When Harmonic Analysis Meets Machine Learning: Lipschitz Analysis of Deep Convolution Networks

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Machine Learning

According to Wikipedia (attributed to Arthur Samuel 1959), "Machine Learning [...] gives computers the ability to learn without being explicitly programmed."
While it has been first coined in 1959, today's machine learning, as a field, evolved from and overlaps with a number of other fields: computational statistics, mathematical optimizations, theory of linear and nonlinear systems.

Machine Learning

According to Wikipedia (attributed to Arthur Samuel 1959), "Machine Learning [...] gives computers the ability to learn without being explicitly programmed."

While it has been first coined in 1959, today's machine learning, as a field, evolved from and overlaps with a number of other fields: computational statistics, mathematical optimizations, theory of linear and nonlinear systems.

Types of problems (tasks) in machine learning:

1. Supervised Learning: The machine (computer) is given pairs of inputs and desired outputs and is left to learn the general association rule.
2. Unsupervised Learning: The machine is given only input data, and is left to discover structures (patterns) in data.
3. Reinforcement Learning: The machine operates in a dynamic environment and had to adapt (learn) continuously as it navigates the problem space (e.g. autonomous vehicle).

Example 1: The AlexNet

The ImageNet Dataset

Dataset: ImageNet dataset [DDSLLF09]. Currently (2017): 14.2 mil.images; 21841 categories; image-net.org

Task: Classify an input image, i.e. place it into one category.

Figure: The "ostrich" category "Struthio Camelus" 1393 pictures. From image-net.org

Example 1: The AlexNet

The Supervised Machine Learning

The AlexNet is 8 layer network, 5 convolutive layers plus 3 dense layers. Introduced by (Alex) Krizhevsky, Sutskever and Hinton in 2012 [KSH12]. Trained on a subset of the ImageNet: Part of the ImageNet Large Scale Visual Recognition Challenge 2010-2012: 1000 object classes and 1,431,167 images.

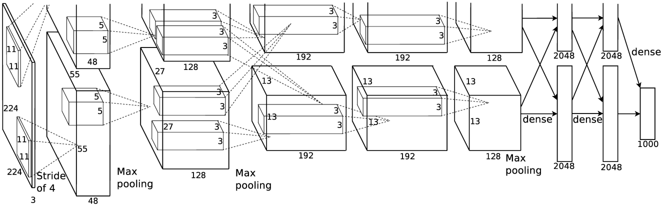


Figure: From Krizhevsky et all 2012 [KSH12]: AlexNet: 5 convolutive layers + 3 dense layers. Input size: 224x224x3 pixels. Output size: 1000.

Example 1: The AlexNet
Adversarial Perturbations

The authors of [SZSBEGF13] (Szegedy, Zaremba, Sutskever, Bruna, Erhan, Goodfellow, Fergus, 'Intriguing properties ...') found small variations of the input, almost imperceptible, that produced completely different classification decisions:

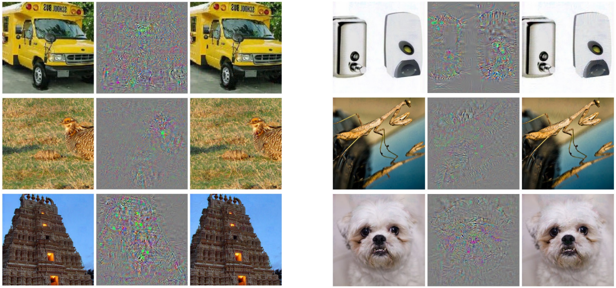


Figure: From Szegedy et all 2013 [SZSBEGF13]: AlexNet: 6 different classes: original image, difference, and adversarial example – all classified as 'ostrich'

Example 1: The AlexNet
Lipschitz Analysis

Szegedy et all 2013 [SZSBEGF13] computed the Lipschitz constants of each layer.

Layer	Size	Sing.Val
Conv. 1	$3 \times 11 \times 11 \times 96$	20
Conv. 2	$96 \times 5 \times 5 \times 256$	10
Conv. 3	$256 \times 3 \times 3 \times 384$	7
Conv. 4	$384 \times 3 \times 3 \times 384$	7.3
Conv. 5	$384 \times 3 \times 3 \times 256$	11
Fully Conn.1	$9216(43264) \times 4096$	3.12
Fully Conn.2	4096×4096	4
Fully Conn.3	4096×1000	4

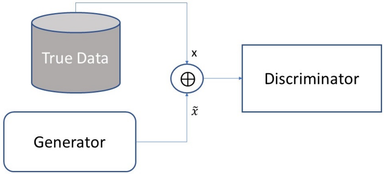
Overall Lipschitz constant:

$$Lip \leq 20 * 10 * 7 * 7.3 * 11 * 3.12 * 4 * 4 = 5,612,006$$

Example 2: Generative Adversarial Networks

The GAN Problem

Two systems are involved: a *generator* network producing synthetic data; a *discriminator* network that has to decide if its input is synthetic data or real-world (true) data:



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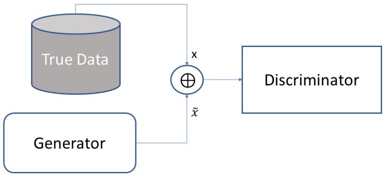

Introduced by Goodfellow et al [GPMXWOCB14] in 2014, GANs solve a minimax optimization problem:

$$\min_G \max_D \mathbb{E}_{x\sim P_r}\left[log(D(x))\right] + \mathbb{E}_{\tilde{x}\sim P_g}\left[log(1 - D(\tilde{x}))\right]$$

where P_r is the distribution of true data, P_g is the generator distribution, and $D : x \mapsto D(x) \in [0, 1]$ is the discriminator map (1 for likely true data; 0 for likely synthetic data).

Example 2: Generative Adversarial Networks

The Wasserstein Optimization Problem

In practice, the training algorithms do not behave well ("saddle point effect").
The Wasserstein GAN (Arjovsky et al [ACB17]) replaces the Jensen-Shannon divergence by the Wasserstein-1 distance:

$$\min_{G} \max_{D \in Lip(1)} \mathbb{E}_{x \sim P_r}\left[D(x)\right] - \mathbb{E}_{\tilde{x} \sim P_g}\left[D(\tilde{x})\right]$$

where $Lip(1)$ denotes the set of Lipschitz functions with constant 1, enforced by weight clipping.

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The Wasserstein Optimization Problem

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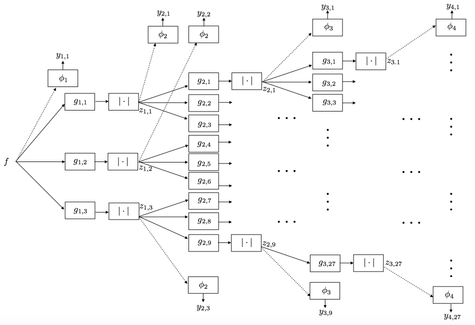
where $Lip(1)$ denotes the set of Lipschitz functions with constant 1, enforced by weight clipping.
Gulrajani et al in [GAADC17] propose to incorporate the Lip(1) condition into the optimization criterion using a soft Lagrange multiplier technique for minimization of:

$$L = \mathbb{E}_{\tilde{x} \sim P_g}[D(x)] - \mathbb{E}_{x \sim P_r}[D(x)] + \lambda \, \mathbb{E}_{\hat{x} \sim P_{\hat{x}}}\left[\|\nabla_{\hat{x}} D(\hat{x})\|_2 - 1)^2 \right]$$

where $\hat{x}$ is sampled uniformly between $x \sim P_r$ and $\tilde{x} \sim P_g$.

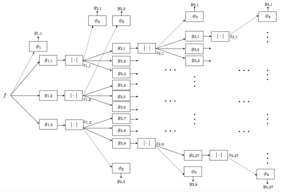
Example 3: The Scattering Network
Topology

Example of Scattering Network; definition and properties: [Mallat12]; this example from [BSZ17]:



Input: f; Outputs: $y = (y_{l,k})$.

Example 3: Scattering Network
Lipschitz Analysis

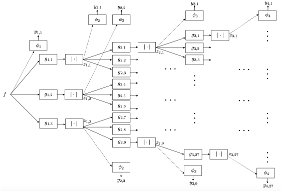


Remarks:

- Outputs from each layer

Example 3: Scattering Network

Lipschitz Analysis



Remarks:

- Outputs from each layer
- Tree-like topology

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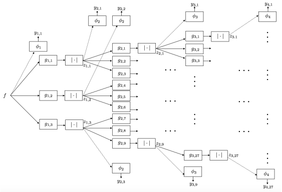
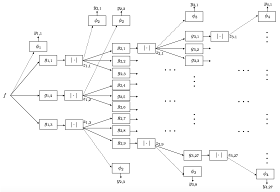


Remarks:

- Outputs from each layer
- Tree-like topology
- Backpropagation/Chain rule: Lipschitz bound 40.

Example 3: Scattering Network

Lipschitz Analysis



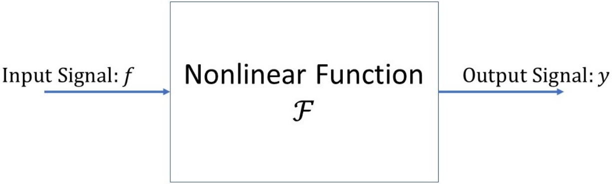
Remarks:

- Outputs from each layer
- Tree-like topology
- Backpropagation/Chain rule: Lipschitz bound 40.
- Mallat's result predicts $Lip = 1$.

Problem Formulation
Nonlinear Maps

Consider a nonlinear function between two metric spaces,

$$\mathcal{F} : (X, d_X) \to (Y, d_Y).$$



Problem Formulation

Lipschitz analysis of nonlinear systems

$$\mathcal{F} : (X, d_X) \to (Y, d_Y)$$

$\mathcal{F}$ is called *Lipschitz* with constant C if for any $f, \tilde{f} \in X$,

$$d_Y(\mathcal{F}(f), \mathcal{F}(\tilde{f})) \leq C \; d_X(f, \tilde{f})$$

The optimal (i.e. smallest) Lipschitz constant is denoted $Lip(\mathcal{F})$. The square C^2 is called Lipschitz bound (similar to the Bessel bound).

$\mathcal{F}$ is called *bi-Lipschitz* with constants $C_1, C_2 > 0$ if for any $f, \tilde{f} \in X$,

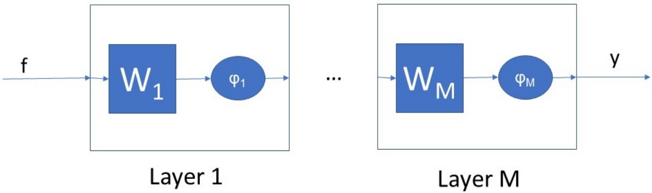
$$C_1 \; d_X(f, \tilde{f}) \leq d_Y(\mathcal{F}(f), \mathcal{F}(\tilde{f})) \leq C_2 \; d_X(f, \tilde{f})$$

The square C_1^2, C_2^2 are called *Lipschitz bounds* (similar to frame bounds).

Problem Formulation

Motivating Examples

Consider the typical neural network as a feature extractor component in a classification system:



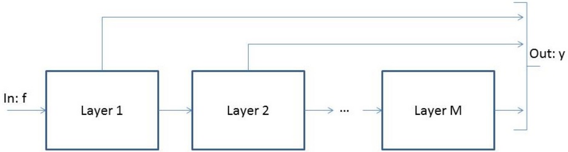
$$g = \mathcal{F}(f) = \mathcal{F}_M(...\mathcal{F}_1(f; W_1, \varphi_1); ...; W_M, \varphi_M)$$

$$\mathcal{F}_m(f; W_m, \varphi_m) = \varphi_m(W_m f)$$

W_m is a linear operator (matrix); φ_m is a Lip(1) scalar nonlinearity (e.g. Rectified Linear Unit).

Problem Formulation
Problem 1

Given a deep network:



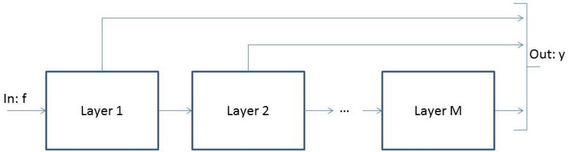
Estimate the Lipschitz constant, or bound:

$$Lip = \sup_{f \neq \tilde{f} \in L^2} \frac{\|y - \tilde{y}\|_2}{\|f - \tilde{f}\|_2} \quad , \quad Bound = \sup_{f \neq \tilde{f} \in L^2} \frac{\|y - \tilde{y}\|_2^2}{\|f - \tilde{f}\|_2^2}.$$

Problem Formulation

Problem 1

Given a deep network:



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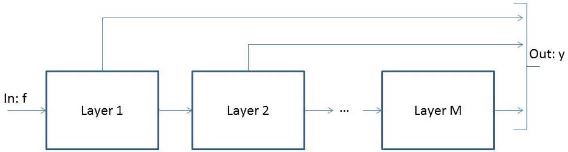
Methods (Approaches):

1. Standard Method: Backpropagation, or chain-rule
2. New Method: Storage function based approach (dissipative systems)
3. Numerical Method: Simulations

Problem Formulation
Problem 2

Given a deep network:

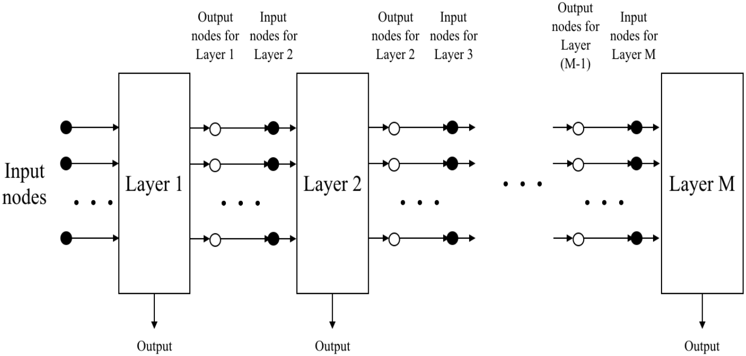


Estimate the stability of the output to specific variations of the input:

1. Invariance to deformations: $\tilde{f}(x) = f(x - \tau(x))$, for some smooth τ.
2. Covariance to such deformations $\tilde{f}(x) = f(x - \tau(x))$, for smooth τ and bandlimited signals f;
3. Tail bounds when f has a known statistical distribution (e.g. normal with known spectral power)

ConvNet
Topology

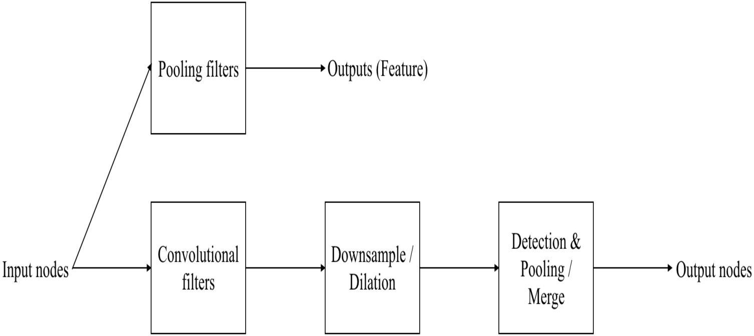
A deep convolution network is composed of multiple layers:



ConvNet

One Layer

Each layer is composed of two or three sublayers: convolution, downsampling, detection/pooling/merge.



ConvNet: Sublayers

Linear Filters: Convolution and Pooling-to-Output Sublayer

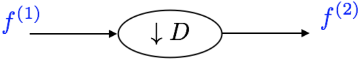
$$f^{(2)} = g * f^{(1)} \quad , \quad f^{(2)}(x) = \int g(x-\xi) f^{(1)}(\xi) d\xi$$

where $g \in \mathcal{B} = \{g \in \mathcal{S}' \ , \ \hat{g} \in L^{\infty}(\mathbb{R}^d)\}$.

$(\mathcal{B}, *)$ is a Banach algebra with norm $\|g\|_{\mathcal{B}} = \|\hat{g}\|_{\infty}$.
Notation: g for regular convolution filters, and Φ for pooling-to-output filters.

ConvNet: Sublayers

Downsampling Sublayer



$$f^{(2)}(x) = f^{(1)}(Dx)$$

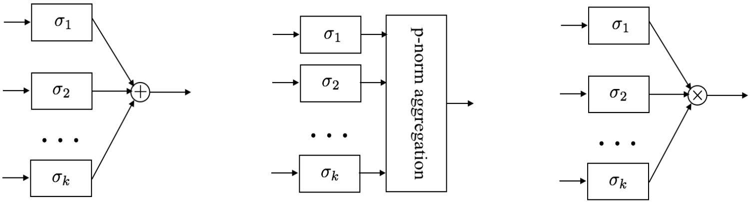
For $f^{(1)} \in L^2(\mathbb{R}^d)$ and $D = D_0 \cdot I$, $f^{(2)} \in L^2(\mathbb{R}^d)$ and

$$\|f^{(2)}\|_2^2 = \int_{\mathbb{R}^d} |f^{(2)}(x)|^2 dx = \frac{1}{|det(D)|} \int_{\mathbb{R}^d} |f^{(1)}(x)|^2 dx = \frac{1}{D_0^d} \|f^{(1)}\|_2^2$$

ConvNet: Sublayers
Detection and Pooling Sublayer

We consider three types of detection/pooling/merge sublayers:

- Type I, τ_1: Componentwise Addition: $z = \sum_{j=1}^{k} \sigma_j(y_j)$
- Type II, τ_2: p-norm aggregation: $z = \left(\sum_{j=1}^{k} |\sigma_j(y_j)|^p\right)^{1/p}$
- Type III, τ_3: Componentwise Multiplication: $z = \prod_{j=1}^{k} \sigma_j(y_j)$

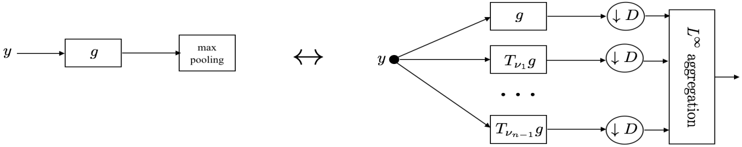


Assumptions: (1) σ_j are scalar Lipschitz functions with $Lip(\sigma_j) \leq 1$; (2) If σ_j is connected to a multiplication block then $\|\sigma_j\|_\infty \leq 1$.

ConvNet: Sublayers

MaxPooling and AveragePooling

MaxPooling can be implemented as follows:



ConvNet: Sublayers

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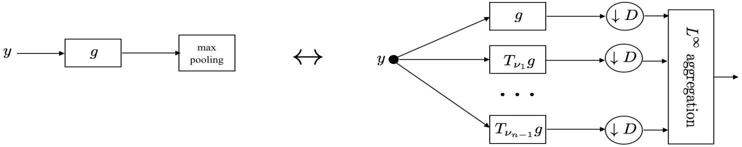
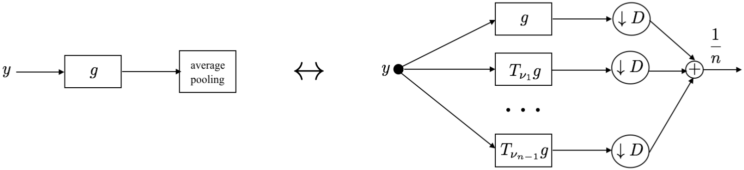
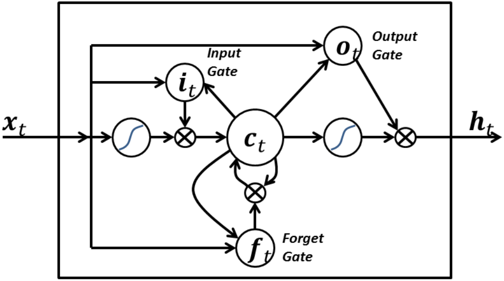


AveragePooling can be implemented as follows:



ConvNet: Sublayers
Long Short-Term Memory



Long Short-Term Memory (LSTM) networks [HS97, GSKSS15].
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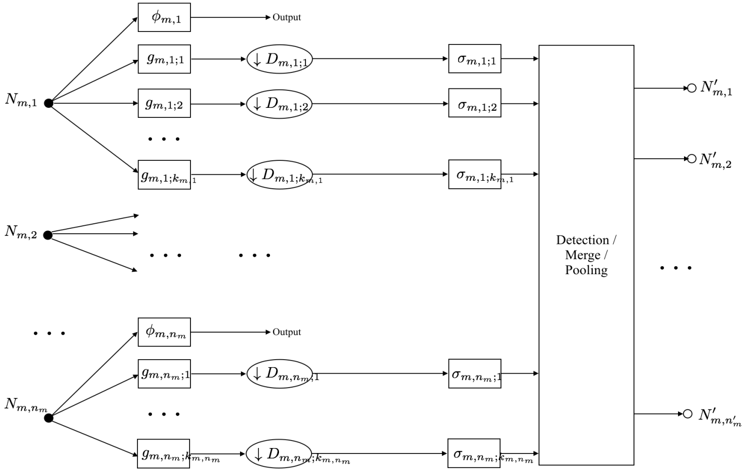
https://commons.wikimedia.org/w/index.php?curid=43992484

ConvNet: Layer m

Components of the m^{th} layer



ConvNet: Layer m

Topology coding of the m^{th} layer

n_m denotes the number of input nodes in the m-th layer:
$\mathcal{I}_m = \{N_{m,1}, N_{m,2}, \cdots, N_{m,n_m}\}$.
Filters:

1. pooling filter: $\phi_{m,n}$ for node n, in layer m;
2. convolution filter: $g_{m,n,k}$ for input node n to output node k, in layer m;

For node n: $G_{m,n} = \{g_{m,n;1}, \cdots g_{m,n;k_{m,n}}\}$.
The set of all convolution filters in layer m: $G_m = \cup_{n=1}^{n_m} G_{m,n}$.

ConvNet: Layer m

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The set of all convolution filters in layer m: $G_m = \cup_{n=1}^{n_m} G_{m,n}$.
$\mathcal{O}_m = \{N'_{m,1}, N'_{m,2}, \cdots, N'_{m,n'_m}\}$ the set of output nodes of the m-th layer.
Note that $n'_m = n_{m+1}$ and there is a one-one correspondence between $\mathcal{O}_m$ and $\mathcal{I}_{m+1}$.
The output nodes automatically partitions G_m into n'_m disjoint subsets $G_m = \cup_{n'=1}^{n'_m} G'_{m,n'}$, where $G'_{m,n'}$ is the set of filters merged into $N'_{m,n'}$.

ConvNet: Layer m

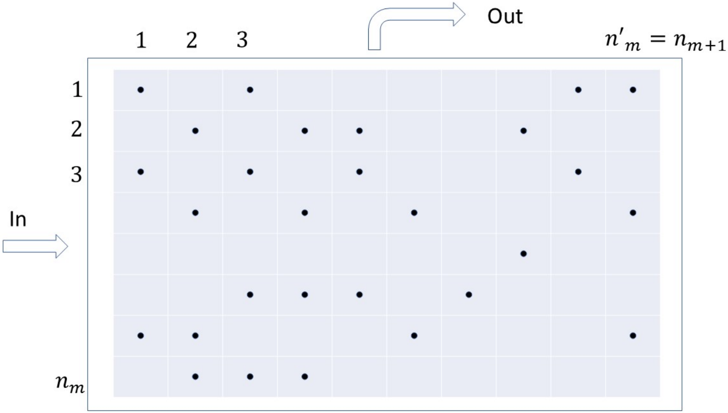
Topology coding of the m^{th} layer

For each filter $g_{m,n;k}$, we define an associated *multiplier* $l_{m,n;k}$ in the following way: suppose $g_{m,n;k} \in G'_{m,k}$, let $K = \left| G'_{m,k} \right|$ denote the cardinality of $G'_{m,k}$. Then

$$l_{m,n;k} = \begin{cases} K & , \text{ if } g_{m,n;k} \in \mathcal{T}_1 \cup \mathcal{T}_3 \\ K^{\max\{0,2/p-1\}} & , \text{ if } g_{m,n;k} \in \mathcal{T}_2 \end{cases} \quad (3.1)$$

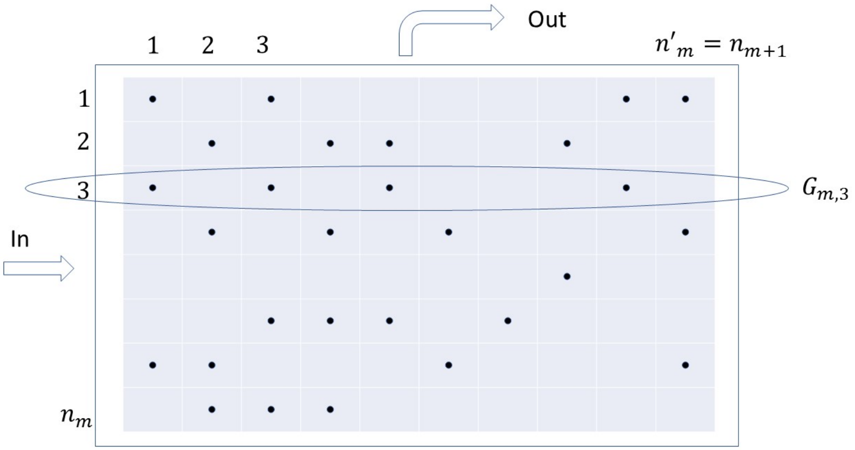
ConvNet: Layer m

Topology coding of the m^{th} layer



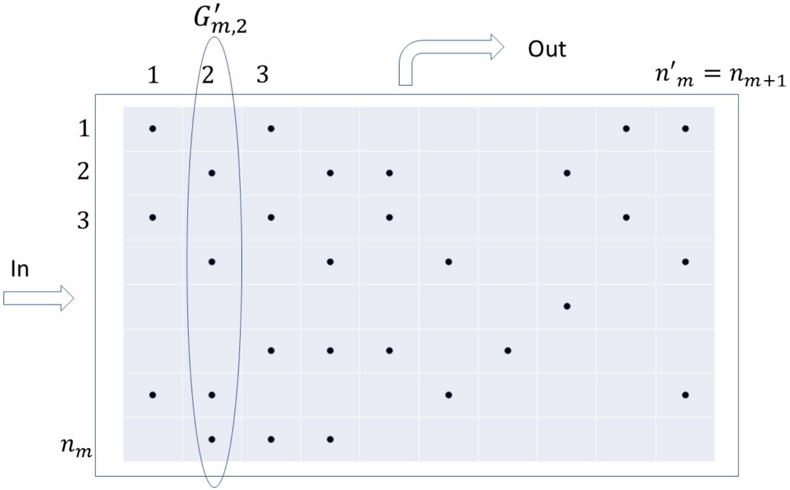
ConvNet: Layer m

Topology coding of the m^{th} layer



ConvNet: Layer m

Topology coding of the m^{th} layer



Layer Analysis
Bessel Bounds

In each layer m and for each *input* node n we define three types of Bessel bounds:

- 1st type Bessel bound:

$$B_{m,n}^{(1)} = \left\| \left|\hat{\phi}_{m,n}\right|^2 + \sum_{g_{m,n;k} \in G_{m,n}} l_{m,n;k} D_{m,n;k}^{-d} \left|\hat{g}_{m,n;k}\right|^2 \right\|_{\infty} \tag{4.2}$$

- 2nd type Bessel bound:

$$B_{m,n}^{(2)} = \left\| \sum_{g_{m,n;k} \in G_{m,n}} l_{m,n;k} D_{m,n;k}^{-d} \left|\hat{g}_{m,n;k}\right|^2 \right\|_{\infty} \tag{4.3}$$

- 3rd type (or generating) bound:

$$B_{m,n}^{(3)} = \left\|\hat{\phi}_{m,n}\right\|_{\infty}^2 . \tag{4.4}$$

Layer Analysis
Bessel Bounds

Next we define the layer m Bessel bounds:

$$\text{1}^{\text{st}} \text{ type Bessel bound} \quad B_m^{(1)} = \max_{1 \leq n \leq n_m} B_{m,n}^{(1)} \tag{4.5}$$

$$\text{2}^{\text{nd}} \text{ type Bessel bound} \quad B_m^{(2)} = \max_{1 \leq n \leq n_m} B_{m,n}^{(2)} \tag{4.6}$$

$$\text{3}^{\text{rd}} \text{ type (generating) Bessel bound} \quad B_m^{(3)} = \max_{1 \leq n \leq n_m} B_{m,n}^{(3)}. \tag{4.7}$$

Remark. These bounds characterize semi-discrete Bessel systems.

Lipschitz Analysis

First Result

Theorem

[BSZ17] Consider a Convolutional Neural Network with M layers as described before, where all scalar nonlinear functions are Lipschitz with $Lip(\varphi_{m,n,n'}) \leq 1$. *Additionally, those* $\varphi_{m,n,n'}$ *that aggregate into a multiplicative block satisfy* $\|\varphi_{m,n,n'}\|_\infty \leq 1$. *Let the m-th layer 1st type Bessel bound be*

$$B_m^{(1)} = \max_{1 \leq n \leq n_m} \left\| \left|\hat{\phi}_{m,n}\right|^2 + \sum_{k=1}^{k_{m,n}} l_{m,n;k} D_{m,n;k}^{-d} \left|\hat{g}_{m,n;k}\right|^2 \right\|_\infty .$$

Then the Lipschitz bound of the entire CNN is upper bounded by $\prod_{m=1}^{M} max(1, B_m^{(1)})$. *Specifically, for any* $f, \tilde{f} \in L^2(\mathbb{R}^d)$:

$$\|\mathcal{F}(f) - \mathcal{F}(\tilde{f})\|_2^2 \leq \left(\prod_{m=1}^{M} max(1, B_m^{(1)})\right) \|f - \tilde{f}\|_2^2,$$

Lipschitz Analysis
Second Result

Theorem

Consider a Convolutional Neural Network with M layers as described before, where all scalar nonlinearities satisfy the same conditions as in the previous result. For layer m, let $B_m^{(1)}$, $B_m^{(2)}$, and $B_m^{(3)}$ denote the three Bessel bounds defined earlier. Denote by L the optimal solution of the following linear program:

$$\begin{aligned}
\Gamma = \max_{y_1,\ldots,y_M,z_1,\ldots,z_M\geq 0} \quad & \sum_{m=1}^{M} z_m \\
s.t. \quad & y_0 = 1 \\
& y_m + z_m \leq B_m^{(1)} y_{m-1}, \quad 1 \leq m \leq M \\
& y_m \leq B_m^{(2)} y_{m-1}, \quad 1 \leq m \leq M \\
& z_m \leq B_m^{(3)} y_{m-1}, \quad 1 \leq m \leq M
\end{aligned} \tag{4.8}$$

Lipschitz Analysis

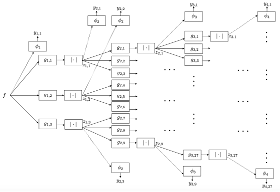
Second Result - cont'd

Theorem

Then the Lipschitz bound satisfies $Lip(\mathcal{F})^2 \leq \Gamma$. *Specifically, for any* $f, \tilde{f} \in L^2(\mathbb{R}^d)$:

$$\|\mathcal{F}(f) - \mathcal{F}(\tilde{f})\|_2^2 \leq \Gamma \|f - \tilde{f}\|_2^2,$$

Example 1: Scattering Network



The Lipschitz constant:

- Backpropagation/Chain rule: Lipschitz bound 40 (hence $Lip \leq 6.3$).

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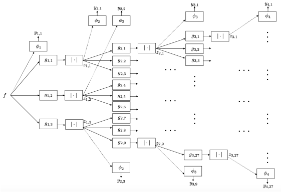
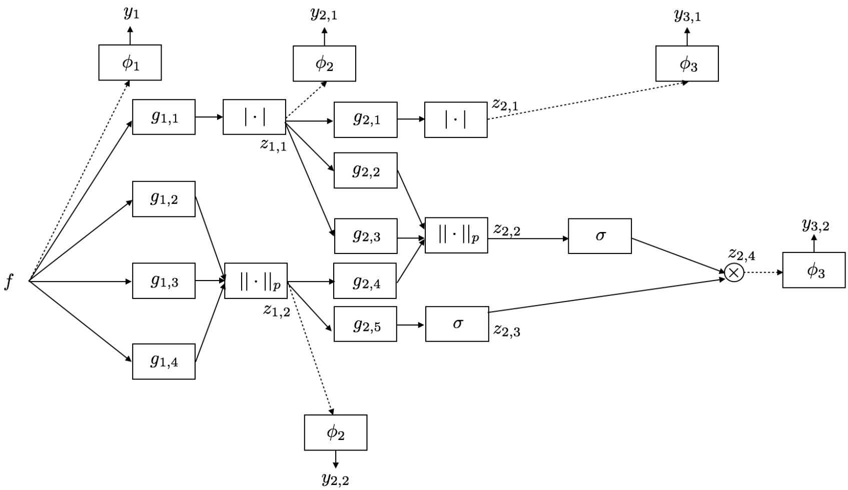


The Lipschitz constant:

- Backpropagation/Chain rule: Lipschitz bound 40 (hence $Lip \leq 6.3$).
- Using our main theorem, $Lip \leq 1$, but Mallat's result: $Lip = 1$.

Filters have been choosen as in a dyadic wavelet decomposition. Thus $B_m^{(1)} = B_m^{(2)} = B_m^{(3)} = 1$, $1 \leq m \leq 4$.

Example 2: A General Convolutive Neural Network



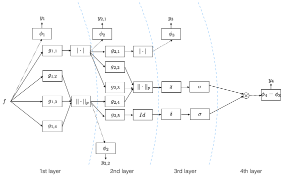
Example 2: A General Convolutive Neural Network

Set $p=2$ and:

$$F(\omega)=\exp(\frac{4\omega^2+4\omega+1}{4\omega^2+4\omega})\chi_{(-1,-1/2)}(\omega)+\chi_{(-1/2,1/2)}(\omega)+\exp(\frac{4\omega^2-4\omega+1}{4\omega^2-4\omega})\chi_{(1/2,1)}(\omega).$$

$$\begin{aligned}
\hat{\phi}_1(\omega) &= F(\omega)\\
\hat{g}_{1,j}(\omega) &= F(\omega+2j-1/2)+F(\omega-2j+1/2)\ ,\ j=1,2,3,4\\
\hat{\phi}_2(\omega) &= \exp(\frac{4\omega^2+12\omega+9}{4\omega^2+12\omega+8})\chi_{(-2,-3/2)}(\omega)+\\
&\quad \chi_{(-3/2,3/2)}(\omega)+\exp(\frac{4\omega^2-12\omega+9}{4\omega^2-12\omega+8})\chi_{(3/2,2)}(\omega)\\
\hat{g}_{2,j}(\omega) &= F(\omega+2j)+F(\omega-2j)\ ,\ j=1,2,3\\
\hat{g}_{2,4}(\omega) &= F(\omega+2)+F(\omega-2)\\
\hat{g}_{2,5}(\omega) &= F(\omega+5)+F(\omega-5)\\
\hat{\phi}_3(\omega) &= \exp(\frac{4\omega^2+20\omega+25}{4\omega^2+20\omega+24})\chi_{(-3,-5/2)}(\omega)+\\
&\quad \chi_{(-5/2,5/2)}(\omega)+\exp(\frac{4\omega^2-20\omega+25}{4\omega^2-20\omega+25})\chi_{(5/2,3)}(\omega).
\end{aligned}$$

Example 2: A General Convolutive Neural Network



Bessel Bounds: $B_m^{(1)} = 2e^{-1/3} = 1.43$, $B_m^{(2)} = B_m^{(3)} = 1$.

The Lipschitz bound:

- Using backpropagation/chain-rule: $Lip^2 \leq 5$.
- Using Theorem 1: $Lip^2 \leq 2.9430$.
- Using Theorem 2 (linear program): $Lip^2 \leq 2.2992$.

Example 3: Lipschitz constant as an optimization criterion

Nonlinear Discriminant Analysis

In Linear Discriminant Analysis (LDA), the objective is to maximize the "separation" between two classes, while controlling the variances within class.

A similar nonlinear *discriminant* can be defined:

$$S = \frac{\|\mathbb{E}[\mathcal{F}(f)|f \in C_1] - \mathbb{E}[\mathcal{F}(f)|f \in C_2]\|^2}{\|Cov(\mathcal{F}(f)|f \in C_1)\|_F + \|Cov(\mathcal{F}(f)|f \in C_2)\|_F}.$$

Example 3: Lipschitz constant as an optimization criterion
Nonlinear Discriminant Analysis

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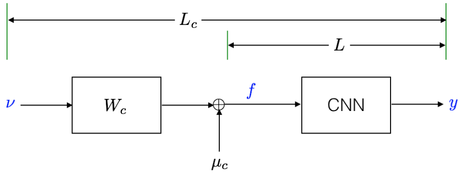
Replace the statistics $\|Cov\|_F$ by Lipschitz bounds:
Lipschitz bound based separation:

$$\tilde{S} = \frac{\|\mathbb{E}[\mathcal{F}(f)|f \in C_1] - \mathbb{E}[\mathcal{F}(f)|f \in C_2]\|^2}{Lip_1^2 + Lip_2^2}.$$

Example 3: Lipschitz constant as an optimization criterion
Nonlinear Discriminant Analysis

The Lipschitz bounds Lip_1^2, Lip_2^2 are computed using Gaussian generative models for the two classes: $(\mu_c, W_c W_c^T)$, where W_c represents the whitening filter for class $c \in \{1,2\}$.



Example 3: Lipschitz constant as an optimization criterion

Numerical Results

Dataset: MNIST database; input images: 28×28 pixels. Two classes: "3" and "8"

Classifier: 3 layer and 4 layer random CNN, followed by a trained SVM.

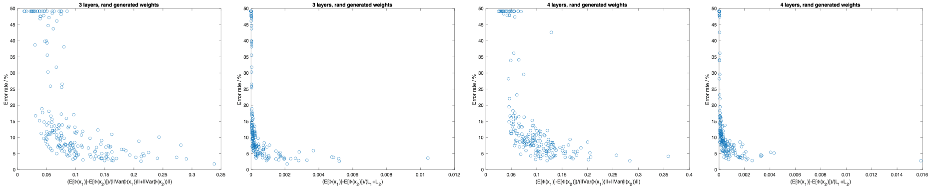


Figure: Results for uniformly distributed random weights

Conclusion: The error rate decreases as the Lipschitz bound separation increases. The discriminant spread is wider.

Example 3: Lipschitz constant as an optimization criterion

Numerical Results

Dataset: MNIST database; input images: 28×28 pixels. Two classes: "3" and "8"
Classifier: 3 layer and 4 layer random CNN, followed by a trained SVM.

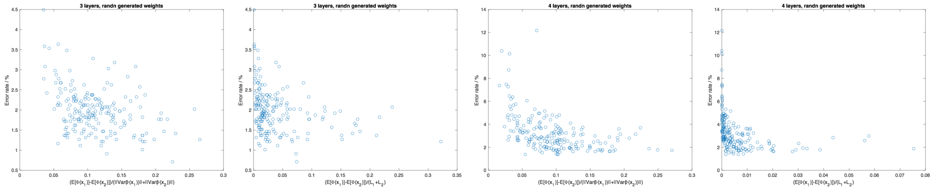


Figure: Results for normaly distributed random weights

References

- [ACB17] M. Arjovsky, S. Chintala, L. Bottou, *Wasserstein GAN*, online arXive:1701.07875, 2017.
- [BSZ17] Radu Balan, Manish Singh, Dongmian Zou, *Lipschitz Properties for Deep Convolutional Networks*, available online arXiv 1701.01527 [cs.LG], 18 Jan 2017.
- [BZ15] Radu Balan, Dongmian Zou, *On Lipschitz Analysis and Lipschitz Synthesis for the Phase Retrieval Problem*, available online arXiv 1506.02092v1 [mathFA], 6 June 2015; Lin. Alg. and Appl. 496 (2016), 152-181.
- [BGC16] Yoshua Bengio, Ian Goodfellow and Aaron Courville, *Deep learning*, MIT Press, 2016.
- [BM13] Joan Bruna and Stephane Mallat, *Invariant scattering convolution networks*, IEEE Transactions on Pattern Analysis and Machine Intelligence **35** (2013), no. 8, 1872–1886.

[BCLPS15] Joan Bruna, Soumith Chintala, Yann LeCun, Serkan Piantino, Arthur Szlam, and Mark Tygert, *A theoretical argument for complex-valued convolutional networks*, CoRR **abs/1503.03438** (2015).

[DDSLLF09] Jia Deng,Wei Dong, Richard Socher, Li-Jia Li, Kai Li, and Li Fei-Fei *Imagenet: A large-scale hierarchical image database*, IEEE Conference on Computer Vision and Pattern Recognition, CVPR 2009, 248–255. 2009.

[GPMXWOCB14] I. Goodfellow, J. Pouget-Abadie, M. Mirza, B. Xu, W. Warde-Farley, S. Ozair, A. Courville, Y. Bengio, *Generative Adverserial Nets*, Advances in Neural Information Processing Systems, 2672–2680, 2014.

[GSKSS15] Klaus Greff, Rupesh K. Srivastava, Jan Koutnik, Bas R. Steunebrink, Jurgen Schmidhuber, *LSTM: A Search Space Odyssey*, arXiv:1503.04069v1 [cs.NE], 13 Mar. 2015.

- [GAADC17] I. Gulrajani, F. Ahmed, M. Arjovsky, V. Dumoulin, A. Courville, *Improved Training of Wasserstein GANs*, online arXiv:1704.00028 [cs.LG], 29 May 2017.
- [HS97] Sepp Hochreiter and Jürgen Schmidhuber, *Long short-term memory*, Neural Comput. **9** (1997), no. 8, 1735–1780.
- [KSH12] Alex Krizhevsky, Ilya Sutskever, and Geoff Hinton, *Imagenet classification with deep convolutional neural networks*, Advances in Neural Information Processing Systems 25, 1106–1114, 2012.
- [LBH15] Yann Lecun, Yoshua Bengio, and Geoffrey Hinton, *Deep learning*, Nature **521** (2015), no. 7553, 436–444.
- [LSS14] Roi Livni, Shai Shalev-Shwartz, and Ohad Shamir, *On the computational efficiency of training neural networks*, Advances in Neural Information Processing Systems 27 (Z. Ghahramani, M. Welling, C. Cortes, N.d. Lawrence, and K.q. Weinberger, eds.), Curran Associates, Inc., 2014, pp. 855–863.

[Mallat12] Stephane Mallat, *Group invariant scattering*, Communications on Pure and Applied Mathematics **65** (2012), no. 10, 1331–1398.

[SVS15] Tara N. Sainath, Oriol Vinyals, Andrew W. Senior, and Hasim Sak, *Convolutional, long short-term memory, fully connected deep neural networks*, 2015 IEEE International Conference on Acoustics, Speech and Signal Processing, ICASSP 2015, South Brisbane, Queensland, Australia, April 19-24, 2015, 2015, pp. 4580–4584.

[SLJSRAEVR15] Christian Szegedy, Wei Liu, Yangqing Jia, Pierre Sermanet, Scott Reed, Dragomir Angelov, Dumitru Erhan, Vincent Vanhoucke, and Andrew Rabinovich, *Going deeper with convolutions*, CVPR 2015, 2015.

[SZSBEGF13] Christian Szegedy, Wojciech Zaremba, Ilya Sutskever, Joan Bruna, Dumitru Erhan, Ian J. Goodfellow, and Rob Fergus, *Intriguing properties of neural networks*, CoRR **abs/1312.6199** (2013).

[WB15a] Thomas Wiatowski and Helmut Bölcskei, *Deep convolutional neural networks based on semi-discrete frames*, Proc. of IEEE International Symposium on Information Theory (ISIT), June 2015, pp. 1212–1216.

[WB15b]Thomas Wiatowski and Helmut Bölcskei, *A mathematical theory of deep convolutional neural networks for feature extraction*, IEEE Transactions on Information Theory (2015).