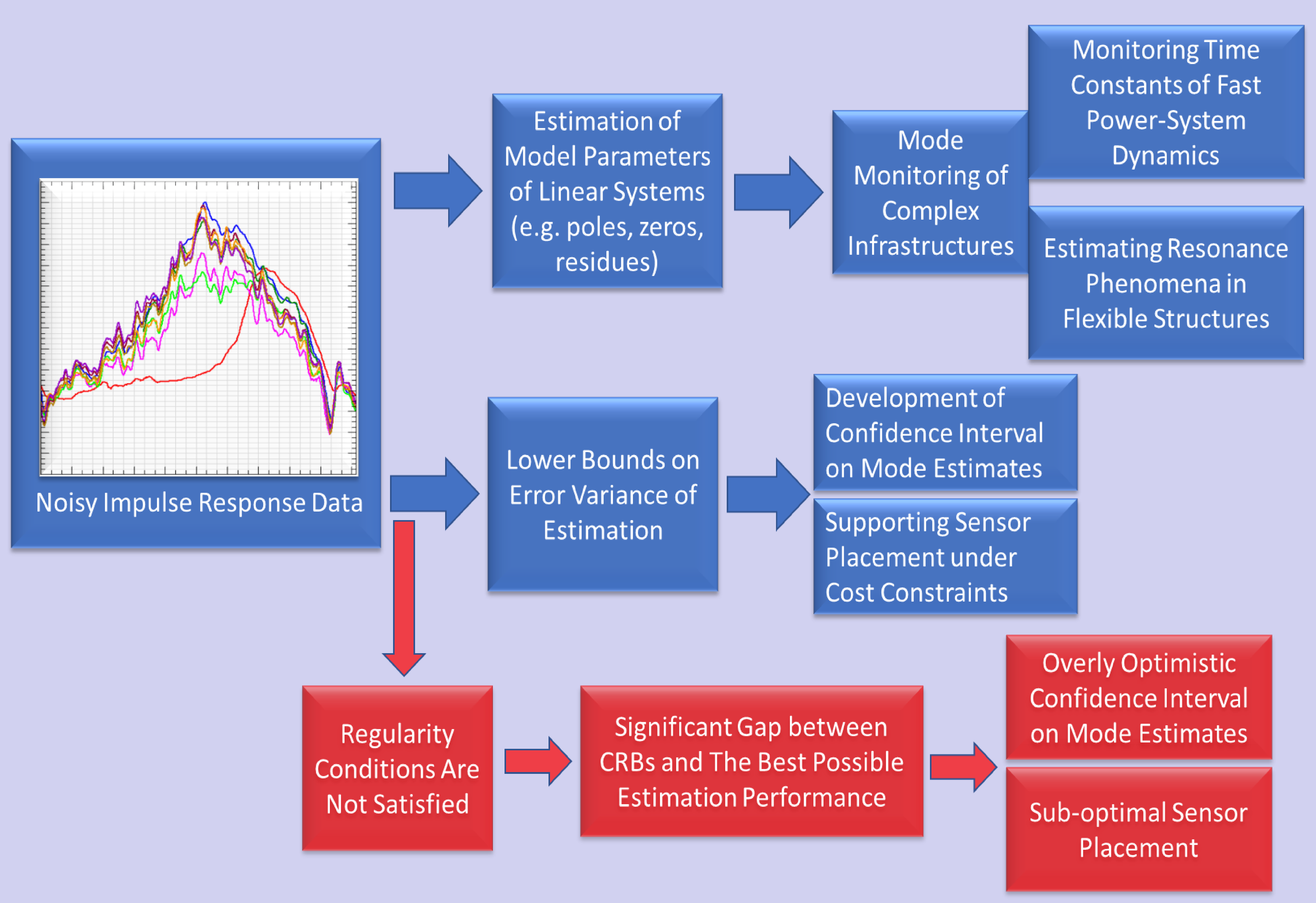
HAMMERSLEY-CHAPMAN-ROBBINS LOWER BOUNDS ON POLE AND RESIDUE ESTIMATES FROM IMPULSE RESPONSE DATA

Abdullah Al Maruf and Sandip Roy
Washington State University

Abstract

The estimation of nonrandom pole and residue parameters from impulse-response data is studied. Specifically, the Hammersley-Chapman-Robbins lower bound (HCRB) on the estimation error variance is analyzed for single-input single-output systems with multiple but distinct poles. The HCRB is compared with the widely used Cramer-Rao lower bound (CRB) in examples. The HCRB is found to be significantly tighter than the CRB when noise levels are high compared to the impulse response signal, while the bounds become close for small noise levels (equivalently, large residues).

Motivation



Objective

Development and characterization of tighter Hammersley-Chapman-Robbins lower bound (HCRB) on estimation of nonrandom parameter vector $\boldsymbol{\theta} = [a_1\ a_2 \dots a_r\ A_1\ A_2 \dots A_r]^T$ from impulse response data of r distinct pole system corrupted by zero mean Gaussian white noise with variance σ^2 given by:

$$y(k) = \sum_{l=1}^{r} A_l\, a_l^k + w(k) \qquad k = 0,1,\dots,n$$

Hammersley-Chapman-Robbins Lower Bound (HCRB)

$$HCRB = \sup\left(\left([\boldsymbol{v}_1 \dots \boldsymbol{v}_k]\, E_{\boldsymbol{\theta}}\left\{\left[\frac{f_{\boldsymbol{\theta}+h_1\boldsymbol{v}_1} - f_{\boldsymbol{\theta}}}{f_{\boldsymbol{\theta}}\, h_1} \dots \frac{f_{\theta+h_k v_k} - f_{\boldsymbol{\theta}}}{f_{\boldsymbol{\theta}}\, h_k}\right]^T \left[\frac{f_{\boldsymbol{\theta}+h_1\boldsymbol{v}_1} - f_{\boldsymbol{\theta}}}{f_{\boldsymbol{\theta}}\, h_1} \dots \frac{f_{\theta+h_k v_k} - f_{\boldsymbol{\theta}}}{f_{\boldsymbol{\theta}}\, h_k}\right]\right\}[\boldsymbol{v}_1 \dots \boldsymbol{v}_k]^T\right)^{\dagger}\right)$$

where $f_{\boldsymbol{\theta}}$ is the joint density function of the observations with parameter value $\boldsymbol{\theta}$, and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k \in \mathbb{R}^k$ are k mutually independent directions, and $h_1, \dots, h_k$ are scalers and supremum are taken over them.

Development

The $HCRB$ for the pole and residue estimation problem is:

$$HCRB = \sup\ ({\boldsymbol{G}_{HCRB}}^{\dagger}) = \sup\left(\begin{bmatrix}\boldsymbol{G}^{aa} & \boldsymbol{G}^{aA} \\ \boldsymbol{G}^{Aa} & \boldsymbol{G}^{AA}\end{bmatrix}^{\dagger}\right)$$

Where $\boldsymbol{G}^{aa}, \boldsymbol{G}^{aA}, \boldsymbol{G}^{Aa}$ and $\boldsymbol{G}^{AA}$ are $r \times r$ matrices whose (i,j) entries are given by

$$G_{ij}^{aa} = \frac{1}{h_{a_i}h_{a_j}}\left[\exp\left(\frac{A_i\, A_j}{\sigma^2}\left[\frac{1-(a_i+h_{a_i})^{n+1}(a_j+h_{a_j})^{n+1}}{1-(a_i+h_{a_i})(a_j+h_{a_j})} + \frac{1-(a_i a_j)^{n+1}}{1-a_i a_j} - \frac{1-{a_i}^{n+1}(a_j+h_{a_j})^{n+1}}{1-a_i(a_j+h_{a_j})} - \frac{1-{a_j}^{n+1}(a_i+h_{a_i})^{n+1}}{1-a_j(a_i+h_{a_i})}\right]\right)-1\right]$$

$$G_{ij}^{aA} = G_{ji}^{Aa} = \frac{1}{h_{a_i}h_{A_j}}\left[\exp\left(\frac{A_i\, h_{A_j}}{\sigma^2}\left[\frac{1-{a_j}^{n+1}(a_i+h_{a_i})^{n+1}}{1-a_j(a_i+h_{a_i})} - \frac{1-(a_i a_j)^{n+1}}{1-a_i a_j}\right]\right)-1\right]$$

$$G_{ij}^{AA} = \frac{1}{h_{A_i}h_{A_j}}\left[\exp\left(\frac{h_{A_i}\, h_{A_j}}{\sigma^2}\,\frac{1-(a_i a_j)^{n+1}}{1-a_i a_j}\right)-1\right]$$

Here, the supremum is found with respect to $h_{a_1}, \dots, h_{a_r}$ and $h_{A_1}, \dots, h_{A_r}$, and any scalar quadratic form $\boldsymbol{z}^T({\boldsymbol{G}_{HCRB}}^{\dagger})\boldsymbol{z}$ may be supremized to obtain a bound.

Numerical Examples

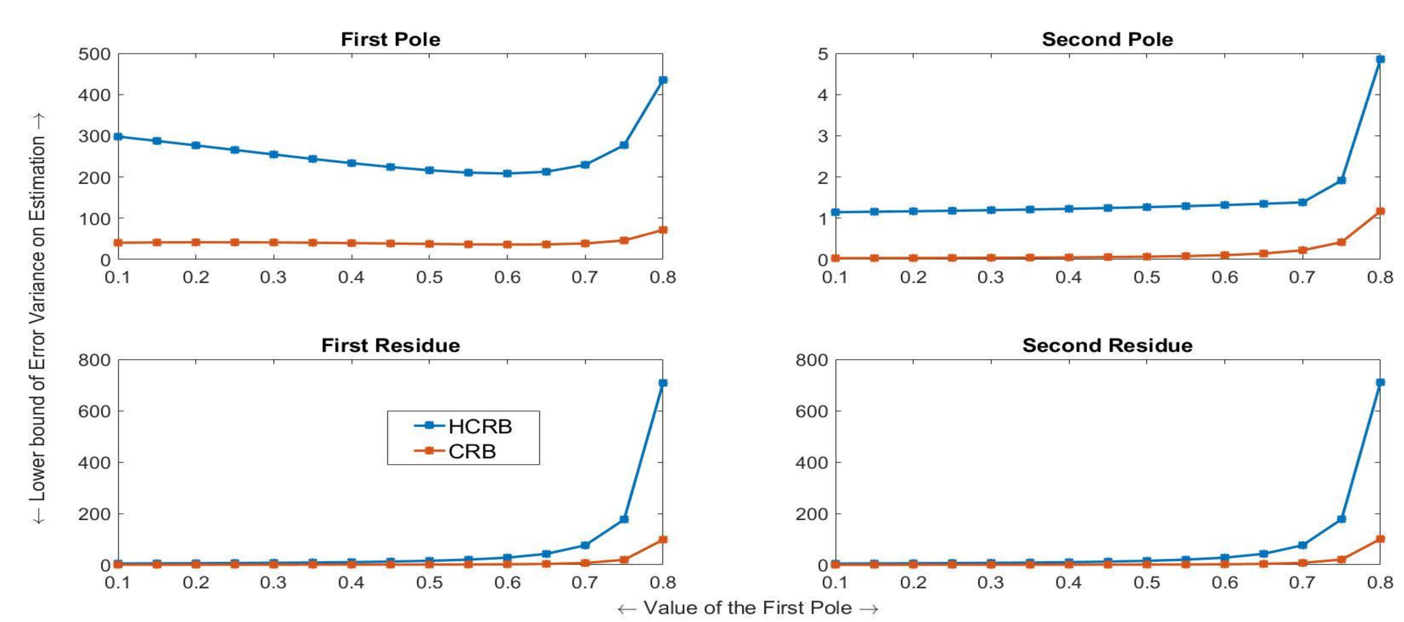


Fig. 1. **Two-pole system:** Lower bounds on the pole and residue estimation error variances as a function of the location of pole 1 (a_1), with $a_2 = 0.9$, $A_1 = 0.1$, $A_2 = 0.3$, $N = 10000$, $\sigma = 0.5$.

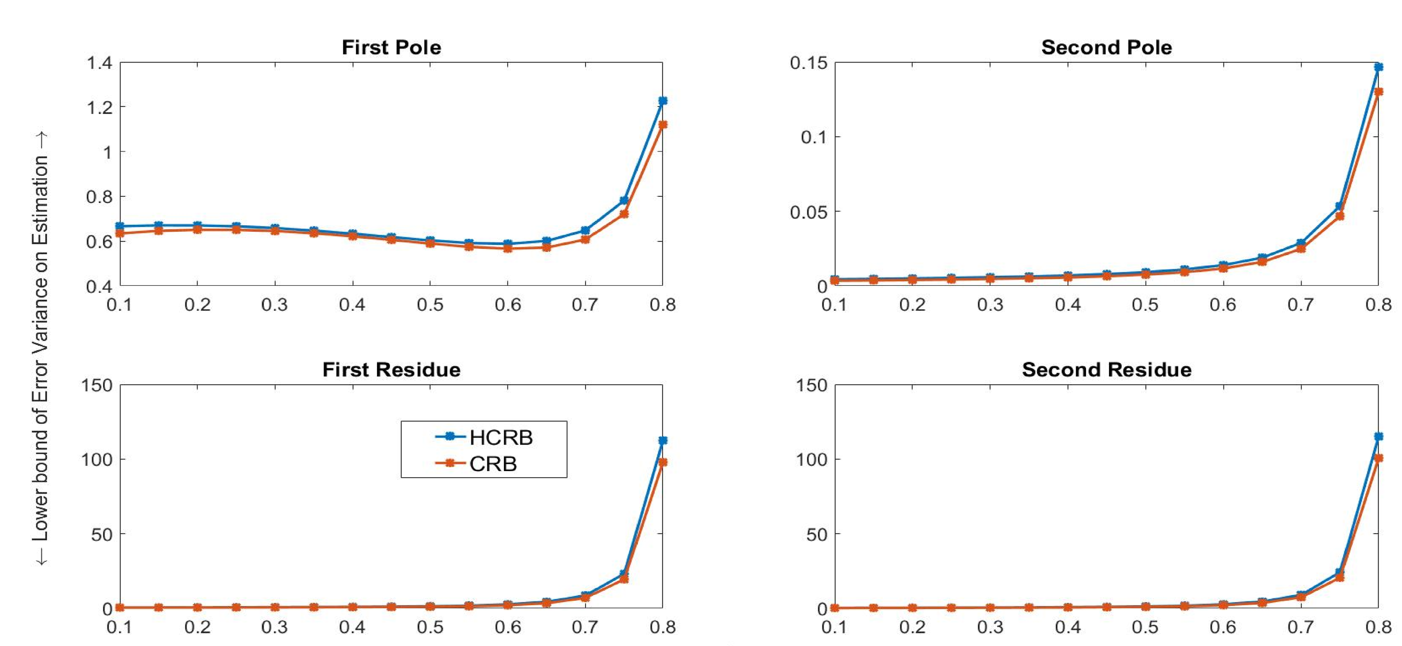


Fig. 2. **Two-pole system:** Lower bounds on the pole and residue estimation error variances as a function of the location of pole 1 (a_1), with $a_2 = 0.9$, $A_1 = 0.8$, $A_2 = 0.9$, $N = 10000$, $\sigma = 0.5$.

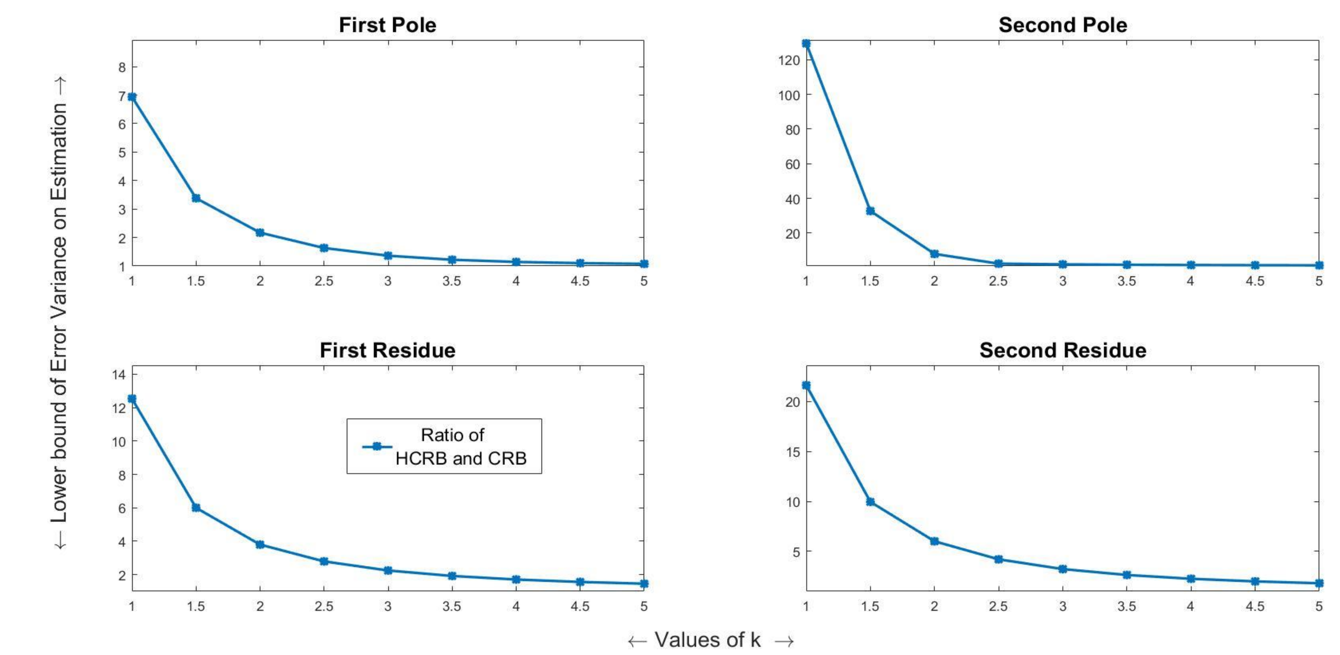


Fig. 3. **Two-pole system**: Ratio between the HCRB and CRB as the residues are scaled as $A_1 = 0.1k$ and $A_2 = 0.3k$, for the pole locations $a_1 = 0.2$, $a_2 = 0.9$ and $N = 10000$, $\sigma = 0.5$.

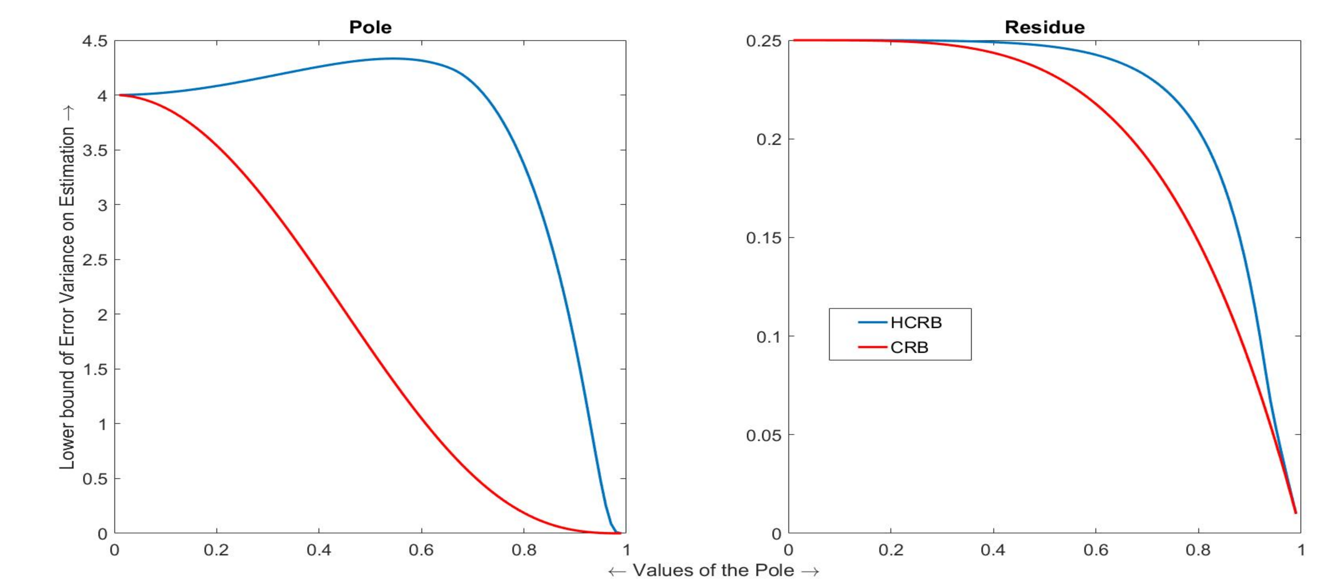


Fig. 4. **Single pole system**: lower bounds on the pole and residue estimation error variance as a function of the location of pole (a) for $A = 0.25$, $N = 10000$, $\sigma = 0.5$.

Observations and Future Work

Observations:

1. CRB is close to the HCRB if the noise level is sufficiently small (equivalently, the residues are sufficiently large), i.e. the signal-to-noise ratio is high.
2. The ratio between the HCRB and CRB increases and reaches an asymptote as the signal-to-noise ratio is decreased.

Future work: Trying to prove the observations by expressing the Taylor expansion of the exponential functions in the HCRB expressions in terms of Hadamard powers. The result has already been proved for the single-pole system.

Acknowledgement

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