Sparse Phase Retrieval Using Partial Nested Fourier Samplers

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Overview

1. Problem Formulations and Related Work
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    - Related Work

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2. PNFS and Iterative Recovery Algorithm
   - Partial Nested Fourier Sampler (PNFS)

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Phase Retrieval and Applications

Definition

Phase Retrieval: Recovery of a signal given the magnitude of its measurements.

Applications:

- X-ray crystallography: recover Bragg peaks from missing-phase data
- Diffraction imaging, optics, astronomical imaging, microscopy
- Acoustics, blind channel estimation, interferometry, quantum information

Mathematical Formulation

Consider data $\mathbf{x} \in \mathbb{C}^N$, sampler set $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_M\}$ and measurements $\mathbf{y} = [y_1, y_2, \cdots, y_M]^T \in \mathbb{R}_+^M$

$$y_i = |\langle \mathbf{x}, \mathbf{f}_i \rangle|$$

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$$y_i = |\langle \mathbf{x}, \mathbf{f}_i \rangle|$$
$$\Leftrightarrow y_i^2 = \mathbf{f}_i^H \mathbf{x} \mathbf{x}^H \mathbf{f}_i$$

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& y_i = |\langle \mathbf{x}, \mathbf{f}_i \rangle| \\
\Leftrightarrow\ & y_i^2 = \mathbf{f}_i^H \mathbf{x}\mathbf{x}^H \mathbf{f}_i \\
\Leftrightarrow\ & y_i^2 = \left(\mathbf{f}_i^T \otimes \mathbf{f}_i^H\right) \mathrm{Vec}\left(\mathbf{x}\mathbf{x}^H\right)
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$\mathcal{F}$ can consist of either Fourier or general samplers [1].

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   - If $\mathcal{F}$ consists of DFT samplers, $M \geq s^2 - s + 1$ with Collision Free Condition [2].

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1. For general data $\mathbf{x}$ and samplers $\mathcal{F}$, $M = 4N - 4$ is sufficient.
2. For s-sparse data $\mathbf{x}$:
   - If $\mathcal{F}$ consists of DFT samplers, $M \geq s^2 - s + 1$ with Collision Free Condition [2].
   - If $\mathcal{F}$ consists of random samplers, $M = O(s \log N)$ is sufficient via convex program.

Coupling Difficulty and Collision Free Condition

If the samplers in $\mathcal{F}$ are drawn from DFT of proper dimension, phase retrieval can be formulated as recovering data from its autocorrelation $\mathbf{r_x} \in \mathbb{C}^{2N-1}$ defined as

$$[\mathbf{r_x}]_l = \sum_{k=\max\{1,1-l\}}^{\min\{N,N-l\}} x_k \bar{x}_{k+l} \quad 0 \leq |l| \leq N-1$$

The pair-wise products are coupled together which hides the sparse support of $\mathbf{x}$. To avoid this, Collision Free Condition is proposed [2].

Definition

(Collision-Free Condition) [2] A sparse vector $\mathbf{x}$ has collision-free property if for pairs of distinct entries $(p, q), (m, n)$ in the support of $\mathbf{x}$, $p - q \neq m - n$ unless $(p, q) = (m, n)$.

Objectives of this paper

- $\mathcal{F}$ consists of Fourier samplers.
- The sufficient measurement number M should be $O(s \log N)$ with convex program.
- The Collision Free Condition on the sparse support should be relaxed.

Partial Nested Fourier Sampler

PNFS is a generalization of DFT-based sampler which with nested index array instead of consecutive one

Definition

(Partial Nested Fourier Sampler:) We define a Partial Nested Fourier Sampler (PNFS) as

$$\mathbf{f}_i = \alpha \left[z_i^1, z_i^2, \cdots, z_i^{N-1}, z_i^{2N-2} \right]^T$$

where $\alpha = (4N-5)^{-1/4}$ and $z_i = e^{j2\pi(i-1)/(4N-5)}$.

Decoupling Effect of Nested Index Set

The nested index set $\mathcal{N} = \{1, 2, \cdots, N-1, 2N-2\}$ can resolve the coupling difficulty by exploiting the second-order difference set.

Example

Consider $N = 3$ and two different index set $\mathcal{N}_1 = \{0, 1, 2\}$ and $\mathcal{N}_2 = \{0, 1, 3\}$. $\mathcal{N}_2$ is a nested index set.
For $\mathcal{N}_1$, ignoring the negative part, we have

$$\{z_i^0 : x_1\bar{x}_1, x_2\bar{x}_2, x_3\bar{x}_3\}\,\{z_i^1 : x_1\bar{x}_2, x_2\bar{x}_3\}\,\{z_i^2 : x_1\bar{x}_3\}$$

For $\mathcal{N}_2$ we have

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Advantage: The sparse support is revealed in vectorized measurements model.

Measurement Structure with PNFS

Plugging the PNFS sampler $\mathbf{f}_i$ into vectorized measurement model, we have

$$y_i^2 = \frac{1}{\sqrt{4N-5}} \left[z_i^{-(2N-3)}, \cdots, z_i^{-1}, 1, z_i^1, \cdots, z_i^{2N-3} \right] \tilde{\mathbf{x}} \quad (1)$$

where $\tilde{\mathbf{x}} \in \mathbb{C}^{4N-5}$ is the corresponding rearranged version of $\text{Vec}(\mathbf{x}\mathbf{x}^H)$ with following form

$$[\tilde{\mathbf{x}}]_m = \begin{cases} \sum_{k=1}^{N} |x_k|^2 & m = 0 \\ \sum_{k=1}^{N-1-m} x_k \bar{x}_{k+m} & m = 1, 2, \cdots, N-2 \\ x_{2N-2-m} \bar{x}_N & N-1 \leq m \leq 2N-3 \\ \overline{[\tilde{\mathbf{x}}]_{-m}} & m < 0 \end{cases} \quad (2)$$

Permuted Version of PNFS

The support of $\mathbf{x}$ is easily identified in $\tilde{\mathbf{x}}$ if x_N is nonzero. If no prior knowledge available, we will need column-permuted version of PNFS defined as

$$\mathbf{f}_i^{(l)} = \frac{1}{\sqrt[4]{4N-5}} \left[z_i^1, z_i^2, \cdots, z_i^{N-1}, z_i^{2N-2} \right] \mathbf{\Pi}^{(l)} \tag{3}$$

$\mathbf{\Pi}^{(l)}$ is a permuting matrix such that the vector $\mathbf{x}^{(l)} = \mathbf{\Pi}^{(l)}\mathbf{x}$ satisfies $[\mathbf{x}^{(l)}]_l = x_N, [\mathbf{x}^{(l)}]_N = x_l, [\mathbf{x}^{(l)}]_i = x_i, i \neq l, N$.
For each l, we collect $\tilde{M}$ phaseless measurements $y_i^{(l)}, i = 1, 2, \cdots, \tilde{M}$ using the permuted PNFS vector (3) and obtain

$$\tilde{\mathbf{y}}^{(l)} = \mathbf{Z}\tilde{\mathbf{x}}^{(l)} \tag{4}$$

where $[\tilde{\mathbf{y}}^{(l)}]_i = (y_i^{(l)})^2, [\mathbf{Z}]_{i,m} = \frac{1}{\sqrt{4N-5}} e^{j2\pi \frac{(i-1)m}{4N-5}}, 1 \leq i \leq \tilde{M}, \quad -2N+3 \leq m \leq 2N-3$.
Objective: If $\mathbf{x}$ is non-zero, we will finally find $\mathbf{\Pi}^{(l)}$ such that $[\mathbf{x}^{(l)}]_N \neq 0$.

Iterative Algorithm

Input: data $\mathbf{x}$ *Output:* estimation $\mathbf{x}^{\#}$
Initialization: $l = N$
Loop:

1. **Step S1:** Using the permuted PNFS vectors (3), obtain $4N-5$ phaseless measurements

$$y_i^{(l)} = | < \mathbf{x}, \mathbf{f}_i^{(l)} > |, i = 1, 2, \cdots 4N-5$$

   Recover $\tilde{\mathbf{x}}^{(l)} = \mathbf{Z}^{-1}\tilde{\mathbf{y}}^{(l)}$

2. **Step S2:** If $[\tilde{\mathbf{x}}^{(l)}]_m = 0, \forall |m| \geq N-1$, declare $x_l = 0$. Assign $l \rightarrow l-1$ and go back to Step **S1**.
   If $[\tilde{\mathbf{x}}^{(l)}]_m \neq 0$ for some m with $|m| \geq N-1$, proceed to the recovery stage.

Iterative Algorithm: Continued

Recovery:

1. Choose $m^* \in \{1, 2, \cdots, N-2\}$ such that $[\tilde{\mathbf{x}}^{(l)}]_{m^*} \neq 0$ and compute

$$|x_N^{(l)}| = \sqrt{[\tilde{\mathbf{x}}^{(l)}]_{m^*}/\beta}$$

& $\beta = \sum_{k=1}^{N-1-m^*} [\tilde{\mathbf{x}}^{(l)}]_{2N-2-k} \overline{[\tilde{\mathbf{x}}^{(l)}]}_{2N-2-k-m^*}$

2. Obtain estimate $\mathbf{x}^{\#}$ as

$$[\mathbf{x}^{\#}]_q = \begin{cases} \left(\frac{[\tilde{\mathbf{x}}^{(l)}]_{2N-2-q}}{|x_N^{(l)}|} \right) & q \neq \{l, N\} \\ |x_N^{(l)}| & q = l \\ \frac{[\tilde{\mathbf{x}}^{(l)}]_{2N-2-l}}{|x_N^{(l)}|} & q = N \end{cases}$$

Performance of Iterative Algorithm

The complexity of the algorithm mainly depends on the number of trials to find $[\mathbf{x}^{(l)}]_N \neq 0$.

Theorem

Let $\mathbf{x} \in \mathbb{C}^N$ be s-sparse with $s \geq 3$. The estimate $\mathbf{x}^{\#}$ produced by the iterative algorithm described in Table 1 is equal to $\mathbf{x}$ (in the sense of $\mathbb{C} \setminus \mathbb{T}$) if the total number of phaseless measurements M equals $4N-5$ for the best case and $(N-s+1)(4N-5)$ for the worst case.

Corollary

If $\mathbf{x}$ is not sparse (i.e. $s = N$), the number of measurements needed for recovering $\mathbf{x}$ is $M = 4N - 5$.

Sketch of Proof

The main idea in the proof is to show the existence of m^* such that $[\tilde{\mathbf{x}}^{(l)}]_{m^*} \neq 0$. Denote $\breve{\mathbf{x}} = [x_1, x_2, \cdots, x_{N-1}]^T$ and let $\mathbf{r}_{\breve{\mathbf{x}}} \in \mathbb{C}^{2N-3}$ be the autocorrelation vector of $\breve{\mathbf{x}}$. Suppose m^* does not exist, implying $[\tilde{\mathbf{x}}]_m = 0$ for $1 \leq |m| \leq N-2$. Hence, $[\mathbf{r}_{\breve{\mathbf{x}}}]_n = \gamma\delta(n)$ where $\gamma = [\tilde{\mathbf{x}}]_0 - |x_N|^2$ and $\delta(n)$ is Kronecker delta. This means that $\hat{\mathbf{r}}_{\breve{\mathbf{x}}}(e^{j\omega}) \triangleq \sum_{n=-N+2}^{N-2}[\mathbf{r}_{\breve{\mathbf{x}}}]_n e^{-j\omega n}$ is an all-pass filter. However, $\hat{\mathbf{r}}_{\breve{\mathbf{x}}}(e^{j\omega}) = |\hat{\breve{\mathbf{x}}}(e^{j\omega})|^2$ where $\hat{\breve{\mathbf{x}}}(e^{j\omega}) \triangleq \sum_{n=-N+2}^{N-2}[\breve{\mathbf{x}}]_n e^{-j\omega n}$. This implies $\hat{\breve{\mathbf{x}}}(e^{j\omega})$ is also an all-pass filter. Since $\hat{\breve{\mathbf{x}}}(e^{j\omega})$ is an FIR filter, this is not possible unless we have

$$[\breve{\mathbf{x}}]_n = \lambda\delta(n - n_0) \tag{5}$$

for some n_0 satisfying $1 \leq n_0 \leq N-1$ and λ is a constant. However, since $s \geq 3$, $\breve{\mathbf{x}}$ has at least two non zero entries which contradicts (5). Therefore, the existence of m^* is guaranteed.

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Observation: PNFS hits the lower bound $4N - 5$ if $\mathbf{x}$ has no zero entries.

Cancellation of Measurements

If we have some prior knowledge of $\mathbf{x}$ that x_N is nonzero, PNFS can achieve better bound for sparse phase retrieval. This is based on the idea of cancellation via two sets of measurements, $\tilde{\mathbf{y}}, \tilde{\mathbf{y}}' \in \mathbb{C}^{\tilde{M}}$ as

$$[\tilde{\mathbf{y}}]_i = | < \mathbf{x}, \mathbf{f}_i > |^2 \tag{6}$$

$$[\tilde{\mathbf{y}}']_i = | < \mathbf{x}, \mathbf{f}_i^{'} > |^2 \tag{7}$$

where $\mathbf{f}_i$ denotes the PNFS vector (as in Def. 3) and $\mathbf{f}_i^{'}$ is defined as

$$\mathbf{f}_i^{'} = \frac{1}{\sqrt[4]{4N-5}} \left[z_i^1, z_i^2, \cdots, z_i^{N-1}, 0 \right] \tag{8}$$

where $z_i = e^{j2\pi(i-1)/(4N-5)}$.

Cancellation of Measurements: Continued

Denoting $\hat{\mathbf{y}} = \tilde{\mathbf{y}} - \tilde{\mathbf{y}}'$, we have

$$\hat{\mathbf{y}} = \mathbf{Z}\hat{\mathbf{x}} \tag{9}$$

where

$$[\hat{\mathbf{x}}]_m = \begin{cases} |x_N|^2 & m = 0 \\ 0 & m = 1, 2, \cdots, N-2 \\ x_{2N-2-m}\bar{x}_N & m = N-1, \cdots, 2N-3 \\ \overline{[\hat{\mathbf{x}}]_{-m}} & m < 0 \end{cases}$$

and $\mathbf{Z} \in \mathbb{C}^{\tilde{M}, 4N-5}$ defined as in (4). Notice that $\tilde{\mathbf{x}}$ has sparsity $2s - 1$ and support of $\mathbf{x}$ (except the Nth entry) is identical to that of the subvector of $\tilde{\mathbf{x}}$ indexed by $m = N-1, \cdots, 2N-3$.

Number of Measurements

The power of cancellation is revealing the sparse support of **x** and then convex program is applicable.
We can recover $\hat{\mathbf{x}}$ by solving the l_1 minimization:

$$\min_{\theta} \|\theta\|_1 \quad \text{subject to } \hat{\mathbf{y}} = \mathbf{Z}\theta \quad (\mathbf{P1})$$

The vector **x** can then be recovered from the solution of (**P1**).

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Theorem

Let $\mathbf{x} \in \mathbb{C}^N$ be a sparse vector with s non zero elements and $x_N \neq 0$. Suppose we construct the difference measurement vector $\hat{\mathbf{y}}$ as in (9) using $\tilde{M}$ pairs of sampling vectors $\{\mathbf{f}_{i_k}, \mathbf{f}'_{i_k}\}_{k=1}^{\tilde{M}}$ where indices i_k are selected uniformly at random between 1 and $4N - 5$. Then $\mathbf{x}$ can be recovered (in sense of $\mathbb{C} \setminus \mathbb{T}$) by solving (P1) if $\tilde{M} = Cs\log N$ for some constant C.

Inefficiency of Collision Free Condition

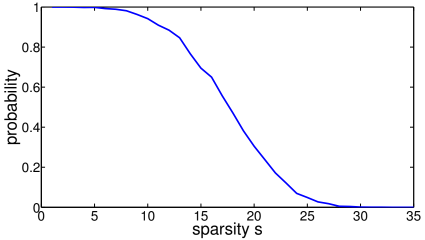

Figure: The probability of "no-collision" as a function of sparsity s. The ambient dimension is $N = 10000$ and the result is averaged over 2000 runs.

Validation of the Theorem 2

The global phase ambiguity is $\rho = x_N / x_N^{\#}$. Using ρ we can compute the entry-wise estimation error as $|x_i - \rho x_i^{\#}|$ for $1 \leq i \leq N$.

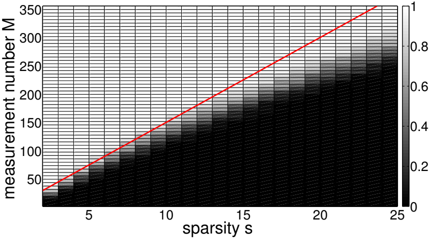

Figure: The phase transition plot for Theorem 2. $M = 2\tilde{M}$ is the total number of measurements needed and $N = 150$. The red line represents $3s\log N$. The color bar denotes probability of success from 0 to 1. The white cells denote successful recoveries (i.e. $|x_i - \rho x_i^{\#}| \leq 10^{-6}$ for all entries) and black cells denote failures.The results are averaged over 100 runs.

Contribution Summary

- The PNFS are general Fourier samplers and can be easily implemented via DFT plus coded diffraction [3].
- The recovery algorithm is deterministic for general case and hits lower bound for nowhere vanishing data $\mathbf{x}$.
- If prior knowledge available, $O(s \log N)$ is possible with cancellation process and convex program.

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