-\infty}^{\infty} \left| x(t) e^{-\sigma t} \right| dt = \int_{T_1}^{T_2} \left| x(t) e^{-\sigma t} \right| dt < \infty$$

We consider three cases, namely, $\sigma = 0$, $\sigma > 0$ and $\sigma < 0$.

The convergence condition is satisfied at $\sigma = 0$ because $x(t)$ is absolutely integrable.

For $\sigma > 0$, $e^{-\sigma T_1} > e^{-\sigma t}$ for $t \in (T_1, T_2)$, and we have:

$$\int_{-\infty}^{\infty} \left| x(t) e^{-\sigma t} \right| dt = \int_{T_1}^{T_2} \left| x(t) e^{-\sigma t} \right| dt < e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)| \, dt < \infty$$

because $e^{-\sigma T_1}$ is bounded and $x(t)$ is absolutely integrable.

Similarly, for $\sigma < 0$, $e^{-\sigma T_2} > e^{-\sigma t}$ for $t \in (T_1, T_2)$, and we have:

$$\int_{-\infty}^{\infty} \left|x(t)e^{-\sigma t}\right| dt = \int_{T_1}^{T_2} \left|x(t)e^{-\sigma t}\right| dt < e^{-\sigma T_2} \int_{T_1}^{T_2} |x(t)|\, dt < \infty$$

because $e^{-\sigma T_2}$ is bounded and $x(t)$ is absolutely integrable.

As for all values of σ, (9.7) is satisfied, hence the ROC is the entire s-plane.

If $x(t)$ is not of finite-duration, it is an infinite-duration signal:

- Right-sided: if $x(t) = 0$ for $t < T_1 < \infty$ (e.g., Example 9.2 or $x(t) = e^{-at}u(t)$ with $T_1 = 0$; $x(t) = e^{-at}u(t-2.2)$ with $T_1 = 2.2$; $x(t) = e^{-at}u(t+3.3)$ with $T_1 = -3.3$).

- Left-sided: if $x(t) = 0$ for $t > T_2 > -\infty$ (e.g., Example 9.3 or $x(t) = e^{-at}u(-t)$ with $T_2 = 0$; $x(t) = e^{-at}u(-t+2.2)$ with $T_2 = 2.2$).

- Two-sided: neither right-sided nor left-sided (e.g., Example 9.4).

Signal	Transform	ROC
$\delta(t)$	1	All s
$\delta(t-T)$	e^{-sT}	All s
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\Re\{s\} > -a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\Re\{s\} < -a$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$\Re\{s\} > -a$
$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$\Re\{s\} < -a$
$e^{-at}\cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$	$\Re\{s\} > -a$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$	$\Re\{s\} > -a$

Table 9.1: Laplace transforms for common signals

Summary of ROC Properties

P1. The ROC of $X(s)$ consists of a region parallel to the $j\Omega$-axis in the s-plane. There are four possible cases, namely, the entire region, right-half plane (region includes ∞), left-half plane (region includes $-\infty$) and single strip (region bounded by two poles).

P2. The Fourier transform of a signal $x(t)$ exists if and only if the ROC of the Laplace transform of $x(t)$ includes the $j\Omega$-axis (e.g., Examples 9.2 and 9.3).

P3: For a rational $X(s)$, its ROC cannot contain any poles (e.g., Examples 9.2 to 9.4).

P4: When $x(t)$ is of finite-duration and is absolutely integrable, the ROC is the entire s-plane (e.g., Example 9.7).

P5: When $x(t)$ is right-sided, the ROC is the right-half plane to the right of the rightmost pole (e.g., Example 9.2).

P6: When $x(t)$ is left-sided, the ROC is left-half plane to the left of the leftmost pole (e.g., Example 9.3).

P7: When $x(t)$ is two-sided, the ROC is of the form $\Re\{p_a\} > \Re\{s\} > \Re\{p_b\}$ where p_a and p_b are two poles of $X(s)$ with the successive values in real part (e.g., Example 9.4).

P8: The ROC must be a connected region.

Example 9.9

Consider a Laplace transform $X(s)$ contains three real poles, namely, a, b and c with $a < b < c$. Determine all possible ROCs.

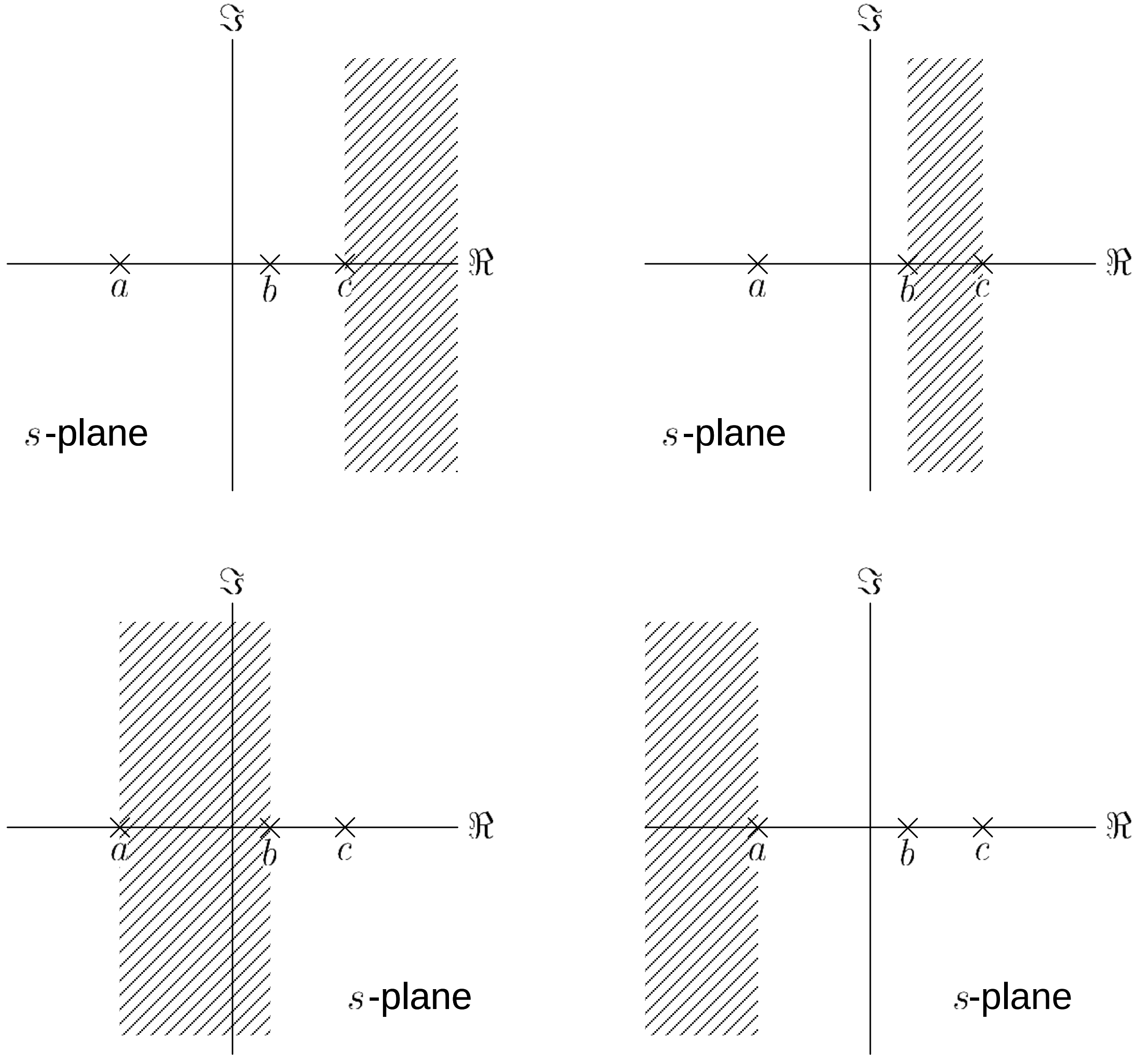


Fig.9.5: ROC possibilities for three poles

Properties of Laplace Transform

Linearity

Let $x_1(t) \leftrightarrow X_1(s)$ and $x_2(t) \leftrightarrow X_2(s)$ be two Laplace transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$, respectively, we have

$$ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s) \qquad (9.10)$$

Its ROC is denoted by $\mathcal{R}$, which includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$ where $\cap$ is the intersection operator. That is, $\mathcal{R}$ contains at least the intersection of $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$.

Example 9.10

Determine the Laplace transform of $y(t)$:

$$y(t) = x_1(t) - x_2(t)$$

where $x_1(t) = 3e^{-2t}u(t)$ and $x_2(t) = 2e^{-t}u(t)$. Find also the pole and zero locations.

From Table 9.1, we have:

$$e^{-2t}u(t) \leftrightarrow \frac{1}{s+2}, \quad \Re\{s\} > -2$$

and

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+1}, \quad \Re\{s\} > -1$$

According to the linearity property, the Laplace transform of $y(t)$ is

$$Y(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{s^2+3s+2}, \quad \Re\{s\} > -1$$

There are two poles, namely -2 and -1 and there is one zero at 1.

Example 9.11

Determine the ROC of the Laplace transform of $y(t)$ which is expressed as:

$$y(t) = x_1(t) - x_2(t)$$

The Laplace transforms of $x_1(t)$ and $x_2(t)$ are:

$$X_1(s) = \frac{1}{s+1},\ \Re\{s\} > -1 \quad \text{and} \quad X_2(s) = \frac{1}{(s+1)(s+2)},\ \Re\{s\} > -1$$

We have:

$$Y(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}$$

We can deduce that the ROC of $y(t)$ is $\Re\{s\} > -2$, which contains the intersection of the ROCs of $X_1(s)$ and $X_2(s)$ which is $\Re\{s\} > -1$. Note also that the pole at $s = -1$ is cancelled by the zero at $s = -1$.

Time Shifting

A time-shift of t_0 in $x(t)$ causes a multiplication of e^{-st_0} in $X(s)$

$$x(t) \leftrightarrow X(s) \Rightarrow x(t-t_0) \leftrightarrow e^{-st_0}X(s) \qquad (9.11)$$

The ROC for $x(t-t_0)$ is identical to that of $X(s)$.

Example 9.12

Find the Laplace transform of $x(t)$ which has the form of:

$$x(t) = e^{-at}u(t-10)$$

Employing the time shifting property with $t = 10$ and:

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, \quad \Re\{s\} > -a$$

we easily obtain

$$e^{-10a} \cdot e^{-a(t-10)}u(t-10) \leftrightarrow e^{-10a} \cdot e^{-10s}\frac{1}{s+a} = \frac{e^{-10(s+a)}}{s+a}, \quad \Re\{s\} > -a$$

Multiplication by an Exponential Signal

If we multiply $x(t)$ by $e^{s_0 t}$ in the time domain, the variable s will be changed to $s - s_0$ in the Laplace transform domain:

$$x(t) \leftrightarrow X(s) \Rightarrow e^{s_0 t}x(t) \leftrightarrow X(s - s_0) \tag{9.12}$$

If the ROC for $x(t)$ is $\mathcal{R}$, then the ROC for $e^{s_0 t}x(t)$ is $\mathcal{R} + \Re\{s_0\}$, that is, shifted by $\Re\{s_0\}$. Note that if $X(s)$ has a pole (zero) at $s = a$, then $X(s - s_0)$ has a pole (zero) at $s = a + s_0$.

Example 9.13

With the use of the following Laplace transform pair:

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, \quad \Re\{s\} > -a$$

Find the Laplace transform of $x(t)$ which has the form of:

$$e^{-at}\cos(bt)u(t)$$

Noting that $\cos(bt) = (e^{jbt} + e^{-jbt})/2$, $x(t)$ can be written as:

$$x(t) = \frac{1}{2}e^{(-a+jb)t}u(t) + \frac{1}{2}e^{(-a-jb)t}u(t)$$

By means of the property of (9.12) with the substitution of $s_0 = jb$ and $s_0 = -jb$, we obtain:

$$\frac{1}{2}e^{jbt}[e^{-at}u(t)] \leftrightarrow \frac{1}{2}\frac{1}{(s-jb)+a}, \quad \Re\{s\} > -a$$

and

$$\frac{1}{2}e^{-jbt}[e^{-at}u(t)] \leftrightarrow \frac{1}{2}\frac{1}{(s+jb)+a}, \quad \Re\{s\} > -a$$

By means of the linearity property, it follows that

$$X(s) = \frac{1}{2}\frac{1}{(s-jb)+a} + \frac{1}{2}\frac{1}{(s+jb)+a} = \frac{s+a}{(s+a)^2+b^2}, \quad \Re\{s\} > -a$$

which agrees with Table 9.1.

Differentiation in s Domain

Differentiating $X(s)$ with respect to s corresponds to multiplying $x(t)$ by $-t$ in the time domain:

$$x(t) \leftrightarrow X(s) \Rightarrow -tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad (9.13)$$

The ROC for $tx(t)$ is identical to that of $X(s)$.

Example 9.14

Determine the Laplace transform of $x(t) = te^{-at}u(t)$.

We start with using:

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, \quad \Re\{s\} > -a$$

and

$$\frac{d}{ds}\left(\frac{1}{s+a}\right) = -\frac{1}{(s+a)^2}$$

Applying (9.13), we obtain:

$$te^{-at}u(t) \leftrightarrow \frac{1}{(s+a)^2}, \quad \Re\{s\} > -a$$

Further differentiation yields:

$$\tfrac{t^2}{2}e^{-at}u(t) \leftrightarrow \frac{1}{(s+a)^3}, \quad \Re\{s\} > -a$$

The result can be generalized as:

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \leftrightarrow \frac{1}{(s+a)^n}, \quad \Re\{s\} > -a$$

which agrees with Table 9.1.

Conjugation

The Laplace transform pair for $x^*(t)$ is:

$$x(t) \leftrightarrow X(s) \Rightarrow x^*(t) \leftrightarrow X^*(s^*) \tag{9.14}$$

The ROC for $x^*(t)$ is identical to that of $X(s)$.

Hence when $x(t)$ is real-valued, $X(s) = X^*(s^*)$.

Time Reversal

The Laplace transform pair for $x(-t)$ is:

$$x(t) \leftrightarrow X(s) \Rightarrow x(-t) \leftrightarrow X(-s) \tag{9.15}$$

The ROC will be reversed as well. For example, if the ROC for $x(t)$ is $\Re\{s\} > -a$, then the ROC for $x(-t)$ is $\Re\{s\} < a$.

Example 9.15

Determine the Laplace transform of $x(t) = e^{at}u(-t)$.

We start with using:

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, \quad \Re\{s\} > -a$$

Applying (9.15) yields

$$e^{at}u(-t) \leftrightarrow \frac{1}{-s+a} = -\frac{1}{s-a}, \quad \Re\{s\} < a$$

Convolution

Let $x_1(t) \leftrightarrow X_1(s)$ and $x_2(t) \leftrightarrow X_2(s)$ be two Laplace transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$, respectively. Then we have:

$$x_1(t) \otimes x_2(t) \leftrightarrow X_1(s)X_2(s) \tag{9.16}$$

and its ROC includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$. The proof is similar to (5.22).

Differentation in Time Domain

Differentiating $x(t)$ with respect to t corresponds to multiplying $X(s)$ by s in the s-domain:

$$x(t) \leftrightarrow X(s) \Rightarrow \frac{dx(t)}{dt} \leftrightarrow sX(s) \tag{9.17}$$

Its ROC includes the ROC for $x(t)$.

Repeated application of (9.17) yields the general form:

$$\frac{d^k x(t)}{dt^k} \leftrightarrow s^k X(s) \tag{9.18}$$

Example 9.16

Use the Laplace transform of $u(t)$ to determine the Laplace transform of $x(t) = \delta(t)$.

According to (2.24):

$$\delta(t) = \frac{du(t)}{dt}$$

Substituting $a = 0$ into Example 9.2 or Table 9.1, we have:

$$u(t) \leftrightarrow \frac{1}{s}, \quad \Re\{s\} > 0$$

Employing (9.17) and (2.24) yields

$$\delta(t) \leftrightarrow s \cdot \frac{1}{s} = 1$$

where the ROC is the entire s-plane.

Note that the result can be easily extended to the derivative of $\delta(t)$. For example,

$$\frac{d\delta(t)}{dt} \leftrightarrow s \cdot 1 = s$$

Extension using (9.18) yields:

$$\frac{d^n\delta(t)}{dt^n} \leftrightarrow s^n$$

Integration

On the other hand, if we perform integration on $x(t)$, this corresponds to dividing $X(s)$ by s in the s-domain:

$$x(t) \leftrightarrow X(s) \Rightarrow \int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}X(s) \qquad (9.19)$$

If the ROC for $x(t)$ is $\mathcal{R}$, then the ROC for $\int_{-\infty}^{t} x(\tau)d\tau$ includes $\mathcal{R} \cap \{\Re\{s\} > 0\}$.

Example 9.17

Prove (9.19), that is, the integration property of Laplace transform.

We first notice that

$$x(t) \otimes u(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)d\tau$$

because $u(t-\tau)=1$ only for $\tau \in (-\infty, t)$.

Applying the convolution property of (9.16) and noting from Example 9.16 that

$$u(t) \leftrightarrow \frac{1}{s}, \quad \Re\{s\} > 0$$

We then have:

$$x(t) \otimes u(t) = \int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow X(s) \cdot \frac{1}{s}$$

where the ROC includes the intersection of ROC of $X(s)$ and $\Re\{s\} > 0$.

Example 9.18

Determine the Laplace transform of $x(t) = u(t) \otimes u(t)$.

From Example 9.17, we know that

$$u(t) \otimes u(t) = \int_{-\infty}^{t} u(\tau) d\tau$$

Employing (9.19) and

$$u(t) \leftrightarrow \frac{1}{s}, \quad \Re\{s\} > 0$$

We then have:

$$u(t) \otimes u(t) \leftrightarrow \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}, \quad \Re\{s\} > 0$$

Alternatively, this can be easily obtained using (9.16). Note that its generalization is:

$$\underbrace{u(t) \otimes \cdots \otimes u(t)}_{n \text{ times}} = \frac{1}{s^n}, \quad \Re\{s\} > 0$$

Causality and Stability Investigation with ROC

Suppose $h(t)$ is the impulse response of a continuous-time linear time-invariant (LTI) system. Recall (3.18), which is the causality condition:

$$h(t) = 0, \quad t < 0 \tag{9.20}$$

If the system is causal and $h(t)$ is of infinite duration, the ROC must be the right-half plane, i.e., the region of the right of the rightmost pole, indicating it is right-sided. Note that causality implies right-half plane ROC but the converse may not be true.

Nevertheless, if $H(s)$ is rational and its ROC is the right-half plane, then the system must be causal.

Example 9.19

Discuss the causality of the two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$. Their Laplace transforms are:

$$H_1(s) = \frac{1}{s+1}, \quad \Re\{s\} > -1, \qquad H_2(s) = \frac{e^s}{s+1}, \quad \Re\{s\} > -1$$

For $H_1(s)$, we use Table 9.1 or Example 9.2 to obtain:

$$h_1(t) = e^{-t}u(t)$$

which corresponds to a causal system. We can also know its causality because $H_1(s)$ is rational and its ROC is the right-half plane.

On the other hand, using the time-shifting property and the above result, we have:

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+1}, \quad \Re\{s\} > -1 \Rightarrow e^{-(t+1)}u(t+1) \leftrightarrow \frac{e^s}{s+1}, \quad \Re\{s\} > -1$$

That is,

$$h_2(t) = e^{-(t+1)}u(t+1)$$

which corresponds to a non-causal system. This also aligns with the above discussion because $H_2(s)$ is not rational although its ROC is also right-half plane.

Recall the stability condition in (3.20):

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty \quad (9.21)$$

(9.21) corresponds to the existence condition of the Fourier transform of $h(t)$. According to P2, this means that the ROC of $H(s)$ includes the $j\Omega$-axis.

That is, a LTI system is stable if and only if the ROC of $H(s)$ includes the $j\Omega$-axis.

Example 9.20

Discuss the causality and stability of a LTI system with impulse response $h(t)$. The Laplace transform of $h(t)$ is:

$$H(s) = \frac{3}{s+1} + \frac{2}{s-2}$$

As the ROC of $H(s)$ is not specified, we investigate all possible cases, i.e., $\Re\{s\} < -1$, $-1 < \Re\{s\} < 2$ and $\Re\{s\} > 2$.

For $\Re\{s\} < -1$, we use Table 9.1 to obtain:

$$-e^{-t}u(-t) \leftrightarrow \frac{1}{s+1}, \quad \Re\{s\} < -1$$

and

$$-e^{2t}u(-t) \leftrightarrow \frac{1}{s-2}, \quad \Re\{s\} < 2$$

where both ROCs agree with $\Re\{s\} < -1$. Combining the results yields:

$$h(t) = -\left[3e^{-t} + 2e^{2t}\right]u(-t)$$

Because of $u(-t)$ and e^{-t} is approaching unbounded as $t \to -\infty$, this system is non-causal and unstable.

Similarly we obtain for $-1 < \Re\{s\} < 2$:

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+1}, \quad \Re\{s\} > -1$$

and

$$-e^{2t}u(-t) \leftrightarrow \frac{1}{s-2}, \quad \Re\{s\} < 2$$

Combining the results yields:

$$h(t) = 3e^{-t}u(t) - 2e^{2t}u(-t)$$

Due to $u(-t)$, the system is not causal. While e^{-t} is absolutely integrable in $t \in (0, \infty)$ and e^{2t} is absolutely integrable in $t \in (-\infty, 0)$, the system is stable.

Finally, for $\Re\{s\} > 2$, we use:

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+1}, \quad \Re\{s\} > -1$$

and

$$e^{2t}u(t) \leftrightarrow \frac{1}{s-2}, \quad \Re\{s\} > 2$$

Combining the results yields:

$$h(t) = 3e^{-t}u(t) + 2e^{2t}u(t)$$

This system is causal but not stable due to $e^{2t}u(t)$.

To summarize, a causal system with rational $H(s)$ is stable if and only if all of the poles of $H(s)$ lies in the left-half of the s -plane, i.e., all of the poles have negative real parts.

Inverse Laplace Transform

Inverse Laplace transform corresponds to finding $x(t)$ given $X(s)$ and its ROC.

The Laplace transform and inverse Laplace transform are one-to-one mapping provided that the ROC is given:

$$x(t) \leftrightarrow X(s) \quad (9.22)$$

There are 3 commonly used techniques to perform the inverse Laplace transform. They are

1. Inspection

2. Partial Fraction Expansion

3. Contour Integration

Inspection

When we are familiar with certain transform pairs, we can do the inverse Laplace transform by inspection.

Example 9.21

Find $x(t)$ if its Laplace transform has the form of:

$$X(s) = \frac{s-1}{s+1}, \quad \Re\{s\} < -1$$

Reorganizing $X(s)$ as:

$$X(s) = \frac{s+1-2}{s+1} = 1 - \frac{2}{s+1}, \quad \Re\{s\} < -1$$

Using Table 9.1 and linearity property, we get:

$$x(t) = \delta(t) - 2e^{-t}u(t)$$

Partial Fraction Expansion

The technique is identical to that in inverse z transform but now we consider that $X(s)$ is a rational function in s:

$$X(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \tag{9.23}$$

To obtain the partial fraction expansion from (9.23), the first step is to determine N nonzero poles, $c_1, c_2, \cdots, c_N$.

There are 4 cases to be considered:

Case 1: $M < N$ and all poles are of first order

$X(s)$ can be decomposed as:

$$X(s) = \sum_{k=1}^{N} \frac{A_k}{s - c_k} \qquad (9.24)$$

For each first-order term of $A_k / (s - c_k)$, its inverse Laplace transform can be easily obtained by inspection.

The A_k can be computed as:

$$A_k = (s - c_k) X(s)\big|_{s=c_k} \qquad (9.25)$$

Case 2: $M \geq N$ and all poles are of first order

In this case, $X(s)$ can be expressed as:

$$X(s) = \sum_{l=0}^{M-N} B_l s^l + \sum_{k=1}^{N} \frac{A_k}{s - c_k} \qquad (9.26)$$

- B_l are obtained by long division of the numerator by the denominator, with the division process terminating when the remainder is of lower degree than the denominator.
- A_k can be obtained using (9.25).

Case 3: $M < N$ with multiple-order pole(s)

Assuming that $X(s)$ has a r-order pole at $s = c_i$ with $r \geq 2$, then $X(s)$ can be decomposed as:

$$X(s) = \sum_{k=1, k \neq i}^{N} \frac{A_k}{s - c_k} + \sum_{m=1}^{r} \frac{C_m}{(s - c_i)^m} \tag{9.27}$$

- When there are two or more multiple-order poles, we include a component like the second term for each corresponding pole

- A_k can be computed according to (9.25)
- C_m can be calculated from:

$$C_m = \frac{1}{(r-m)!} \cdot \frac{d^{r-m}}{ds^{r-m}} \left[(s-c_i)^r X(s) \right] \Big|_{s=c_i} \qquad (9.28)$$

Case 4: $M \geq N$ with multiple-order pole(s)

Assuming that $X(s)$ has a r-order pole at $s = c_i$ with $r \geq 2$, then $X(s)$ can be decomposed as:

$$X(s) = \sum_{l=0}^{M-N} B_l s^l + \sum_{k=1, k \neq i}^{N} \frac{A_k}{s - c_k} + \sum_{m=1}^{r} \frac{C_m}{(s - c_i)^m} \qquad (9.29)$$

The A_k, B_l and C_m can be calculated as in Cases 1, 2 and 3.

Example 9.22

Find $x(t)$ if its Laplace transform has the form of:

$$X(s) = \frac{2s^2+9s-11}{(s+1)(s-2)(s+3)}, \quad \Re\{s\} > 2$$

We can express $X(s)$ as:

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s-2} + \frac{A_3}{s+3}$$

Employing (9.25), A_1, A_2 and A_3 are:

$$A_1 = \left. \frac{2s^2+9s-11}{(s-2)(s+3)} \right|_{s=-1} = 3$$

$$A_2 = \left. \frac{2s^2 + 9s - 11}{(s+1)(s+3)} \right|_{s=2} = 1$$

and

$$A_3 = \left. \frac{2s^2 + 9s - 11}{(s+1)(s-2)} \right|_{s=-3} = -2$$

Together with the ROC of $\Re\{s\} > 2$, we obtain:

$$x(t) = 3e^{-t}u(t) + e^{2t}u(t) - 2e^{-3t}u(t)$$

<u>Example 9.23</u>

Find $x(t)$ if its Laplace transform has the form of:

$$X(s) = \frac{2s^3 + 9s^2 + 11s + 2}{s^2 + 4s + 3}, \quad \Re\{s\} > -1$$

First we perform long division to obtain:

$$X(s) = 2s + 1 + \frac{s-1}{s^2 + 4s + 3}$$

The last term can be further decomposed as:

$$\frac{s-1}{(s+1)(s+3)} = \frac{A_1}{s+1} + \frac{A_2}{s+3}$$

Employing (9.25), A_1 and A_2 are:

$$A_1 = \left.\frac{s-1}{s+3}\right|_{s=-1} = -1$$

and

$$A_2 = \left.\frac{s-1}{s+1}\right|_{s=-3} = 2$$

Together with the ROC of $\Re\{s\} > -1$, we obtain:

$$x(t) = 2\frac{d\delta(t)}{dt} + \delta(t) - e^{-t}u(t) + 2e^{-3t}u(t)$$

<u>Example 9.24</u>

Find $x(t)$ if its Laplace transform has the form of:

$$X(s) = \frac{s+2}{(s+1)^2(s+3)}, \quad \Re\{s\} > -1$$

Accordingly to (9.27), we can express $X(s)$ as:

$$X(s) = \frac{A_1}{s+3} + \frac{C_1}{s+1} + \frac{C_2}{(s+1)^2}$$

Employing (9.25), A_1 is:

$$A_1 = \left.\frac{s+2}{(s+1)^2}\right|_{s=-3} = -\frac{1}{4}$$

Applying (9.28), C_1 and C_2 are:

$$C_1 = \frac{1}{(2-1)!} \cdot \frac{d}{ds}\left(\frac{s+2}{s+3}\right)\bigg|_{s=-1} = \left.\frac{s+3-(s+2)}{(s+3)^2}\right|_{s=-1} = \frac{1}{4}$$

and

$$C_2 = \frac{1}{(2-2)!} \cdot \left.\frac{s+2}{s+3}\right|_{s=-1} = \frac{1}{2}$$

Together with the ROC of $\Re\{s\} > -1$, we obtain:

$$x(t) = -0.25e^{-3t}u(t) + 0.25e^{-t}u(t) + 0.5te^{-t}u(t)$$

Transfer Function of Linear Time-Invariant System

A LTI system can be characterized by the transfer function, which is a Laplace transform expression.

Starting with the differential equation in (3.25) which describes the continuous-time LTI system:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{9.30}$$

Applying Laplace transform on (9.30) with the use of the linearity property and (9.18), we have:

$$Y(s) \sum_{k=0}^{N} a_k s^k = X(s) \sum_{k=0}^{M} b_k s^k \tag{9.31}$$

The transfer function, denoted by $H(s)$, is defined as:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \tag{9.32}$$

The system impulse response $h(t)$ is given by the inverse Laplace transform of $H(s)$ with an appropriate ROC, that is, $h(t) \leftrightarrow H(s)$, such that $y(t) = x(t) \otimes h(t)$. This suggests that we can first take the Laplace transforms of $x(t)$ and $h(t)$, then multiply $X(s)$ by $H(s)$, and finally perform the inverse Laplace transform of $X(s)H(s)$ to obtain $y(t)$.

Comparing with (5.29), we see that the system frequency response can be obtained as $H(s)|_{s=j\Omega} = H(j\Omega)$ if it exists.

Example 9.25

Determine the transfer function for a LTI system whose input $x(t)$ and output $y(t)$ are related by:

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

Taking Laplace transform on the both sides with the use of the linearity and differentiation properties, $H(s)$ is:

$$Y(s)(s+3) = X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+3}$$

Note that there are two ROC possibilities, namely, $\Re\{s\} > -3$ and $\Re\{s\} < -3$, and we cannot uniquely determine $h(t)$. However, if it is known that the system is causal, $h(t)$ can be uniquely found because the ROC should be $\Re\{s\} > -3$.

Example 9.26

Find the differential equation corresponding to a continuous-time LTI system whose transfer function is given by

$$H(s) = \frac{s+3}{(s+1)(s+2)}$$

Let $H(s) = Y(s)/X(s)$. Performing cross-multiplication and inverse Laplace transform, we obtain:

$$\begin{aligned}
&(s+1)(s+2)Y(z) = (s+3)X(z) \\
&\Rightarrow \left(s^2+3s+2\right)Y(z) = (s+3)X(z) \\
&\Rightarrow \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t)
\end{aligned}$$

Examples 9.25 and 9.26 imply the equivalence between the differential equation and transfer function.

Example 9.27

Compute the impulse response $h(t)$ for a LTI system which is characterized by the following equation:

$$y(t) = x(t) - x(t-1)$$

Applying Laplace transform on the input-ouput equation using the linearity and time shifting properties, $H(s)$ is:

$$Y(s) = X(s)\left(1 - e^{-s}\right) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = 1 - e^{-s}$$

From Table 9.1, there is only one ROC possibility, i.e., entire s-plane. Taking the inverse Laplace transform on $H(s)$ yields:

$$h(t) = \delta(t) - \delta(t-1)$$

which agrees with Example 3.10.

Example 9.28

Compute the impulse response $h(t)$ for a LTI system which is characterized by the following equation:

$$y(t) = \frac{1}{10}\int_0^{10} x(t-\tau)d\tau$$

Noting that

$$\begin{aligned}\frac{1}{10}\int_0^{10} x(t-\tau)d\tau &= 0.1\int_{-\infty}^{\infty}[u(\tau)-u(\tau-10)]x(t-\tau)d\tau \\ &= 0.1\left[x(t)\otimes u(t) - x(t)\otimes u(t-10)\right]\end{aligned}$$

Taking the Laplace transform on the input-output relationship and using convolution as well as time-shifting properties, we get:

$$Y(s) = 0.1\left[X(s)\cdot\frac{1}{s} - X(s)\cdot\frac{e^{-10s}}{s}\right] \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1-e^{-10s}}{10s}$$

Due to the convolution property, we can deduce that the ROC of $H(s)$ is $\Re\{s\} > 0$.

Finally, taking the inverse Laplace transform on $H(s)$ yields:

$$h(t) = 0.1[u(t) - u(t-10)]$$

which agrees with Example 3.11.

Example 9.29

Compute the output $y(t)$ if the input is $x(t) = e^{-at}u(t)$ with $a > 0$ and the LTI system impulse response is $h(t) = u(t)$.

The Laplace transforms of $x(t)$ and $h(t)$ are

$$X(s) = \frac{1}{s+a}, \quad \Re\{s\} > -a$$

and

$$H(s) = \frac{1}{s}, \quad \Re\{s\} > 0$$

As a result, we have:

$$Y(s) = X(s)H(s) = \frac{1}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right], \quad \Re\{s\} > 0$$

Taking the inverse Laplace transform of $Y(s)$ with the ROC of $\Re\{s\} > 0$ yields:

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right)u(t)$$

which agrees with Example 3.16.

Concluding Remarks

Signals in Time Domain

For signals which are functions of time, there are two main types: continuous-time and discrete-time.

A continuous-time signal $x(t)$ is defined on a continuous range of time $t \in [T_1, T_2]$, i.e., $x(t)$ has a value for any $t \in [T_1, T_2]$. It can be observed in real world and examples include speech, music, power line and ECG.

A discrete-time signal $x[n]$ is defined only at discrete instants of time where n is integer. It can be obtained from sampling a continuous-time signal or generated using computer.

Continuous-Time and Discrete-Time Signal Conversion

$x[n]$ can be obtained from a continuous-time signal $x(t)$ via sampling:

$$x[n] = x(t)|_{t=nT} = x(nT), \quad n = \cdots - 1, 0, 1, 2, \cdots \tag{10.1}$$

If $x(t)$ is bandlimited such that $X(j\Omega) = 0$ for $|\Omega| \geq \Omega_b$ and if the sampling frequency $\Omega_s > 2\Omega_b$, then $x(t)$ can be reconstructed from $x[n]$:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \mathrm{sinc}\left(\frac{t - nT}{T}\right) \tag{10.2}$$

Signal Representation in other Domains

Apart from the time domain, we can also study signals in other domains.

For $x(t)$, it can be converted to $X(s)$ and $X(j\Omega)$.

In the Laplace transform domain, the conversion is:

$$x(t) \leftrightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \quad (10.3)$$

Together with the region of convergence (ROC), $x(t)$ and $X(s)$ correspond to a one-to-one mapping. That is, both $x(t)$ and $X(s)$ with ROC are equivalent.

There are at least two advantages of Laplace transform:

- It generalizes the Fourier transform, that is, substituting $s = j\Omega$ yields $X(j\Omega)$. We can see whether the ROC includes the $j\Omega$-axis to check the existence of Fourier transform. The inverse Laplace transform techniques can be applied to convert $X(j\Omega)$ back to $x(t)$.

- It facilitates the analysis of linear time-invariant (LTI) systems. In the time domain, the input $x(t)$, output $y(t)$ and impulse response $h(t)$ are characterized by convolution but in the Laplace transform, they have simpler relation:

$$y(t) = x(t) \otimes h(t) \leftrightarrow Y(s) = X(s)H(s) \qquad (10.4)$$

If $x(t)$ is periodic, then it can be represented as Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \qquad t \in (-\infty, \infty) \tag{10.5}$$

which is a linear combination of harmonically related complex sinusoids. The Fourier series coefficients are:

$$a_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-jk\Omega_0 t} dt, \quad k = \cdots -1, 0, 1, 2, \cdots \tag{10.6}$$

where T_p is the fundamental period and $\Omega_0 = 2\pi / T_p$ is the fundamental frequency.

We can write this pair as:

$$x(t) \leftrightarrow X(j\Omega) \quad \text{or} \quad x(t) \leftrightarrow a_k \tag{10.7}$$

because $\{a_k\}$ contain the amplitude information of all frequency components of $x(t)$. For example, we know the strength of $e^{jk\Omega_0 t}$ from $|a_k|$.

If $x(t)$ is aperiodic, then it can be represented as Fourier transform as:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega = x(t) \leftrightarrow X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \tag{10.8}$$

where $X(j\Omega)$ indicates the amplitude at frequency Ω.

Even if $x(t)$ is periodic, it can also be represented using Fourier transform as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \leftrightarrow X(j\Omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0) \quad (10.9)$$

Nevertheless, we still see that $X(j\Omega)$ is characterized by $\{a_k\}$ as in the Fourier series in (10.7).

The Fourier transform is related to Laplace transform via:

$$X(j\Omega) = X(s)\big|_{s=j\Omega} \quad (10.10)$$

Hence we can use the techniques in Laplace transform to compute Fourier transform and inverse Fourier transform.

It is worth mentioning that although $X(j\Omega)$ does not naturally arise in real world, its magnitude $|X(j\Omega)|$ can be observed using electronic equipment, namely, spectrum analyzer.

<u>Example 10.1</u>

Given the frequency response of a continuous-time LTI system:

$$H(j\Omega) = \frac{1}{j\Omega + a}, \quad a > 0$$

Find the system impulse response $h(t)$.

Although inverse Fourier transform in (10.8) can be employed to determine $h(t)$, integration is needed.

Another approach which is computationally simpler is to make use of Laplace transform. Via the substitution of $j\Omega = s$, the system transfer function is:

$$H(s) = \frac{1}{s+a}$$

As $H(j\Omega)$ exists, we know that the ROC should include the $j\Omega$ -axis and hence is $\Re\{s\} > -a$. From Table 9.1, we easily obtain:

$$h(t) = e^{-at}u(t)$$

This is consistent with Examples 5.3, 5.6 and 9.2.

Example 10.2

Determine the continuous-time signal $x(t)$ if its Fourier transform has the form of:

$$X(j\Omega) = \frac{j\Omega + 4}{-\Omega^2 + 5j\Omega + 6}$$

Via substitution of $j\Omega = s$, the Laplace transform of $x(t)$ is:

$$X(s) = \frac{s+4}{s^2+5s+6} = \frac{s+4}{(s+2)(s+3)}$$

As $X(j\Omega)$ exists, we know that the ROC should include the $j\Omega$ -axis and hence is $\Re\{s\} > -2$.

By means of partial fraction expansion, we obtain:

$$X(s) = \frac{2}{s+2} - \frac{1}{s+3}, \quad \Re\{s\} > -2$$

Taking the inverse Laplace transform yields:

$$x(t) = 2e^{-2t}u(t) - e^{-3t}u(t)$$

As the Laplace transform generalizes the Fourier transform, the properties of the Laplace transform are similar to those of Fourier transform and Fourier series.

For $x[n]$, it can be converted to $X(z)$ and $X(e^{j\omega})$.

In the z transform domain, the conversion is:

$$x[n] \leftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{10.11}$$

Together with the ROC, $x[n]$ and $X(z)$ correspond to a one-to-one mapping. That is, both $x[n]$ and $X(z)$ with ROC are equivalent.

There are at least two advantages of z transform:

- It generalizes the discrete-time Fourier transform, (DTFT), that is, substituting $z = e^{j\omega}$ yields $X(e^{j\omega})$. We can see whether the ROC includes the unit circle or $|z| = 1$ to check the existence of DTFT. The inverse z transform techniques can be applied to convert $X(e^{j\omega})$ back to $x[n]$.

- It facilitates the analysis of LTI systems. In the time domain, the input $x[n]$, output $y[n]$ and impulse response $h[n]$ are characterized by convolution but in the z transform, they have simpler relation:

$$y[n] = x[n] \otimes h[n] \leftrightarrow Y(z) = X(z)H(z) \qquad (10.12)$$

We use DTFT to convert $x[n]$ to frequency domain:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = x[n] \leftrightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad (10.13)$$

where $X(e^{j\omega})$, which is periodic with a period of 2π, indicates the amplitude at frequency ω.

The DTFT is related to z transform via:

$$X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}} \tag{10.14}$$

Hence we can use the techniques in z transform to compute DTFT and inverse DTFT.

<u>Example 10.3</u>

Given the frequency response of a discrete-time LTI system:

$$H(e^{j\omega}) = \frac{1+e^{-j\omega}}{1-0.1e^{-j\omega}}$$

Find the system impulse response $h[n]$.

Although inverse DTFT in (10.13) can be employed to determine $h[n]$, integration is needed.

Another approach which is computationally simpler is to make use of z transform. Via the substitution of $e^{j\omega} = z$, the system transfer function is:

$$H(z) = \frac{1 + z^{-1}}{1 - 0.1z^{-1}} = \frac{1}{1 - 0.1z^{-1}} + \frac{z^{-1}}{1 - 0.1z^{-1}}$$

As $H(e^{j\omega})$ exists, we know that the ROC should include the unit circle and hence is $|z| > 0.1$. Using Table 8.1 and time-shifting property, we easily obtain:

$$h[n] = (0.1)^n u[n] + (0.1)^{n-1} u[n-1]$$

Example 10.4

Find the discrete-time signal $x[n]$ if its DTFT has the form of:

$$X(e^{j\omega}) = \frac{1}{20}\sum_{n=0}^{19} e^{-j\omega n}$$

Via substitution of $e^{j\omega} = z$, the z transform of $x[n]$ is:

$$X(z) = \frac{1}{20}\sum_{n=0}^{19} z^{-n}$$

Clearly there is only one ROC, which is $|z| > 0$. Applying inverse z transform on $X(z)$ yields:

$$x[n] = \frac{1}{20}(\delta[n] + \delta[n-1] + \cdots + \delta[n-19])$$
$$= \frac{1}{20}\sum_{k=0}^{19}\delta[n-k] = \frac{1}{20}(u[n] - u[n-20])$$

which aligns with Example 6.7.

As the z transform generalizes the DTFT, the properties of the z transform are similar to those of DTFT.

LTI System Analysis with Transforms

In the time domain, LTI system is characterized by convolution:

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \qquad (10.15)$$

or

$$y[n] = x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \qquad (10.16)$$

In the Laplace (or Fourier) transform and z transform (or DTFT) domains, (10.15) and (10.16) become multiplication:

$$Y(s) = X(s)H(s) \qquad (10.17)$$

and

$$Y(z) = X(z)H(z) \tag{10.18}$$

Equations (10.17) and (10.18) indicate that we may obtain $Y(s)$ (or $Y(z)$), $X(s)$ (or $X(z)$) and $H(s)$ (or $H(z)$) in an easier manner.

Note that even if the LTI systems are not stable, $H(s)$ and $H(z)$ still exist and their ROCs will not include the $j\Omega$-axis and unit circle, respectively, while $H(j\Omega)$ and $H(e^{j\omega})$ do not converge.

Example 10.5

Determine the transfer functions of the continuous-time and discrete-time LTI systems with impulse responses:

$$h(t) = e^{2t}u(t)$$

and

$$h[n] = 2^n u[n]$$

It is clear from (3.20) and (3.21) that the systems are unstable because they are not absolutely summable and integrable:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

and

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Taking Laplace transform on $h(t)$ yields:

$$H(s) = \frac{1}{s-2}, \quad \Re\{s\} > 2$$

As $\Re\{s\} > 2$ does not include the $j\Omega$-axis, $H(j\Omega)$ does not exist. This conclusion can also be obtained because (9.9) is not satisfied.

Taking z transform on $h[n]$ yields:

$$H(z) = \frac{1}{1 - 2z^{-1}}, \quad |z| > 2$$

As $|z| > 2$ does not include the unit circle, $H(e^{j\omega})$ does not exist. This conclusion can also be obtained because (8.9) is not satisfied.

Example 10.6

Consider a continuous-time LTI system with impulse response $h(t)$, input $x(t)$ and output $y(t)$. Calculate $y(t)$ when $x(t) = h(t) = e^{at}u(t)$.

The Laplace transforms of both $x(t)$ and $h(t)$ are

$$X(s) = H(s) = \frac{1}{s-a}, \quad \Re\{s\} > a$$

As a result, we have:

$$Y(s) = X(s)H(s) = \frac{1}{(s-a)^2}, \quad \Re\{s\} > a$$

According to Table 9.1, we obtain:

$$y(t) = te^{at}u(t)$$

Example 10.7

Consider a discrete-time LTI system with impulse response $h[n]$, input $x[n]$ and output $y[n]$. Calculate $y[n]$ when $x[n] = h[n] = a^n u[n]$.

The z transforms of both $x[n]$ and $h[n]$ are

$$X(z) = H(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

As a result, we have:

$$Y(z) = X(z)H(z) = \frac{1}{(1 - az^{-1})^2}, \quad |z| > |a|$$

According to Table 8.1 and time-shifting property, we obtain:

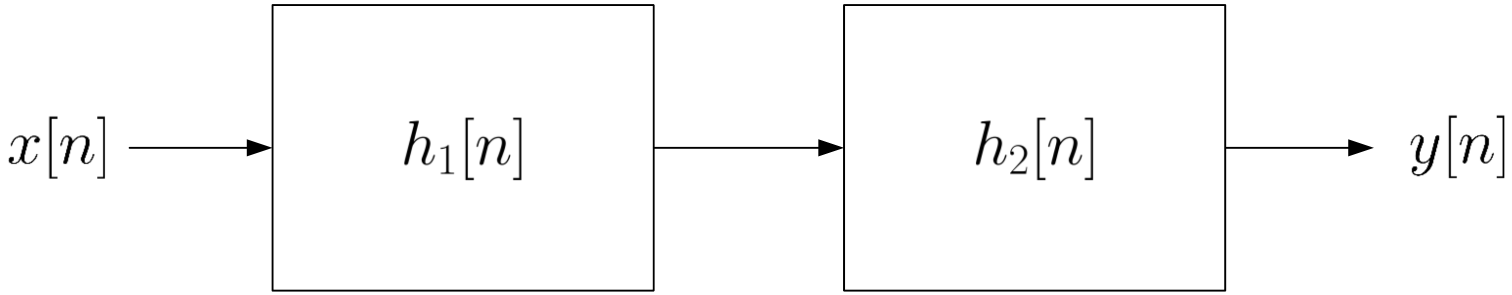
$$na^n u[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2} \Rightarrow (n+1)a^{n+1}u[n+1] \leftrightarrow \frac{a}{(1-az^{-1})^2}$$

Finally, we have:

$$y[n] = (n+1)a^n u[n+1] = (n+1)a^n u[n]$$

<u>Example 10.8</u>

Consider a cascade system of two discrete-time LTI systems with impulse responses $h_1[n]$ and $h_2[n]$. Let the system input and output be $x[n]$ and $y[n]$, respectively. Determine the overall impulse response $h[n]$ and transfer function $H(z)$ if $h_1[n] = h_2[n] = a^n u[n]$. Find the difference equation that relates $x[n]$ and $y[n]$.



The overall impulse response is:

$$h[n] = h_1[n] \otimes h_2[n]$$

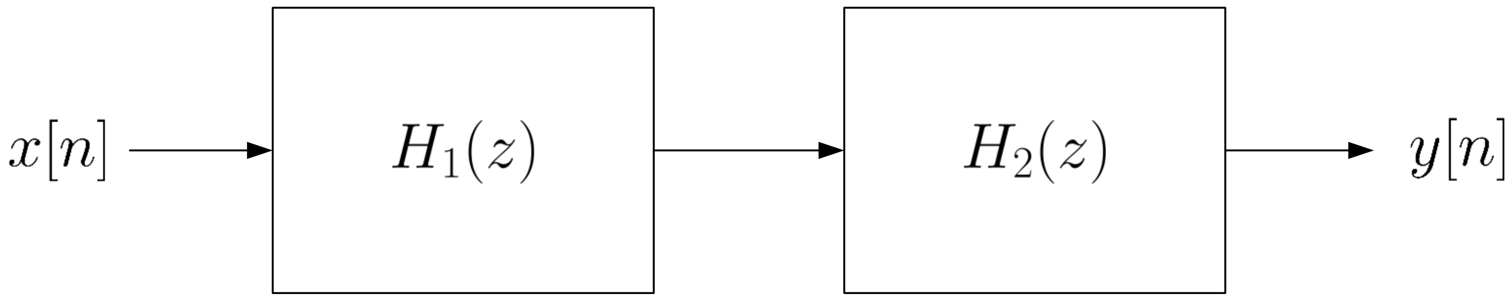
Using the result in Example 10.7, we have:

$$h[n] = (n+1)a^n u[n]$$

From Example 10.7 again, the overall transfer function is:

$$H(z) = H_1(z)H_2(z) = \frac{1}{(1-az^{-1})^2}, \quad |z| > |a|$$

Note that it is equivalent to use $H_1(z)$ and $H_2(z)$ in the block diagram:

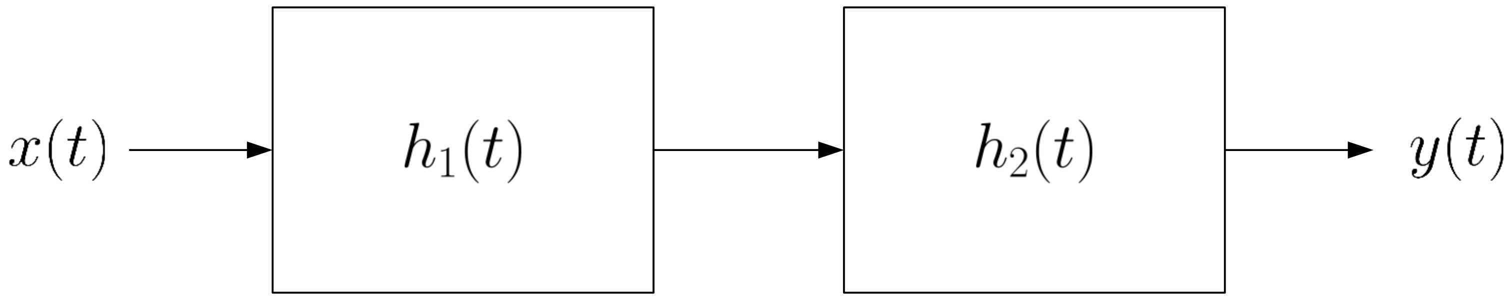


As $H(z)=Y(z)/X(z)$, we perform cross-multiplication and inverse z transform to obtain:

$$\left(1-az^{-1}\right)^2 Y(z)=X(z)$$
$$\Rightarrow \left(1-2az^{-1}+a^2z^{-2}\right)Y(z)=X(z)$$
$$\Rightarrow y[n]-2ay[n-1]+a^2y[n-2]=x[n]$$

<u>Example 10.9</u>

Consider a cascade system of two continuous-time LTI systems with impulse responses $h_1(t)$ and $h_2(t)$. Let the system input and output be $x(t)$ and $y(t)$, respectively. Determine the overall impulse response $h(t)$ and transfer function $H(s)$ if $h_1(t) = h_2(t) = e^{at}u(t)$. Find the differential equation that relates $x(t)$ and $y(t)$.



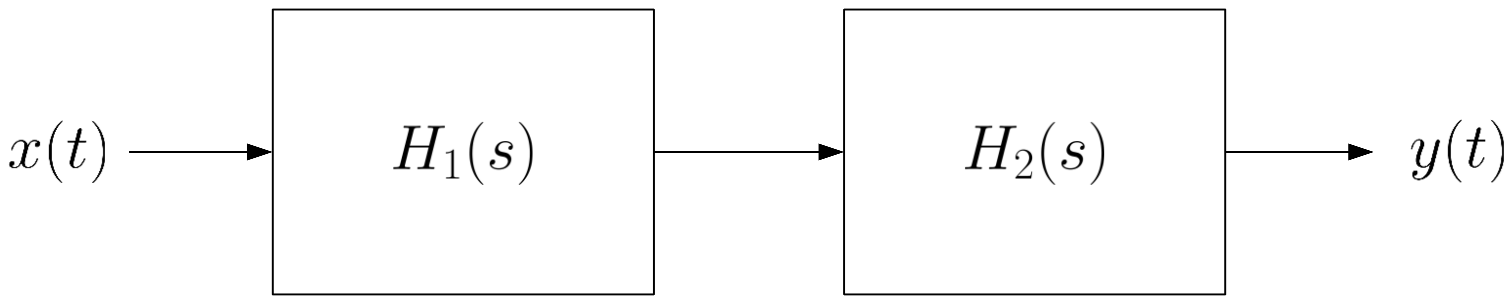
Using Example 10.6, the overall impulse response is:

$$h(t) = h_1(t) \otimes h_2(t) = te^{at}u(t)$$

and the overall transfer function is:

$$H(s) = H_1(s)H_2(s) = \frac{1}{(s-a)^2}, \quad \Re\{s\} > a$$

We can also use $H_1(s)$ and $H_2(s)$ in the block diagram:



As $H(s) = Y(s)/X(s)$, we have:

$$\begin{aligned}(s-a)^2 Y(s) = X(s) &\Rightarrow \left(s^2 - 2as + a^2\right) Y(s) = X(s) \\ &\Rightarrow \frac{d^2y(t)}{dt^2} - 2a\frac{dy(t)}{dt} + a^2 y(t) = x(t)\end{aligned}$$

References:

1. A. V. Oppenheim and A. S. Willsky, ***Signals & Systems***, 2nd Edition, Prentice Hall, 1997

2. S. Haykin and B. Van Veen, ***Signals and Systems***, 2nd Edition, Wiley, 2003