Design of Unimodular Sequences with Good Autocorrelation and Good Complementary Autocorrelation Properties

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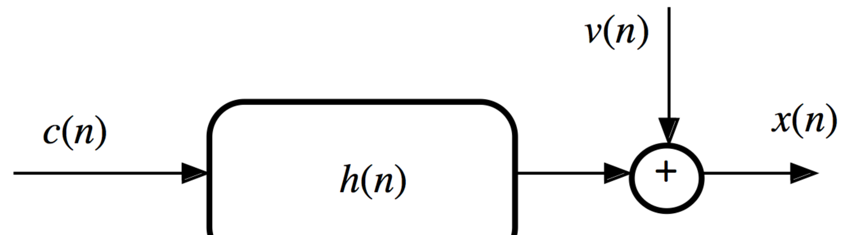
Abstract

The construction of constant magnitude sequences whose aperiodic autocorrelation and aperiodic complementary autocorrelation vanish for a given set of lags is here addressed. The design criterion is based upon the minimization of a generalized weighted integrated sidelobe level.

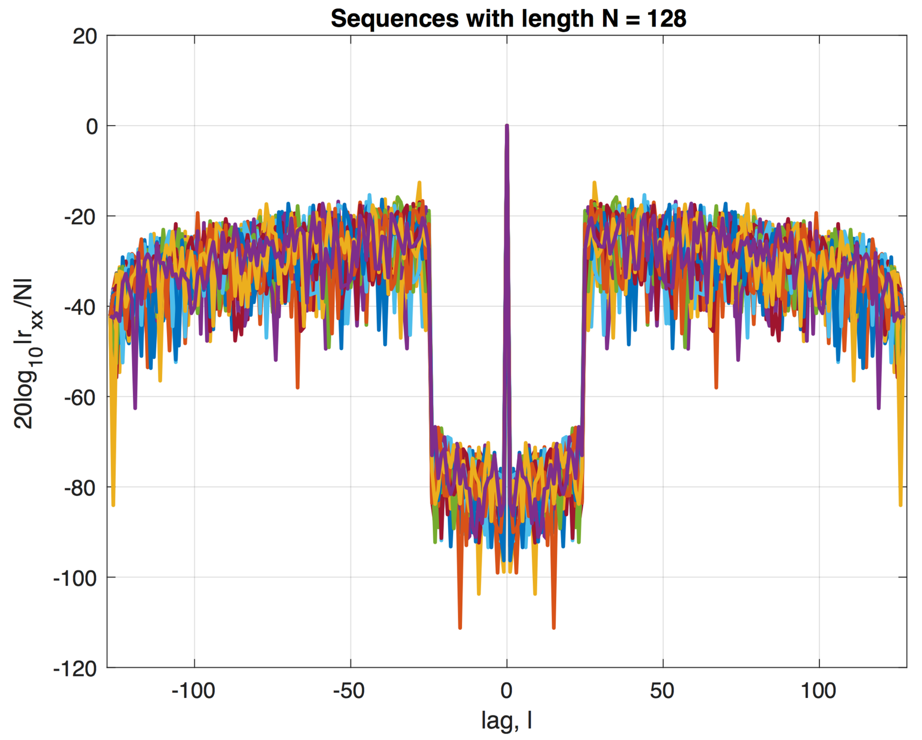
Introduction

- Sequences with good autocorrelation properties are of great interest due to their applications: channel estimation, signal processing for medical applications, radar and active sensing systems among others [1].



$$\mathbf{x} = \mathbf{C}\mathbf{h} + \boldsymbol{v}$$

- The design criterion so far has been based on the minimization of the Weighted Integrated Sidelobe Level (WISL) [2-8]



$$r_{xx}(l) = \sum_{n=l}^{N-1} x(n)x^*(n-l)$$

$$\text{WISL} = \sum_{l=1}^{N-1} \alpha_l |r_{xx}(l)|^2,$$

$$\alpha_l \in \mathbb{R}^+$$

- For strictly linear (SL) processing, it is sufficient to consider only the autocorrelation function, $r_{xx}(l)$.

Problem Statement

- Let $x(n)$ be a complex sequence of length N with $\gamma_{xx}(l)$ denoting its complementary autocorrelation function, which is defined by,

$$\gamma_{xx}(l) = \sum_{n=l}^{N-1} x(n)x(n-l),$$

for $l = -(N-1), -(N-2), \ldots, N-2, N-1$.

- The problem here addressed is finding the elements of $x(n)$, that minimize the **generalized WISL** given by,

$$\text{WISL} = \sum_{l=1}^{N-1} \alpha_l |r_{xx}(l)|^2 + \sum_{l=0}^{N-1} \beta_l \, |\gamma_{xx}(l)|^2, \qquad (1)$$

$$\{\alpha_l\}_{l=1}^{N-1} \cup \{\beta_l\}_{l=0}^{N-1} \subset \mathbb{R}^+$$

subject to the restriction $|x(n)|^2 = 1$ for $n = 0, 1, \ldots, N-1$.

Generalized WISL

- Let $\mathbf{x} = [x(0), x(1), \ldots, x(N-1)]^T$ denote the N dimensional vector containing the elements of $x(n)$. Additionally, define the extended vector $\underline{\mathbf{x}} = [\mathbf{x}^T \; \mathbf{0}^T_{N\mathrm{x}1}]^T$ with dim $\underline{\mathbf{x}} = 2N \times 1$. Then, the design criterion given by (1) can be written as,

$$f(\underline{\mathbf{x}}) = \underline{\mathbf{r}}^H_{xx}\mathbf{M}^H\mathbf{M}\underline{\mathbf{r}}_{xx} + \underline{\boldsymbol{\gamma}}^H_{xx}\mathbf{M}'^H\mathbf{M}'\underline{\boldsymbol{\gamma}}_{xx} \qquad (2)$$

with

$$\underline{\mathbf{r}}_{xx} = \frac{1}{2N}\mathbf{F}_{2N}\left(\left(\mathbf{F}_{2N}\underline{\mathbf{x}}\right)_{-} \circ \left(\mathbf{F}_{2N}\underline{\mathbf{x}}\right)^*_{-}\right)$$

$$\underline{\boldsymbol{\gamma}}_{xx} = \frac{1}{2N}\mathbf{F}_{2N}\left(\left(\mathbf{F}_{2N}\underline{\mathbf{x}}\right) \circ \left(\mathbf{F}_{2N}\underline{\mathbf{x}}\right)_{-}\right)$$

$$\mathbf{M} = \left[\mathbf{0}_{N\times 1} \sqrt{\alpha_1}\mathbf{e}_1 \cdots \sqrt{\alpha_{N-1}}\mathbf{e}_{N-1} \; \mathbf{0}_{N\times N}\right] \qquad (3)$$

$$\mathbf{M}' = \left[\sqrt{\beta_0}\mathbf{e}_0 \sqrt{\beta_1}\mathbf{e}_1 \cdots \sqrt{\beta_{N-1}}\mathbf{e}_{N-1} \; \mathbf{0}_{N\times N}\right]$$

- The matrix $\mathbf{F}_{2N}$ in (3) denotes the Fourier matrix, with dim $\mathbf{F}_{2N} = 2N \times 2N$. Meanwhile $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{N-1}\}$ denotes the standard basis of $\mathbb{C}^{N\mathrm{x}1}$.

Numerical Solution

- If the parameterization $x(n) = e^{i\phi n}$ is considered, then it is possible to define the vector $\boldsymbol{\Phi} = [\phi_0, \phi_1, \ldots, \phi_{N-1}]^T$ and write (2) as,

$$f(\boldsymbol{\Phi}) = \underline{\mathbf{r}}^H_{xx}(\boldsymbol{\Phi})\mathbf{U}\underline{\mathbf{r}}_{xx}(\boldsymbol{\Phi}) + \underline{\boldsymbol{\gamma}}^H_{xx}(\boldsymbol{\Phi})\mathbf{V}\underline{\boldsymbol{\gamma}}_{xx}(\boldsymbol{\Phi}) \qquad (4)$$

whose gradient is given by

$$\frac{\partial}{\partial\boldsymbol{\Phi}} f(\boldsymbol{\Phi}) = 2\mathrm{Re}\left\{\left(\frac{\partial}{\partial\boldsymbol{\Phi}}\underline{\mathbf{r}}_{xx}(\boldsymbol{\Phi})\right)^T \mathbf{U}\underline{\mathbf{r}}^*_{xx}(\boldsymbol{\Phi}) + \left(\frac{\partial}{\partial\boldsymbol{\Phi}}\underline{\boldsymbol{\gamma}}_{xx}(\boldsymbol{\Phi})\right)^T \mathbf{V}\underline{\boldsymbol{\gamma}}^*_{xx}(\boldsymbol{\Phi})\right\}$$

with $\mathbf{U} = \mathbf{M}^H\mathbf{M}$ and $\mathbf{V} = \mathbf{M}'^H\mathbf{M}'$.

- The cost function given by (4) can be minimized using optimization techniques such as the limited memory Broyden-Fletcher-Goldfarb-Shanno [9].

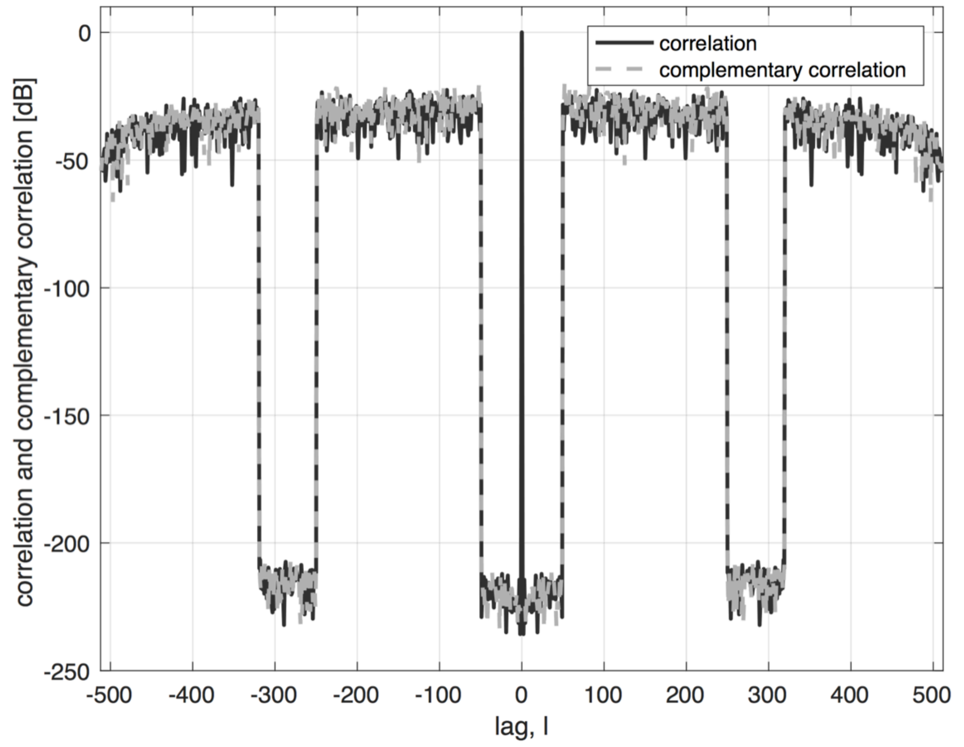
Example 1. Minimization of generalized WISL

- Here we consider the design of unimodular sequences with length $N = 512$ that minimizes the generalized WISL with the following weights

$$\alpha_l = \begin{cases} 1 \; if \; l \in \{1, 2, \ldots, 49\} \cup \{250, 251, \ldots, 319\} \\ 0 \qquad \text{otherwise} \end{cases}$$

$$\beta_l = \begin{cases} 1 \; if \; l \in \{0, 1, \ldots, 49\} \cup \{250, 251, \ldots, 319\} \\ 0 \qquad \text{otherwise} \end{cases}$$

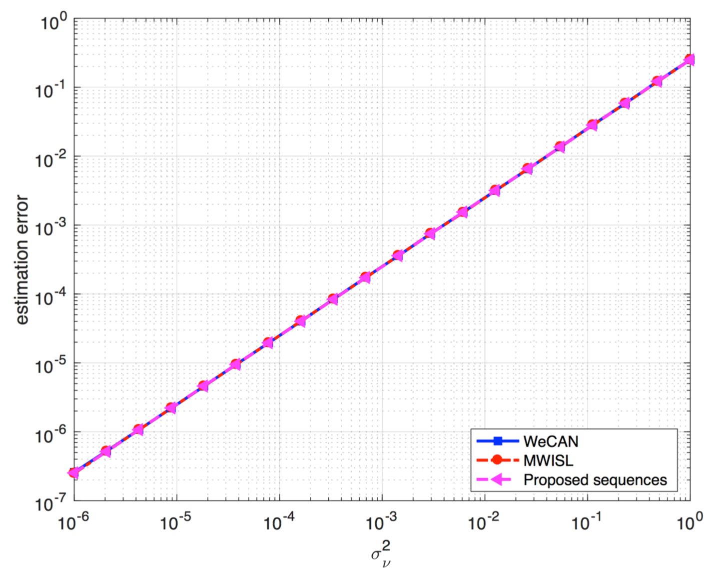
- The minimization of (4) is done using the L-BFGS technique as implemented in Scipy, using a random point $\boldsymbol{\Phi}_0$ to initialize the optimization method.

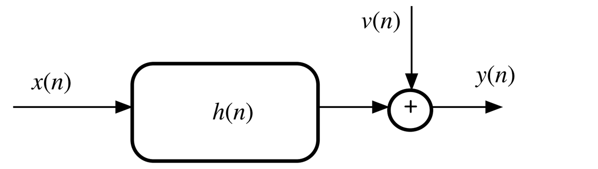


Example 2. SL and WL system identification

- The experiment consists in generating the impulse response $h(n)$ of a system with $L = 40$ coefficients from a complex normal distribution. The output of the system is affected by additive white Gaussian noise (AWGN).

- A sequence of length $N = 160$ is used to probe the system: The sequence was designed using the weights $\alpha_l = 1$ for $l = 1, 2, \ldots, 39$ and $\beta_l = 1$ for $l = 0, 1, \ldots, 39$.





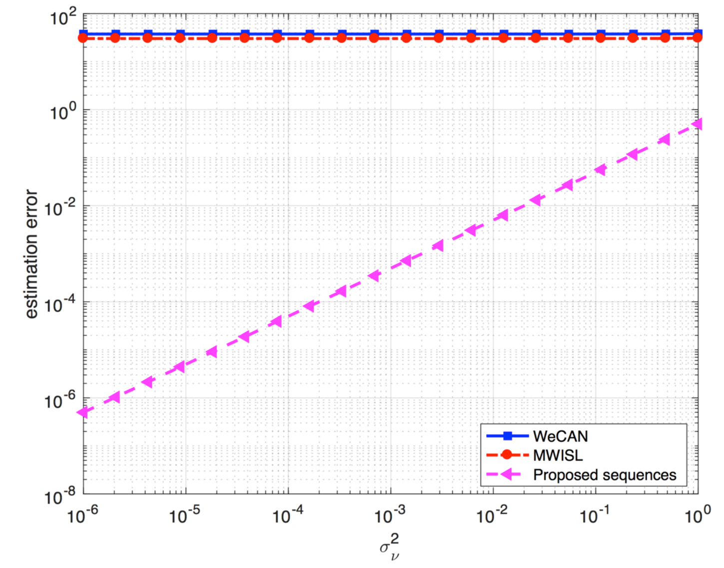
$$\hat{h}(n) = \mathbf{x}^H_n\mathbf{y}, \qquad n = 0, 1, \ldots, L-1$$

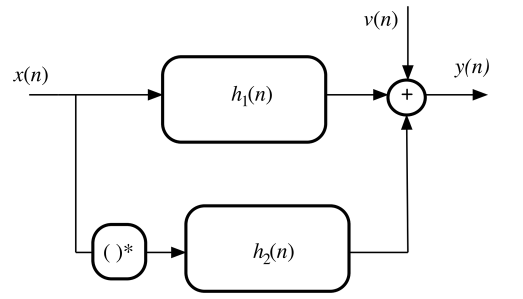
$$\mathbf{x}_n = \left[\mathbf{0}^T_{n\times 1} \, \mathbf{x}^T \, \mathbf{0}^T_{(L-n-1)\times 1}\right]^T$$

$$\mathbf{x} = [x(0), x(1), \ldots, x(N-1)]^T$$

$$\mathbf{y} = [y(0), y(1), \ldots, y(N+L-1)]^T$$

- Same experiment as the previous, now identifying the responses $h_1(n)$ and $h_2(n)$ of a WL system.





$$h_1(n) = \mathbf{x}^H_n\mathbf{y}$$

$$h_2(n) = \mathbf{x}^T_n\mathbf{y}$$

Conclusions

A generalized WISL criterion to design sequences with low autocorrelation and low complementary autocorrelation coefficients in a region of interest has been proposed. They can be used in the estimation of SL and WL systems.

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