Assessing cross-dependencies using bivariate multifractal analysis

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Abstract — Multifractal analysis has become a reference tool to characterize scale-free temporal dynamics in time series. It proved successful in numerous applications very diverse in nature. However, such successes remained restricted to univariate analysis, while many recent applications call for the joint analysis of several components. Surprisingly, multivariate multifractal analysis remained mostly overlooked. The present contribution aims at defining a wavelet-leader-based framework for multivariate multifractal analysis and at studying its properties and estimation performance. To better understand what properties of multivariate data are actually captured, a multivariate multifractal model is used as representative paradigm and permits to show that multivariate multifractal analysis puts in evidence transient and local dependencies that are not well quantified or even evidenced by the classical Pearson correlation coefficient.

Multifractal analysis

Multifractal Spectrum

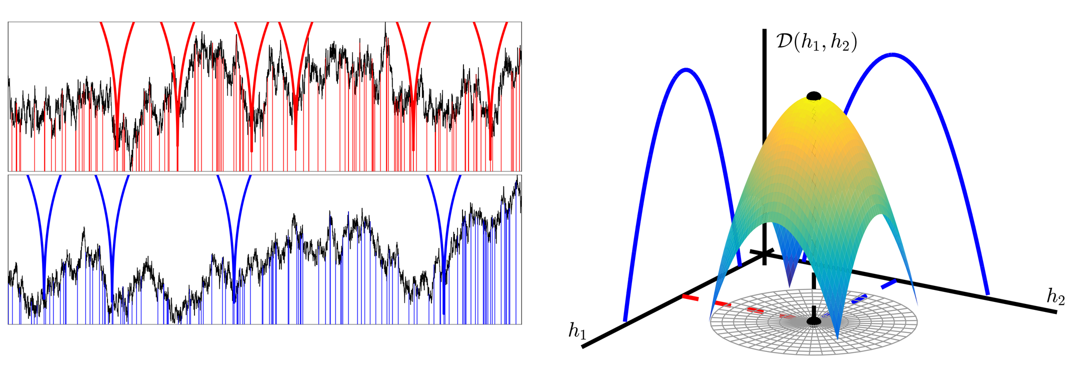
- Local Regularity:
  - Get regularity exponent from function $X(t)$
  - Compare X with local polynomial approximation P_t
  - **Most common**: Hölder exponent $h(t) \geq 0$

$$|X(t+a) - P_t(a)| \overset{a\to 0^+}{\sim} a^{h(t)}$$

- Multifractal Spectrum:
  - Bivariate signal: $\mathbf{X} = (X_1, X_2)$
  - Hölder exponents: $(h_1(t), h_2(t))$
  - Multifractal spectrum:

$$\mathcal{D}(h_1,h_2) = \dim_{\text{Hausdorff}}\{t \;:\; h_1(t) = h_1 \text{ and } h_2(t) = h_2\}$$

$\longrightarrow$ "Quantity" of points with given regularity



- **Problem:** Can not be computed in practice
  $\longrightarrow$ Use *multifractal formalism* to estimate

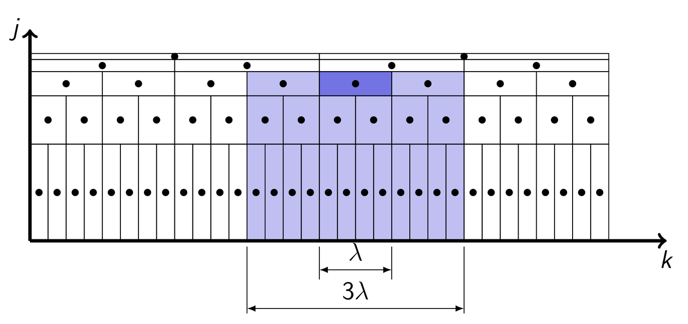
Multiresolution quantities

- $X(t) \longrightarrow T_X(j,k)$ (scale 2^{-j}, position $k = t\,2^j$)
- *Choice 1*: Discrete wavelet transform

$$T_X(j,k) = d_X(j,k) = d_X(\lambda)$$

$\longrightarrow$ Poor performance

- Dyadic intervals:
  $\lambda = \lambda_{j,k} = (2^{-j}(k-1), 2^{-j}k]$ and $3\lambda_{j,k} = \bigcup_{i=-1}^{1} \lambda_{j,k+i}$



- *Choice 2*: Wavelet leaders (Wendt, Abry & Jaffard, 2007)

$$T_X(j,k) = \ell_X(j,k) = \sup_{\lambda' \in 3\lambda} |c_{\lambda'}|$$

$\longrightarrow$ Much better performance

Multifractal Formalism: Definition

- **Goal:** provide estimate of $\mathcal{D}(h_1,h_2)$ from wavelet leaders
- Easily computable in practice. Steps:
  1. Structure functions S:

$$S(q_1,q_2,j) = \frac{1}{n_j}\sum_{k=1}^{n_j} L_{X_1}(j,k)^{q_1} L_{X_2}(j,k)^{q_2}$$

  2. Scaling function ζ:

$$S(q_1,q_2,j) \sim 2^{-j\zeta(q_1,q_2)}, \quad j \to \infty$$

  3. Legendre spectrum $\mathcal{L}$:

$$\mathcal{L}(h_1,h_2) = \inf_{q_1,q_2}\left(1 + q_1h_1 + q_2h_2 - \zeta(q_1,q_2)\right) \geq \mathcal{D}(q_1,q_2).$$

- Upper bound $\mathcal{L}$ used as estimate for $\mathcal{D}$

Multifractal Formalism: Cumulants

- Bivariate cumulants of $(\ln L_{X_1}(j,k), \ln L_{X_2}(j,k))$
  - $C_{p_1p_2}(j) = \mathbb{E}\left[\ln\left(L_{X_1}(j,\cdot)\right)^{p_1} \ln\left(L_{X_2}(j,\cdot)\right)^{p_2}\right]$
  - Order $p_1 + p_2 \geq 1$
- Scale dependence:

$$C_{p_1p_2}(j) = c^0_{p_1p_2} + c_{p_1p_2}\ln 2^j$$

- Polynomial expansion:

$$\mathcal{L}(h_1,h_2) \approx 1 + \frac{c_{02}b}{2}\left(\frac{h_1 - c_{10}}{b}\right)^2 + \frac{c_{20}b}{2}\left(\frac{h_2 - c_{01}}{b}\right)^2 - c_{11}b\left(\frac{h_1 - c_{10}}{b}\right)\left(\frac{h_2 - c_{01}}{b}\right)$$

where $b = c_{20}c_{02} - c_{11}^2 \geq 0$

- Information synthesized in second order: 5 parameters
- Interpretation:
  - c_{01}, c_{10}: average regularity on each component
  - c_{02}, c_{20}: width of regularity fluctuations on each component
  - c_{11}: leading-order joint regularity fluctuation

Practical Estimation

- $\zeta(q_1,q_2) \longrightarrow$ linear regressions $\log_2 S(q_1,q_2,j)$ *vs* $\log_2 2^j$
- $c_{p_1p_2} \longrightarrow$ linear regressions $C_{p_1p_2}(j)$ *vs* $\ln 2^j$

Synthetic Process

Bivariate Multifractal Random Walk

Synthetic process with bivariate multifractal behavior

- Definition
  - Use **two pairs** of stochastic processes
  - **Pair 1:** bivariate fractional Gaussian noise $G_1(t), G_2(t)$
    - $*$ Self-similarity parameters: H_1, H_2
    - $*$ Covariance matrix:

$$\Sigma_{ss} = \begin{pmatrix} 1 & \rho_{ss} \\ \rho_{ss} & 1 \end{pmatrix}$$

  - **Pair 2:** Gaussian processes $\omega_1(t), \omega_2(t)$
    - $*$ Multifractality parameters: λ_1, λ_2
    - $*$ Covariance function: Σ_{mf} such that

$$\{\Sigma_{mf}\}_{ij}(k,l) = \boldsymbol{\rho_{mf}}(i,j)\lambda_i\lambda_j \log\left(\frac{N}{|k-l|+1}\right),\; i = 1,2$$

where

$$\boldsymbol{\rho_{mf}} = \begin{pmatrix} 1 & \rho_{mf} \\ \rho_{mf} & 1 \end{pmatrix}$$

$\longrightarrow$ Logarithmic covariance to induce multifractality

  - $G_i(t), \omega_i(t)$ synthesized following (Helgason, Pipiras & Abry, 2011)
  - **Final process**:

$$X_i(t) = \sum_{k=1}^{t} G_i(k)e^{\omega_i(k)}, \quad i = 1,2$$

- Properties
  - Correlation coefficient of final process:

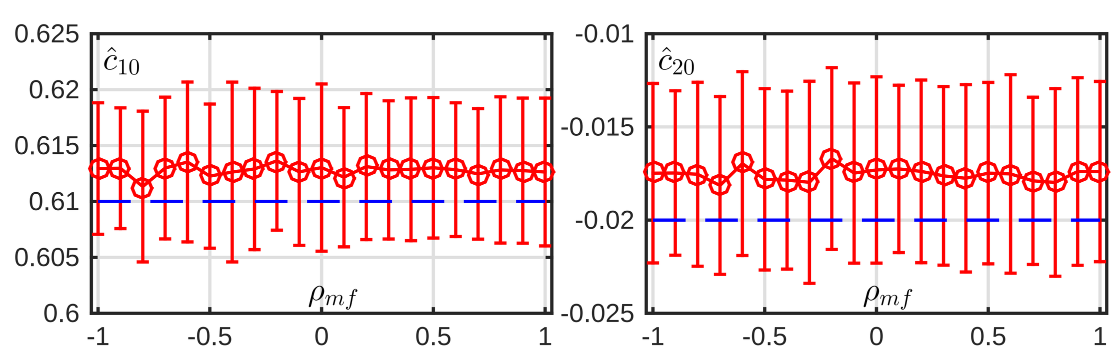
$$\rho_{bMRW} = \rho_{ss} \cdot f(\rho_{mf}, \lambda_1, \lambda_2)$$

$\longrightarrow$ Can have $\rho_{bMRW} = 0$ with $\rho_{mf} \neq 0$!

  - Cumulants:
    - $*$ $c_{10} = H_1 + \lambda_1^2/2$ and $c_{01} = H_2 + \lambda_2^2/2$
    - $*$ $c_{20} = -\lambda_1^2$ and $c_{02} = -\lambda_2^2$
    - $*$ $c_{11} = -\rho_{mf}\lambda_1\lambda_2$

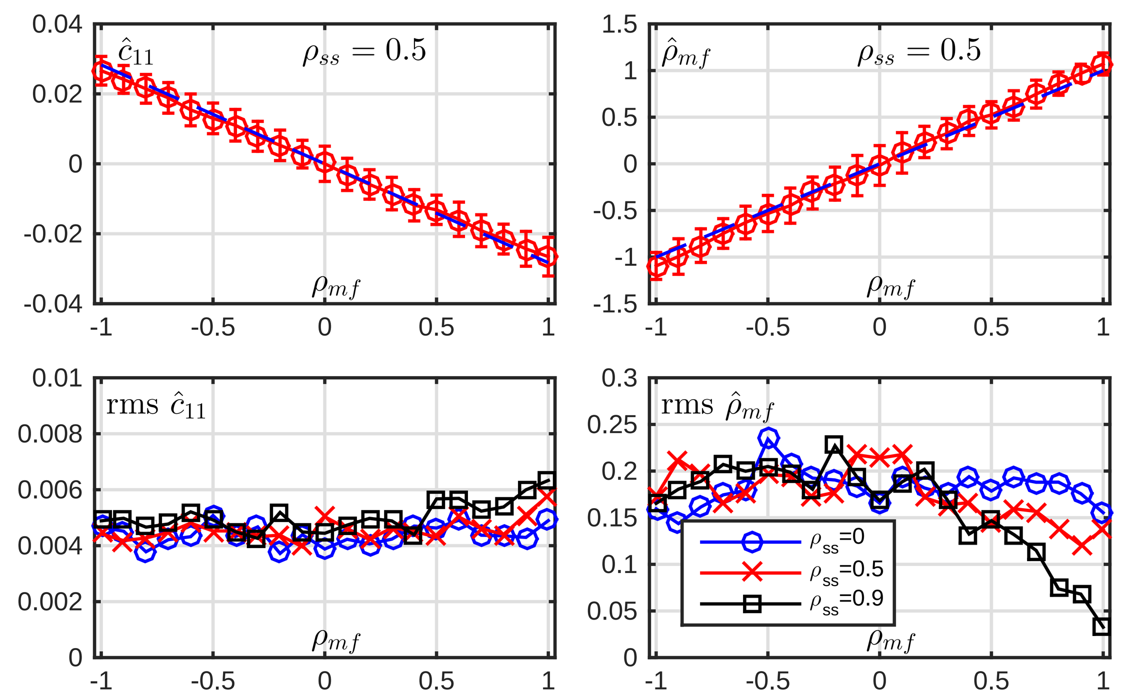
Results

Estimation performance: univariate



- Performance independent of correlation
- Performance independent of multifractal dependence

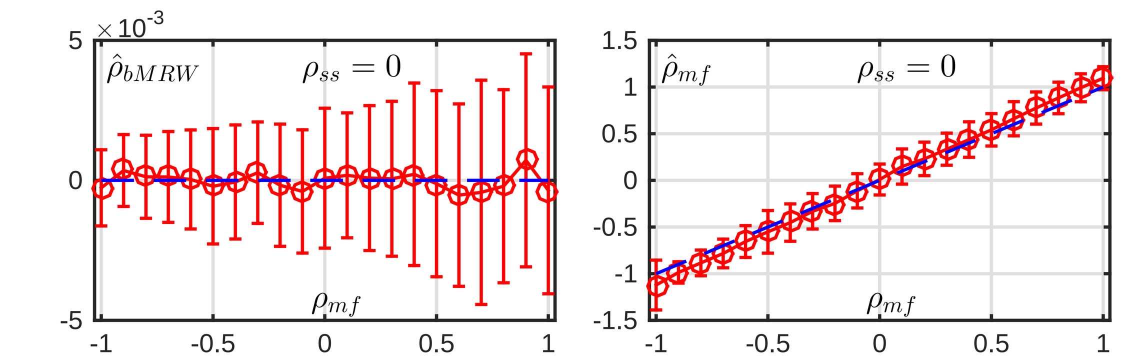
Estimation performance: bivariate



- Excellent estimation performance
- Estimation performance largely independent of ρ_{ss} and ρ_{mf}
- Relevant and robust estimates for bivariate parameters c_{11} and ρ_{mf}

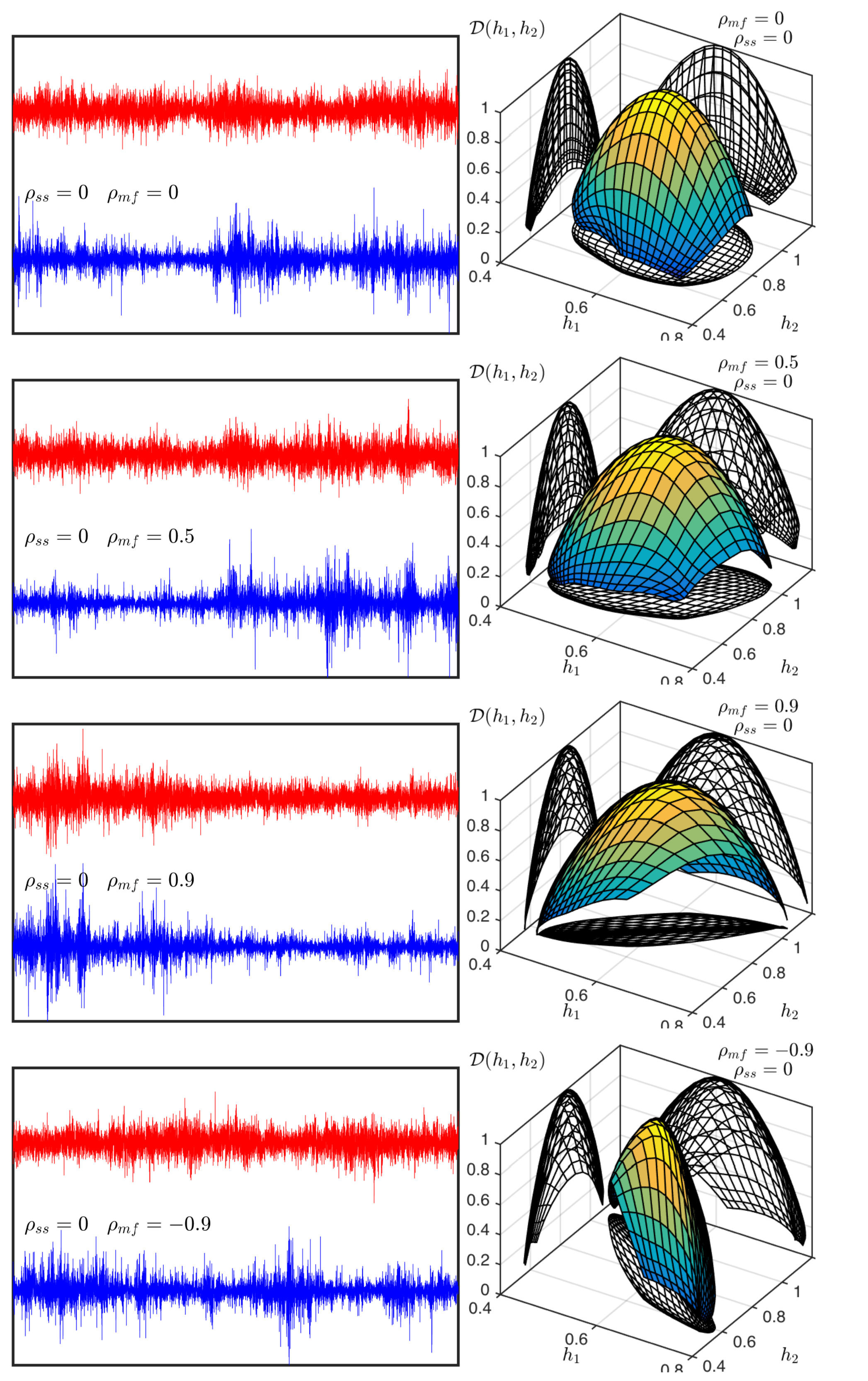
Higher-order dependence

- Set $\rho_{ss} = 0 \implies \rho_{bMRW} = 0$



- Measured Pearson correlation indeed null (left)
- Estimated $\hat{\rho}_{mf} \neq 0 \implies$ dependence beyond correlation

Multifractal spectra



- Orientation and eccentricity of support: higher-order dependence
- $\rho_{ss} = 0, \rho_{mf} = 0 \implies$: uncorrelated, independent
- $\rho_{ss} = 0, \rho_{mf} > 0 \implies$: uncorrelated, positive dependence
- $\rho_{ss} = 0, \rho_{mf} < 0 \implies$: uncorrelated, negative dependence

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