Numerical differentiation of noisy, nonsmooth, multidimensional data

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November 16, 2017

A mathematician is a device for turning coffee into theorems.
–Alfréd Rényi

What we really need is a machine to turn some of those theorems back into coffee.
–A. J. Tolland

Outline

Regularized differentiation

ADMM algorithm

Application: phase unwrapping

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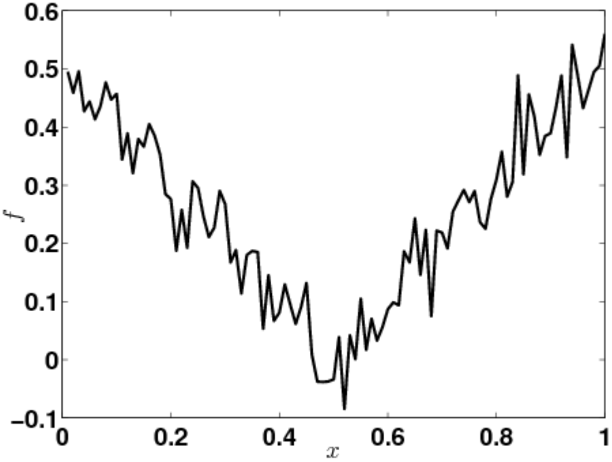
Motivation

Consider a function f defined by noisy data.



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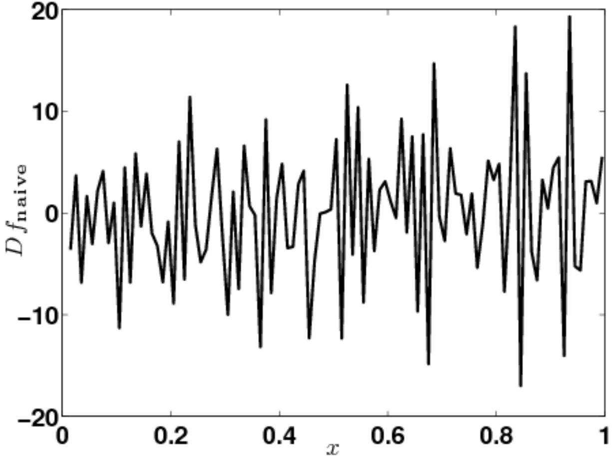
Motivation

Computing the derivative Df in a naive fashion gives a terrible result.



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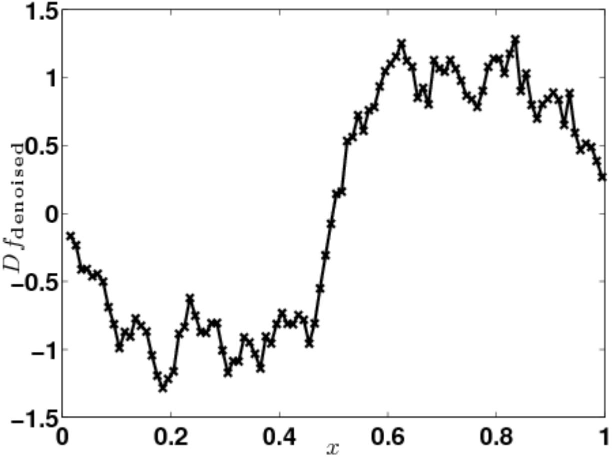
Motivation

Denoising the data first (in this case, using total-variation regularization) and then differentiating improves the result, but we can do better.



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Differentiation as an inverse problem

- We formulate the differentiation as an inverse problem: find a function $\boldsymbol{u}$ whose antiderivative is approximately equal to $\boldsymbol{f}$: $\boldsymbol{Ku \approx f}$.
- By regularizing this inverse problem, we can enforce the condition that $\boldsymbol{u}$ not be noisy.

$$\min_{\boldsymbol{u}} \|\boldsymbol{Du}\|_1 + \frac{\boldsymbol{\mu}}{\mathbf{2}}\|\boldsymbol{Ku - f}\|_2^2.$$

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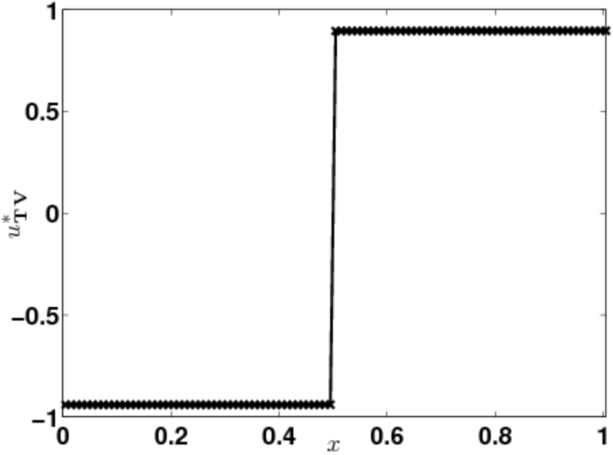
Result

The TV-regularized derivative is noisefree, and captures the essential qualities of the derivative, including the discontinuity.



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2D case

The same framework applies when f has 2 (or more) dimensions.

$$\min_{u} \|Du\|_1 + \frac{\mu}{2}\|Ku - f\|_2^2.$$

However:

- u is vector valued.
- Du is matrix valued.
- Computational efficiency is more important.

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Variable splitting

We introduce a variable w, a proxy for Du.

$$\min_{w,u} \|w\|_1 + \frac{1}{2\lambda}\|w - Du\|_2^2 + \frac{\mu}{2}\|Ku - f\|_2^2.$$

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Variable splitting

We introduce a variable w, a proxy for Du.

$$\min_{w,u} \|w\|_1 + \frac{1}{2\lambda}\|w - Du\|_2^2 + \frac{\mu}{2}\|Ku - f\|_2^2.$$

Then we alternate between solving for each variable, with the other fixed. Each subproblem is much easier than the original problem.

$$\min_{w} \|w\|_1 + \frac{1}{2\lambda}\|w - Du\|_2^2,$$
$$\min_{u} \frac{1}{2\lambda}\|w - Du\|_2^2 + \frac{\mu}{2}\|Ku - f\|_2^2.$$

Shrinkage

The w subproblem has an explicit solution:

$$\arg\min_{w} \|w\|_1 + \frac{1}{2\lambda}\|w - Du\|_2^2 = S_1(Du, \lambda),$$

where

$$S_1(x, \lambda) = \max\{\|x\| - \lambda, 0\}\frac{x}{\|x\|}$$

is known as *soft thresholding*.

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is known as *soft thresholding*.

A modification of the ℓ^1 norm lets us use p-*shrinkage* instead:

$$S_p(x, \lambda) = \max\{\|x\| - \lambda^{2-p}x^{p-1}, 0\}\frac{x}{\|x\|},$$

which for $p < 1$ can give better results.

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Calculus in the Fourier domain

The $\boldsymbol{u}$-subproblem is quadratic, giving us a linear equation to solve:

$$\left(\frac{1}{\lambda}D^TD + \mu K^TK\right)u = \frac{1}{\lambda}D^Tw + \mu K^Tf. \tag{1}$$

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Calculus in the Fourier domain

The u-subproblem is quadratic, giving us a linear equation to solve:

$$(\frac{1}{\lambda}D^T D + \mu K^T K)u = \frac{1}{\lambda}D^T w + \mu K^T f. \quad (1)$$

If we use periodic boundary conditions for the differentiation, then D is diagonalized by the discrete Fourier transform.

Calculus in the Fourier domain

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If we use periodic boundary conditions for the differentiation, then $\boldsymbol{D}$ is diagonalized by the discrete Fourier transform.

We define our antidifferentiation $\boldsymbol{K}$ to also be diagonalized by the discrete Fourier transform. Then (1) can be solved via an FFT, pointwise division by a fixed kernel, and an inverse FFT.

The method of multipliers

We can enforce the equality constraint $\boldsymbol{w = Du}$ by including a Lagrange multiplier,

$$\min_{\boldsymbol{w},\boldsymbol{u}} \|\boldsymbol{w}\|_1 + \frac{1}{2\lambda}\|\boldsymbol{w} - \boldsymbol{D}\boldsymbol{u} - \boldsymbol{\Lambda}\|_2^2 + \frac{\mu}{2}\|\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f}\|_2^2,$$

which is updated each iteration by adding the residual:

$$\boldsymbol{\Lambda}^{n+1} = \boldsymbol{\Lambda}^n + \boldsymbol{D}\boldsymbol{u}^{n+1} - \boldsymbol{w}^{n+1}.$$

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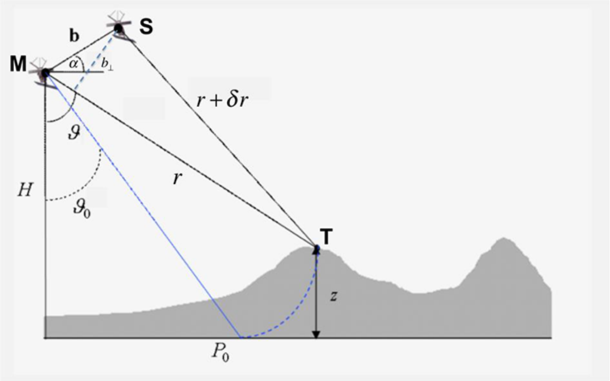
Interferometric SAR

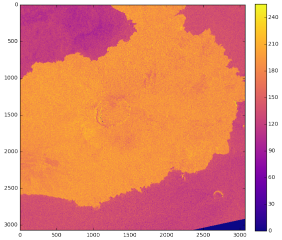
Given synthetic aperture radar images from two satellite passes, the pixelwise difference in phase is a function of the elevation, and any elevation changes between the two passes.



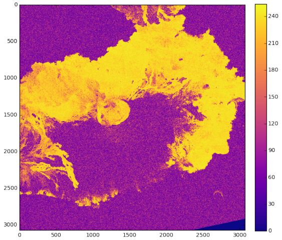
From G. Solaro, P. Imperatore, and A. Pepe, Satellite SAR interferometry for Earth's crust deformation monitoring and geological phenomena analysis, *InTech* 2016.

Sentinel-1 example

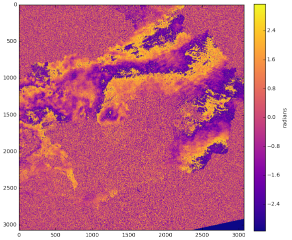
We look at an interferogram from the Descartes Labs platform, using Sentinel-1A imagery of Isla Isabela in the Galápagos Islands on April 7 and April 19, 2017.



amplitude



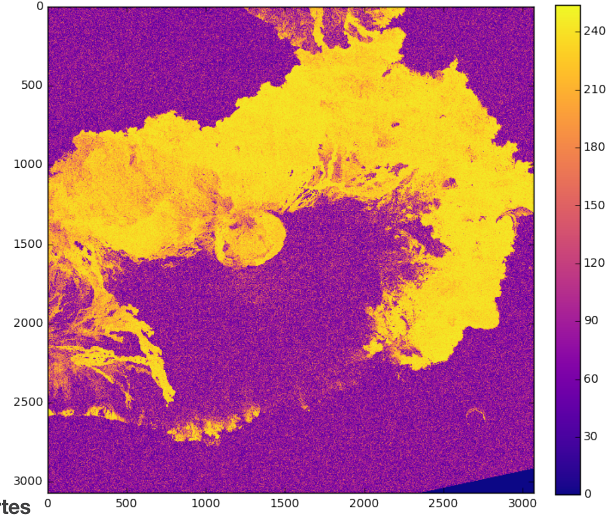
coherence



phase

Interferogram coherence

We can use the coherence band to determine where the phase information is meaningful.



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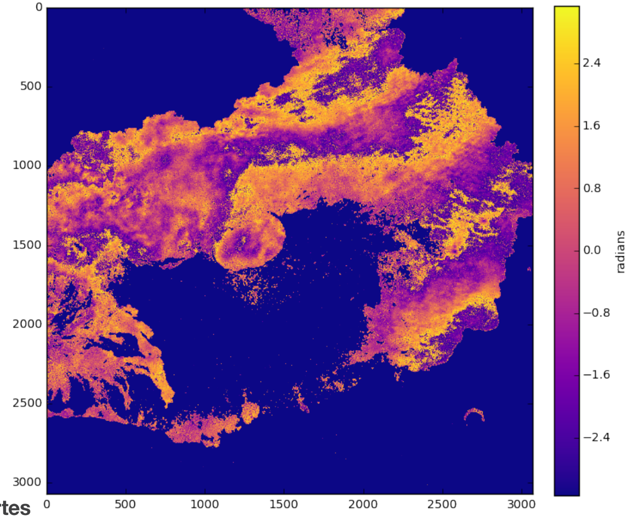
Interferogram phase

The phase difference is noisy, and is only known modulo $\mathbf{2\pi}$ (*phase wrapping*).



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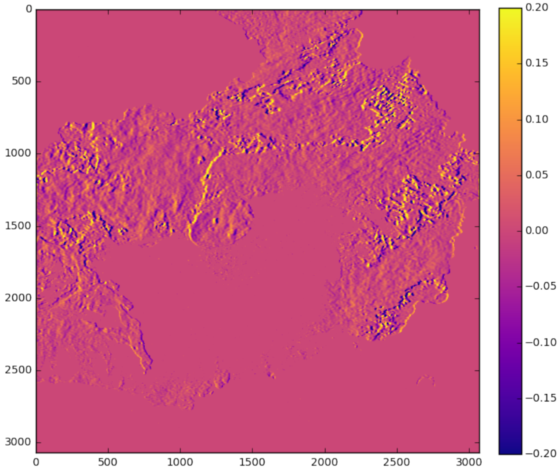
Phase unwrapping

To unwrap the phase, we use the fact that the difference between the true phase and the unwrapped phase is piecewise constant. Our approach:

- Compute the gradient (using our regularization method).
- Where the gradient magnitude is large, fill in gradient values with the mean of nearby non-large gradient values.
- Re-integrate the adjusted gradient (using $\boldsymbol{K}$).

Regularized gradient

Regularization suppresses noise, while preserving discontinuities at phase jumps. Using $p = 1/4$ gives less contrast loss than $p = 1$. The Python implementation of the algorithm ran in 45 seconds.



u_x

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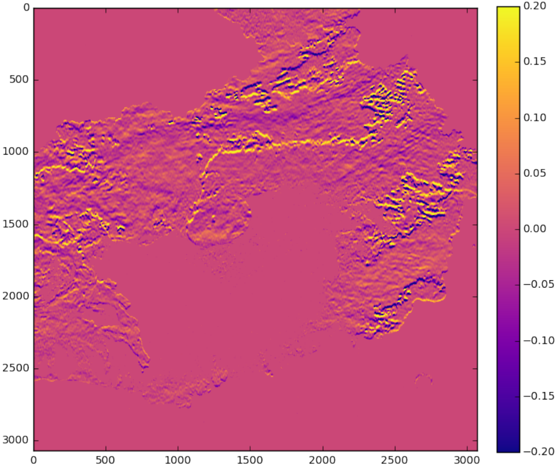

u_y

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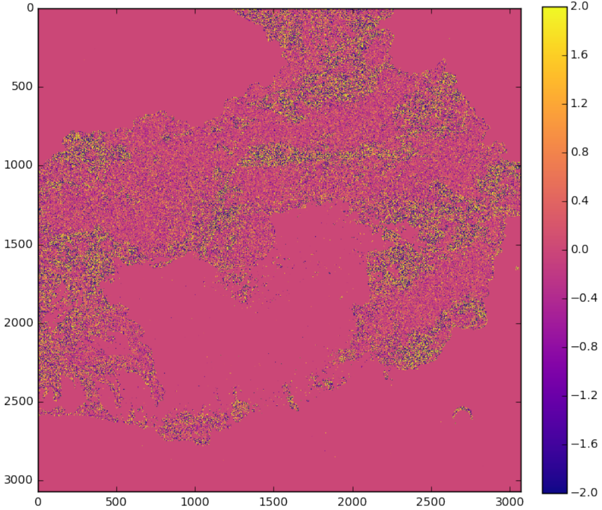
Unregularized gradient

The unregularized gradient is too noisy to be useful.



$D_x f$

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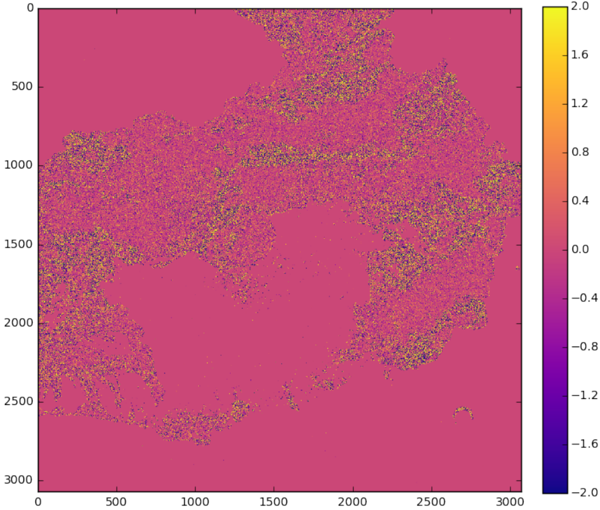
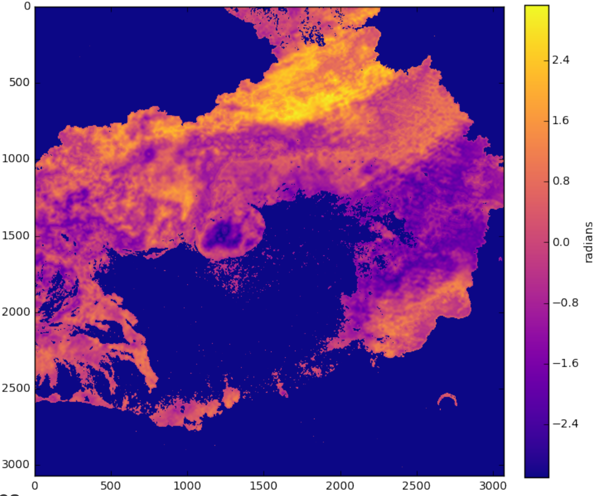


$D_y f$

Result

The unwrapped phase has no discontinuities, and preserves the elevation information.



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Summary

- Regularizing the differentiation process with total variation suppresses noise, while preserving discontinuities.
- The alternating directions, method of multipliers algorithm makes the differentiation very efficient.
- Differentiating interferometry phase lets us identify and remove phase wrapping. Thanks to Mike Warren, Jason Schatz, and the Descartes Labs platform team.
- Descartes Labs: satellite imagery startup. We're hiring! `http://www.descarteslabs.com/jobs/`