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MITSUBISHI ELECTRIC RESEARCH LABORATORIES
Cambridge, Massachusetts

Guided Signal Reconstruction with Application to Image Magnification

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Outline

1. Introduction
   - Problem Definition and Motivation
   - Related Work
2. Reconstruction Set
   - Geometric Interpretation
   - Algorithm for Finding the Reconstruction Set
   - Relation to Regularized Reconstruction
3. Experiments
4. Conclusion and Future Work

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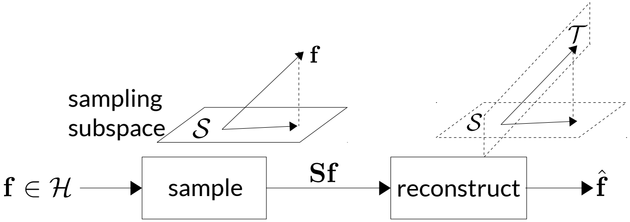
Problem Definition



- *Lossy* measurements
- Prior information about the signal $\Rightarrow$ Guiding subspace $\mathcal{T} \subset \mathcal{H}$

$$\mathbf{f} \in \mathcal{T} \quad \text{or} \quad \|\mathbf{T}^{\perp}\mathbf{f}\| \text{ small}$$

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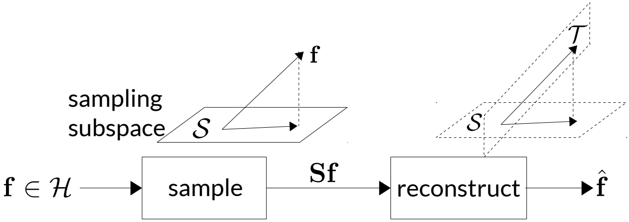

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Questions

Conditions on $\mathcal{S}$ and $\mathcal{T}$ for:

- Uniqueness of reconstruction
- Stability of reconstruction
- Efficient algorithm for reconstruction
- Effect of noise and model mismatch

Motivation

- Image magnification

- $\mathcal{S}$: 2×2 averaging
- $\mathcal{T}$: low pass DCT

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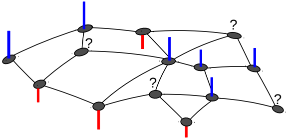
- Semi-supervised learning



- $\mathcal{S} = \{\mathbf{x} | \mathbf{x}_{\mathcal{U}} = \mathbf{0}\}$
- $\mathcal{T}$: low pass GFT

$$\mathcal{T} = \left\{ \sum_{i=1}^{K} c_i \mathbf{u}_i \right\},$$

$\{\mathbf{u}_i\}$ e.v.'s of graph $\mathbf{L}$

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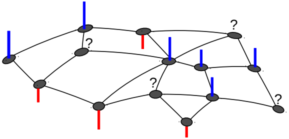

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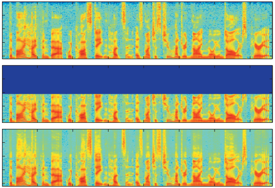
- Bandwidth expansion of speech



- $\mathcal{S}$: low pass DFT
- $\mathcal{T}$: learned from data

Related Work: Consistent Reconstruction

- **Consistent** reconstruction $\hat{\mathbf{f}} \Leftrightarrow \mathbf{S}\hat{\mathbf{f}} = \mathbf{S}\mathbf{f}$ (Unser and Aldroubi'94, Eldar'03)

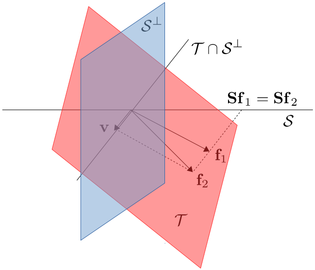
Existence and Uniqueness

- Consistent reconstruction exists in $\mathcal{T}$ for any $\mathbf{f} \in \mathcal{H}$

$$\text{iff } \mathcal{T} + \mathcal{S}^{\perp} = \mathcal{H}$$

- Consistent reconstruction is unique

$$\text{iff } \mathcal{T} \cap \mathcal{S}^{\perp} = \{\mathbf{0}\}$$



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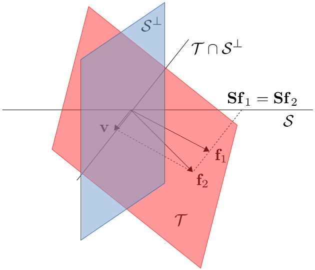
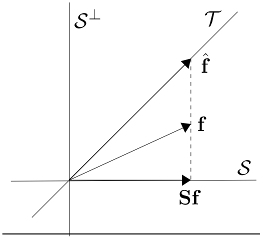




- Under the above assumptions

$$\hat{\mathbf{f}} = \mathbf{P}_{\mathcal{T}\perp\mathcal{S}}\mathbf{f} \quad \text{(oblique projection)}$$

- If $\mathbf{f} \in \mathcal{T}$ then $\hat{\mathbf{f}} = \mathbf{f}$
- If $\mathcal{T} \cap \mathcal{S}^\perp \neq \{\mathbf{0}\}$ (non-unique consistent solutions), pick one by imposing additional constraints

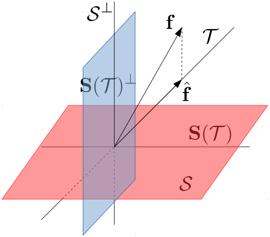
Related Work: Generalized Reconstruction

- Existence of consistent reconstruction needs $\mathcal{T} + \mathcal{S}^{\perp} = \mathcal{H}$
- Can lead to unstable reconstructions (if min. gap between $\mathcal{T}$ and $\mathcal{S}$ is large)
- Oversampling for stability can cause $\mathcal{T} + \mathcal{S}^{\perp} \subset \mathcal{H}$

Generalized reconstruction

- Sample consistent plane $\mathbf{Sf} + \mathcal{S}^{\perp}$
- $\hat{\mathbf{f}} \in \mathcal{T}$ closest to $\mathbf{Sf} + \mathcal{S}^{\perp}$ (relax $\mathbf{Sf} = \mathbf{S}\hat{\mathbf{f}}$)

$$\hat{\mathbf{f}} = \mathbf{P}_{\mathcal{T} \perp \mathbf{S}(\mathcal{T})} \mathbf{f}$$



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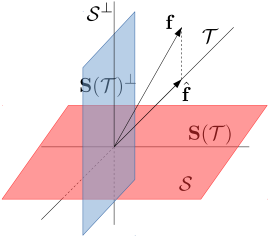

Question: $\hat{\mathbf{f}} \in \mathbf{Sf} + \mathcal{S}^\perp$ (consistent) or $\hat{\mathbf{f}} \in \mathcal{T}$ (generalized) or something else?

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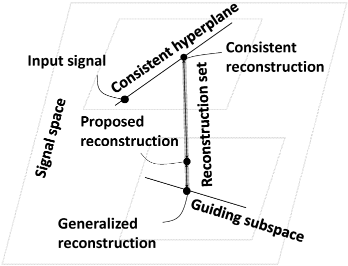
Reconstruction Set



- sample consistent place $\mathbf{Sf} + \mathcal{S}^{\perp}$
- guiding subspace $\mathcal{T}$

Reconstruction set

Shortest pathway between the consistent place and the guiding subspace

$$\min_{\hat{\mathbf{f}} \in \mathbf{Sf}+\mathcal{S}^{\perp}} \min_{\mathbf{t} \in \mathcal{T}} \|\hat{\mathbf{f}} - \mathbf{t}\| = \min_{\substack{\hat{\mathbf{f}} \in \mathbf{Sf}+\mathcal{S}^{\perp} \\ \mathbf{t} \in \mathcal{T}}} \|\hat{\mathbf{f}} - \mathbf{t}\| = \min_{\mathbf{t} \in \mathcal{T}} \min_{\hat{\mathbf{f}} \in \mathbf{Sf}+\mathcal{S}^{\perp}} \|\hat{\mathbf{f}} - \mathbf{t}\|$$

- $\hat{\mathbf{f}}$: consistent reconstruction
- $\mathbf{t}$: generalized reconstruction

Iterative Consistent Reconstruction Using Cojugate Gradient

- Consistent reconstruction

$$\inf_{\hat{\mathbf{f}}} \|\mathbf{T}^{\perp}\mathbf{f}\| \quad \text{subject to} \quad \mathbf{S}\hat{\mathbf{f}} = \mathbf{S}\mathbf{f}$$

Iterative Consistent Reconstruction Using Cojugate Gradient

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Consistent reconstruction using CG

Define $\hat{\mathbf{x}} = (\hat{\mathbf{f}} - \mathbf{S}\mathbf{f}) \in \mathcal{S}^{\perp}$. Then the above problem is equivalent to solving

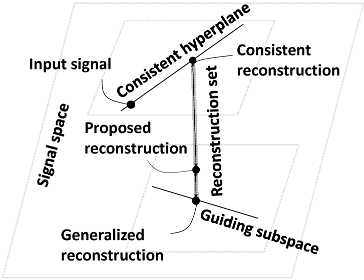
$$(\mathbf{S}^{\perp}\mathbf{T}^{\perp})\big|_{\mathcal{S}^{\perp}}\mathbf{x} = -\mathbf{S}^{\perp}\mathbf{T}^{\perp}\mathbf{S}\mathbf{f} \qquad (\mathbf{S}\mathbf{f}: \text{ measurement})$$

- Restriction of $\mathbf{S}^{\perp}\mathbf{T}^{\perp}$ to $\mathcal{S}^{\perp}$ is self-adjoint
- Use CG with initialization $\mathbf{x}_0 \in \mathcal{S}^{\perp}$

- CG: most efficient iterative method for solving linear systems
- Frame-less algorithm: Needs only the (approximate) projector $\mathbf{T}$

Finding the Reconstruction Set

$$\min_{\hat{\mathbf{f}} \in \mathbf{S}\mathbf{f} + \mathcal{S}^\perp} \min_{\mathbf{t} \in \mathcal{T}} \|\hat{\mathbf{f}} - \mathbf{t}\| = \min_{\substack{\hat{\mathbf{f}} \in \mathbf{S}\mathbf{f} + \mathcal{S}^\perp \\ \mathbf{t} \in \mathcal{T}}} \|\hat{\mathbf{f}} - \mathbf{t}\| = \min_{\mathbf{t} \in \mathcal{T}} \min_{\hat{\mathbf{f}} \in \mathbf{S}\mathbf{f} + \mathcal{S}^\perp} \|\hat{\mathbf{f}} - \mathbf{t}\|$$



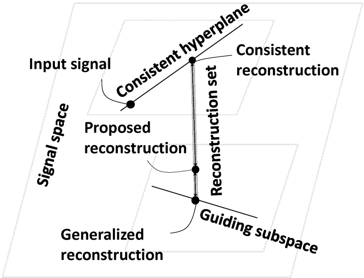
- $\hat{\mathbf{f}}$: consistent reconstruction
- $\mathbf{t}$: generalized reconstruction

- Relation between $\hat{\mathbf{f}}$ and $\mathbf{t}$

$$\mathbf{t} = \mathbf{T}\hat{\mathbf{f}}$$

Finding the Reconstruction Set

$$\min_{\hat{\mathbf{f}} \in \mathbf{Sf} + \mathcal{S}^{\perp}} \min_{\mathbf{t} \in \mathcal{T}} \|\hat{\mathbf{f}} - \mathbf{t}\| = \min_{\substack{\hat{\mathbf{f}} \in \mathbf{Sf} + \mathcal{S}^{\perp} \\ \mathbf{t} \in \mathcal{T}}} \|\hat{\mathbf{f}} - \mathbf{t}\| = \min_{\mathbf{t} \in \mathcal{T}} \min_{\hat{\mathbf{f}} \in \mathbf{Sf} + \mathcal{S}^{\perp}} \|\hat{\mathbf{f}} - \mathbf{t}\|$$



- $\hat{\mathbf{f}}$: consistent reconstruction
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- Relation between $\hat{\mathbf{f}}$ and $\mathbf{t}$

$$\mathbf{t} = \mathbf{T}\hat{\mathbf{f}}$$

$$\text{Reconstruction Set} = \{\alpha\hat{\mathbf{f}} + (1-\alpha)\mathbf{T}\hat{\mathbf{f}}, \quad \text{where } \alpha \in [0,1]\}$$

Connection with Regularization

Reconstruction by regularization

$$\inf_{\hat{\mathbf{f}}_\rho} \left\| \mathbf{S}\hat{\mathbf{f}}_\rho - \mathbf{S}\mathbf{f} \right\|^2 + \rho \left\| \left(\hat{\mathbf{f}}_\rho - \mathbf{T}\hat{\mathbf{f}}_\rho \right) \right\|^2, \quad \rho > 0$$

Connection with Regularization

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Theorem (Reconstruction set and Regularization)

Let $\hat{\mathbf{f}}$ *be the consistent reconstruction given by*

$$\inf_{\hat{\mathbf{f}}} \|\mathbf{T}^\perp \mathbf{f}\| \quad \textit{subject to} \quad \mathbf{S}\hat{\mathbf{f}} = \mathbf{S}\mathbf{f}.$$

The reconstruction set is given by $\{\hat{\mathbf{f}}_\alpha = \alpha\hat{\mathbf{f}} + (1-\alpha)\mathbf{T}\hat{\mathbf{f}}, \quad \textit{where } 0 \leq \alpha \leq 1\}$*. Then* $\hat{\mathbf{f}}_\alpha$ *is a solution of the regularized reconstruction problem with* $\rho = (1-\alpha)/\alpha$*.*

- If a unique $\hat{\mathbf{f}} \in \mathcal{T} \cap (\mathbf{S}\mathbf{f} + \mathcal{S}^\perp)$ exists, then $\hat{\mathbf{f}}_\rho = \hat{\mathbf{f}} = \mathbf{T}\hat{\mathbf{f}} \quad \forall\, \rho > 0$
- No need to re-solve the regularization problem if ρ changes

Reconstruction in the Presence of Noise

- Noisy measurements: $\mathbf{S}\mathbf{f}' = \mathbf{S}\mathbf{f} + \mathbf{e} \Rightarrow$ Original signal $\mathbf{f} \notin (\mathbf{S}\mathbf{f}' + \mathcal{S}^{\perp})$
- Trust the guiding more than the samples
- Let $\hat{\mathbf{f}} \in \mathbf{S}\mathbf{f}' + \mathcal{S}^{\perp}$ be the consistent solution

$\Rightarrow$ Good solution is $\hat{\mathbf{f}}_\alpha = \alpha\hat{\mathbf{f}} + (1-\alpha)\mathbf{T}\hat{\mathbf{f}}$ with $\alpha > 0$

Good choice of α

Noise energy $\|\mathbf{e}\|$. Then pick α such that

$$1 - \alpha = \frac{\|\mathbf{e}\|}{\|\hat{\mathbf{f}} - \mathbf{T}\hat{\mathbf{f}}\|} \Rightarrow \hat{\mathbf{f}}_\alpha = \hat{\mathbf{f}} - \|\mathbf{e}\| \frac{\hat{\mathbf{f}} - \mathbf{T}\hat{\mathbf{f}}}{\|\hat{\mathbf{f}} - \mathbf{T}\hat{\mathbf{f}}\|}$$

- Assumes that noise is orthogonal to $\mathcal{T}$

Outline

Experiment I: Problem Setting

- **Original signal** $\mathbf{f}$: $w \times w$ image
- **Sampling subspace** $\mathcal{S}$: $\mathbf{S}\mathbf{f} = \mathbf{B}_{\mathcal{S}}\mathbf{B}_{\mathcal{S}}^{*}\mathbf{f}$, where

$$\mathbf{B}_{\mathcal{S}}^{*} : r \times r \text{ averaging then downsampling}$$

$$\mathbf{B}_{\mathcal{S}} : \text{ upsampling by copying each pixel in } r \times r \text{ block}$$

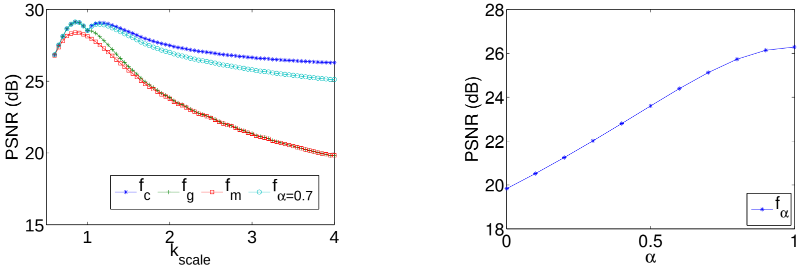
- **Guiding subspace** $\mathcal{T}$: $k \times k$ low pass bandlimited DCT
- $k_{\text{scale}} = (w/r)/k$: relative dimensionality of $\mathcal{S}$ and $\mathcal{T}$

$$k_{\text{scale}} < 1 : \text{ undersampling}$$

$$k_{\text{scale}} > 1 : \text{ oversampling}$$

Experiment II: Noiseless Reconstruction

$\hat{\mathbf{f}}_c$: Consistent, $\hat{\mathbf{f}}_g = \mathbf{T}\hat{\mathbf{f}}_c$: Generalized, $\hat{\mathbf{f}}_m = \mathbf{TSf}$: minimax regret (Eldar *et al*'06)

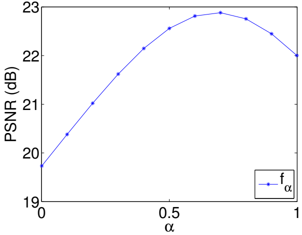


(a) $\alpha = 0.7$ (b) $k_{\text{scale}} = 4$

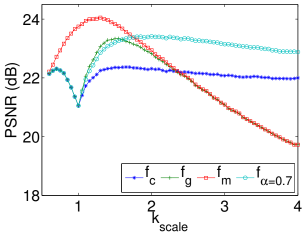
- $k_{\text{scale}} < 1$ (undersampling): $\hat{\mathbf{f}}_c = \hat{\mathbf{f}}_g$ better than $\hat{\mathbf{f}}_m$
- $k_{\text{scale}} > 1$ (oversampling): $\hat{\mathbf{f}}_c$ better than $\hat{\mathbf{f}}_g$ and $\hat{\mathbf{f}}_m$
- Since samples are noiseless, reconstruction improves as α increases

Experiment III: Reconstruction in the Presence of Noise

- $\mathbf{Sf}' = \mathbf{Sf} + \mathbf{e}$, where $\mathbf{e}$ is iid $\mathcal{N}(0, 0.001) \Rightarrow \alpha_{\text{opt}} = 0.7$



(a) $k_{\text{scale}} = 4$



(b) $\alpha = 0.7$

(a) $\mathbf{Sf}'$, 21.69dB

(b) $\hat{\mathbf{f}}_g$, 19.73dB

(c) $\hat{\mathbf{f}}_c$, 22.00dB

(d) $\hat{\mathbf{f}}_{\alpha=0.7}$, 22.88dB

Outline

Conclusion

- Unified view of different reconstruction methods
- Novel formulation of the reconstruction set
- Efficient reconstruction algorithm for finding the reconstruction set
- Connection with regularization and reconstruction with noisy samples

Future work

- Error bounds based on
  - noise
  - model mismatch
  - relative positions of $\mathcal{S}$ and $\mathcal{T}$
- Applications in other areas: speech, video, machine learning

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