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CRAMER-RAO LOWER BOUNDS OF JOINT DELAY-DOPPLER ESTIMATION FOR AN EXTENDED TARGET

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Outline

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Background

- To estimate the time delay and Doppler stretch with wideband signals
- Advantages of wideband waveform: high range resolution and low interception probability
- Differences in modeling echoes between wideband and narrowband signals:
  - narrowband signals: target=point scatterer, Doppler effect=Doppler shift
  - wideband signals: target=distributed target, Doppler effect=Doppler stretch

Background

- The Cramer-Rao Lower Bound (CRLB) is a lower bound for the variance of any unbiased estimator.
- The CRLB can be used to predict the performance of the maximum likelihood estimator (MLE) and optimize waveforms.
- The most studies on the CRLB of the joint time delay and Doppler estimation for wideband signals either consider the target as a point scatterer or deal with some specific waveform
- In this paper, we consider more general wideband sensing systems and derive the CRLB for an arbitrary wideband signal along with an extended target model and discuss the influences of some waveform parameters on the CRLB.

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Signal Model

- The received signal is

$$y(t) = \sum_{p=1}^{P} x_p s(\gamma(t - \tau_p)) + w(t), \qquad (1)$$

  where $s(t)$ is the transmitted signal with a duration T, $\tau_p = \tau + (p-1)\Delta$ is the time delay of the pth scatterer, Δ is the sample interval, γ is the Doppler stretch and x_p is the scattering coefficient.
- The noise $w(t)$ is assumed as a bandlimited complex Gaussian random process, where $\mathrm{Re}\{w(t)\}$ and $\mathrm{Im}\{w(t)\}$ are mutually independent with a bandwidth $1/(2\Delta)$ and power spectral density$N_0/2$.

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- Sample the echoes at the rate of $1/\Delta$, and write the signal in the form of matrices
$$\mathbf{y} = \mathbf{\Phi}\mathbf{x} + \mathbf{w}, \tag{2}$$
where $\mathbf{y} = [y_0, ..., y_{N-1}]^T \in \mathbb{C}^{N\times 1}$ with $y_n = y(n\Delta)$, $\mathbf{\Phi} \in \mathbb{C}^{N\times P}$ with $\Phi_{ij} = s(\gamma((i-1)\Delta - \tau_j))$, and $\mathbf{x} = [x_1, ..., x_P]^T \in \mathbb{R}^{P\times 1}$, $\mathbf{w} = [w_0, ..., w_{N-1}]^T \in \mathbb{C}^{N\times 1}$ is distributed as $\mathbb{C}N(\mathbf{0}, \sigma^2\mathbf{I})$, where $w_n = w(n\Delta)$ and $\sigma^2\Delta = N_0$.
- The parameters under estimation are $\boldsymbol{\theta} = \left[\tau, \gamma, \mathbf{a}^T, \mathbf{b}^T\right]^T$, where $\mathbf{a} = \mathrm{Re}\{\mathbf{x}\}$ and $\mathbf{b} = \mathrm{Im}\{\mathbf{x}\}$.

Signal Model

- We assume that $s(t)$ has derivatives of all orders $s^{(m)}(t) = d^m s/dt^m(t), m \in \mathbb{N}$.
- Denote

$$M_i^{(k)} = \int_{-\infty}^{+\infty} \left| s^{(k)}(t) \right|^2 t^i dt, i = 0, 1, 2, k \in \mathbb{N}$$

$$\widetilde{M}_i^{(k)} = \mathrm{Im}\left\{ \int_{-\infty}^{+\infty} t^i s^{*(k)}(t) s^{(k+1)}(t) dt \right\}, i = 0, 1, 2, k \in \mathbb{N}$$

The parameters $M_0^{(0)}$, $\bar{B} = \left(M_0^{(1)} / M_0^{(0)} \right)^{\frac{1}{2}}$ and $\bar{T} = \left(M_2^{(1)} / M_0^{(1)} \right)^{\frac{1}{2}}$ are the energy, effective bandwidth and effective duration of the transmitted signal, respectively.

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Derivations of the CRLB

The covariance matrix of any unbiased estimator $\hat{\boldsymbol{\theta}}$ satisfies

$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} \triangleq E\left\{(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \cdot (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T\right\} \geq \mathbf{FIM}^{-1} \quad (3)$$

where $\mathbf{FIM} \in \mathbb{R}^{(2P+2)\times(2P+2)}$ is the Fisher information matrix with

$$\mathrm{FIM}_{ij} = \frac{2}{\sigma^2}\mathrm{Re}\left\{\frac{\partial \boldsymbol{\mu}^H(\boldsymbol{\theta})}{\partial \theta_i}\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_j}\right\}$$

where $\boldsymbol{\mu}(\boldsymbol{\theta}) = \boldsymbol{\Phi}\mathbf{x}$. After some calculations, the CRLBs of τ and γ are given as

$$\mathrm{var}(\tau) \geq \mathrm{CRLB}_\tau = a_{22}/\left(a_{11}a_{22} - a_{12}^2\right) \quad (4)$$

$$\mathrm{var}(\gamma) \geq \mathrm{CRLB}_\gamma = a_{11}/\left(a_{11}a_{22} - a_{12}^2\right) \quad (5)$$

where $a_{ij} = F_{ij} - \mathrm{Re}\left\{\mathbf{F}_{3i}^H\mathbf{F}_{33}^{-1}\mathbf{F}_{3j}\right\}$

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$$F_{11} = \sum_{i=1}^{P}\sum_{j=1}^{P} \frac{2x_i^* x_j \gamma}{N_0} \int_{-\infty}^{+\infty} s^{*(1)}(t) s^{(1)}(t+\gamma(\tau_i - \tau_j))dt$$

$$F_{12} = -\sum_{i=1}^{P}\sum_{j=1}^{P} \mathrm{Re}\left\{ \frac{2x_i^* x_j}{\gamma N_0} \int_{-\infty}^{+\infty} t s^{*(1)}(t+\gamma(\tau_j - \tau_i)) s^{(1)}(t)dt \right\}$$

$$F_{22} = \sum_{i=1}^{P}\sum_{j=1}^{P} \frac{2x_i^* x_j}{\gamma^3 N_0} \int_{-\infty}^{+\infty} t(t+\gamma(\tau_i - \tau_j)) s^{*(1)}(t) s^{(1)}(t+\gamma(\tau_i - \tau_j))dt$$

$$[\mathbf{F}_{31}]_{i1} = -\sum_{j=1}^{P} \frac{2x_j}{N_0} \int_{-\infty}^{+\infty} s^*(t) s^{(1)}(t+\gamma(\tau_i - \tau_j))dt$$

$$[\mathbf{F}_{32}]_{i1} = \sum_{j=1}^{P} \frac{2x_j}{\gamma^2 N_0} \int_{-\infty}^{+\infty} t s^*(t+\gamma(\tau_j - \tau_i)) s^{(1)}(t)dt$$

$$[\mathbf{F}_{33}]_{ij} = \frac{2}{\gamma N_0} \int_{-\infty}^{+\infty} s^*(t) s(t+\gamma(\tau_i - \tau_j))dt$$

Furthermore, we can express the Fisher information matrix in the form of series:

$$F_{ij} = \lim_{K\to\infty} F_{ij}^{(K)} = \lim_{K\to\infty} F_{1ij}^{(K)} + \sqrt{-1}F_{2ij}^{(K)} \tag{6}$$

$$\mathbf{F}_{3i} = \lim_{K\to+\infty} \mathbf{F}_{3i}^{(K)} = \lim_{K\to+\infty} \mathbf{F}_{13i}^{(K)} + \sqrt{-1}\mathbf{F}_{23i}^{(K)} \tag{7}$$

where $F_{lij}^{(K)}$ and $\mathbf{F}_{lij}^{(K)}$ are given below, $\mathbf{\Gamma}^{(k)} = \left[\Gamma_{ij}^{(k)}\right] \in \mathbb{R}^{P\times P}$, $1 \le i, j \le P, k \in \mathbb{N}$ with $\Gamma_{ij}^{(k)} = (\tau_i - \tau_j)^k$.

$$F_{111}^{(K)} = \sum_{0\le 2k\le K} \frac{(-1)^k 2\gamma^{2k+1}}{(2k)!N_0} M_0^{(k+1)} \mathbf{x}^H \mathbf{\Gamma}^{(2k)} \mathbf{x}$$

$$F_{211}^{(K)} = \sum_{0\le 2k+1\le K} \frac{(-1)^k 2\gamma^{2k+2}}{(2k+1)!N_0} \widetilde{M}_0^{(k+1)} \mathbf{x}^H \mathbf{\Gamma}^{(2k+1)} \mathbf{x}$$

$$F_{112}^{(K)} = \sum_{0\le 2k\le K} \frac{(-1)^{k+1} 2\gamma^{2k-1}}{(2k)!N_0} M_1^{(k+1)} \mathbf{x}^H \mathbf{\Gamma}^{(2k)} \mathbf{x}$$

$$F_{212}^{(K)} = \sum_{0\le 2k+1\le K} \frac{(-1)^{k+1} 2\gamma^{2k}}{(2k+1)!N_0} \widetilde{M}_1^{(k+1)} \mathbf{x}^H \mathbf{\Gamma}^{(2k+1)} \mathbf{x}$$

$$F_{122}^{(K)} = \sum_{1\le 2k\le K} \frac{(-1)^k (k-1)\gamma^{2k-3}}{(2k-1)!N_0} M_0^{(k)} \mathbf{x}^H \mathbf{\Gamma}^{(2k)} \mathbf{x} + \sum_{0\le 2k\le K} \frac{(-1)^k 2\gamma^{2k-3}}{(2k)!N_0} M_2^{(k+1)} \mathbf{x}^H \mathbf{\Gamma}^{(2k)} \mathbf{x}$$

$$F_{222}^{(K)} = \sum_{0\le 2k+1\le K} \frac{(-1)^k 2k^2\gamma^{2k-2}}{(2k+1)!N_0} \widetilde{M}_0^{(k)} \mathbf{x}^H \mathbf{\Gamma}^{(2k+1)} \mathbf{x} + \sum_{0\le 2k+1\le K} \frac{(-1)^k 2\gamma^{2k-2}}{(2k+1)!N_0} \widetilde{M}_2^{(k+1)} \mathbf{x}^H \mathbf{\Gamma}^{(2k+1)} \mathbf{x}$$

$$\mathbf{F}_{131}^{(K)} = \sum_{0\le 2k-1\le K} \frac{(-1)^{k+1} 2\gamma^{2k-1}}{(2k-1)!N_0} M_0^{(k)} \mathbf{\Gamma}^{(2k-1)} \mathbf{x}, \mathbf{F}_{231}^{(K)} = \sum_{0\le 2k\le K} \frac{(-1)^{k+1} 2\gamma^{2k}}{(2k)!N_0} \widetilde{M}_0^{(k)} \mathbf{\Gamma}^{(2k)} \mathbf{x}$$

$$\mathbf{F}_{132}^{(K)} = \sum_{0\le 2k\le K} \frac{(-1)^k (2k-1)\gamma^{2k-2}}{(2k)!N_0} M_0^{(k)} \mathbf{\Gamma}^{(2k)} \mathbf{x} + \sum_{0\le 2k+1\le K} \frac{(-1)^{k+1} 2\gamma^{2k-1}}{(2k+1)!N_0} M_1^{(k+1)} \mathbf{\Gamma}^{(2k+1)} \mathbf{x}$$

$$\mathbf{F}_{232}^{(K)} = \sum_{0\le 2k+1\le K} \frac{(-1)^k 2k\gamma^{2k-1}}{(2k+1)!N_0} \widetilde{M}_0^{(k)} \mathbf{\Gamma}^{(2k+1)} \mathbf{x} + \sum_{0\le 2k\le K} \frac{(-1)^k 2\gamma^{2k-2}}{(2k)!N_0} \widetilde{M}_1^{(k)} \mathbf{\Gamma}^{(2k)} \mathbf{x}$$

$$\mathbf{F}_{133}^{(K)} = \sum_{0\le 2k\le K} \frac{(-1)^k 2\gamma^{2k-1}}{(2k)!N_0} M_0^{(k)} \mathbf{\Gamma}^{(2k)}, \mathbf{F}_{233}^{(K)} = \sum_{0\le 2k+1\le K} \frac{(-1)^k 2\gamma^{2k}}{(2k+1)!N_0} \widetilde{M}_0^{(k)} \mathbf{\Gamma}^{(2k+1)}$$

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Discussions on waveform parameters

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$$\mathrm{CRLB}_{\tau} = O\left(\frac{1}{M_0^{(0)}\bar{B}^2}\right), \mathrm{CRLB}_{\gamma} = O\left(\frac{1}{M_0^{(0)}\bar{B}^2\bar{T}^2}\right)$$

- There exists a positive correlation between the estimation accuracy of the time delay and the effective bandwidth
- The estimation accuracy of the Doppler stretch is positive correlated to the effective time-bandwidth product

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Wide-band ambiguity function estimators and CRLBs

- The wide-band ambiguity function (WBAF) is defined by

$$W_{s_r s_d}(\tau, \gamma) = \sqrt{\gamma} \int_{-\infty}^{+\infty} s_r(t) s_d^*(\gamma(t-\tau))dt$$

where s_r and s_d are the received and reference signals, respectively.

- Two estimators are used to compare with the CRLBs:

1) *Oracle matched filter* $[\hat{\tau}_*, \hat{\gamma}_*] = \arg\max_{\tau,\gamma} W_{s_r s_d}$ with $s_d = \sum_{p=1}^{P} x_p s(\gamma(t-\tau_p))$

2) *WBAF estimator* $[\hat{\tau}, \hat{\gamma}] = \arg\max_{\tau,\gamma} W_{s_r s_d}$ with $s_d = s(\gamma(t-\tau_p))$

- The source signal is chosen as the chirp signal.

Wide-band ambiguity function estimators and CRLBs

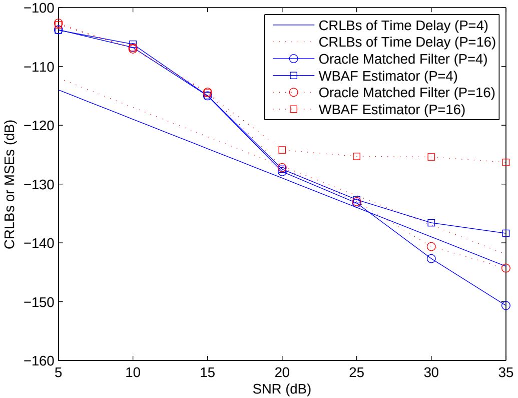

Figure: The CRLBs and MSEs of time delay

Wide-band ambiguity function estimators and CRLBs

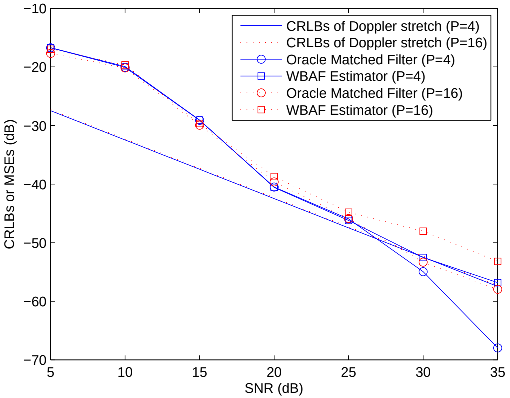

Figure: The CRLBs and MSEs of Doppler stretch

Effective bandwidth and the CRLBs of time delay

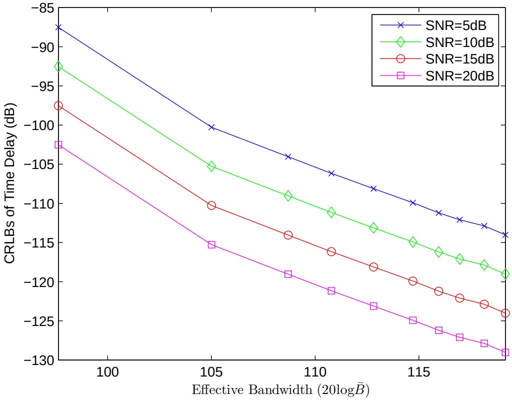

Figure: The influences of effective bandwidth on the CRLBs of time delay

Effective time-bandwidth product and the CRLB of Doppler stretch

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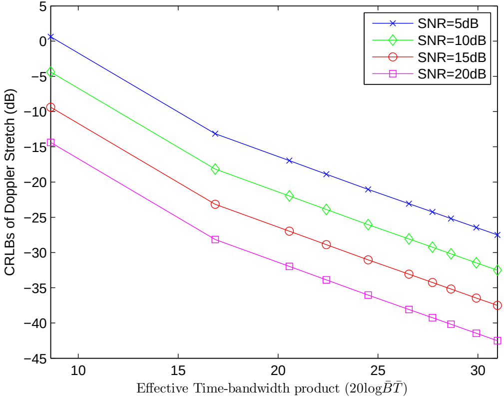

Figure: The impact of effective time-bandwidth product on the CRLB of Doppler stretch. $\bar{T} = (3.7 \pm 0.2) \times 10^{-5}$s and is almost unchanged.

Effective time-bandwidth product and the CRLB of Doppler stretch

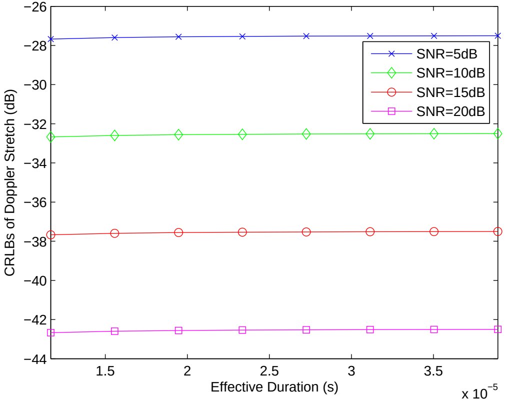

Figure: The impact of effective time-bandwidth product on the CRLB of Doppler stretch. $\bar{B}\bar{T} \equiv 35.3786$.

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Conclusion

- The CRLBs of time delay and Doppler stretch for an extended target are derived.
- The CRLBs of time delay and Doppler stretch are negatively correlated to the effective bandwidth and the effective time-bandwidth product of the transmitted signal, respectively.
- Compared with a point scatterer, an extended target consisting of multiple scatterers leads to higher CRLBs under the same SNR level.

Thanks

Thanks!