Finite-Alphabet NOMA for Two-User Uplink Channel

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1. Background, Motivation and Our Contributions

- NOMA has been a key enabling technology to meet the requirements of 5G on high spectral efficiency, massive connectivity, and low transmission latency;
- Most existing NOMA designs assumed **Gaussian** inputs. The drawbacks are:
  - the implementation in reality will result in huge storage capacity, unaffordable computational complexity and extremely long encoding/decoding delay;
  - the actual transmitted signals in real communication systems are drawn from **finite-alphabet** constellations, such as PAM, QAM, and PSK;
  - Applying the results derived from the Gaussian inputs to the signals with finite-alphabet inputs can lead to significant performance loss.
- We consider the NOMA design for a classical two-user MAC with QAM constellations at both transmitters, whose sizes are not necessarily the same.
  - We aim to maximize the minimum Euclidean distance of the received sum-constellation for a ML receiver where the formulated problem is a **mixed continuous-discrete** optimization problem and is non-trivial to resolve;
  - We discover that Farey sequence can be employed to tackle the formulated problem. However, the existing Farey sequence is not applicable when the constellation sizes of the two users are different;
  - To address this challenge, we define a new type of Farey sequence, termed **punched Farey sequence**. Based on the punched Farey sequence and its properties, we manage to resolve the mixed continuous-discrete optimization problem by providing a neat closed-form optimal solution.

2. System Model and Problem Formulation

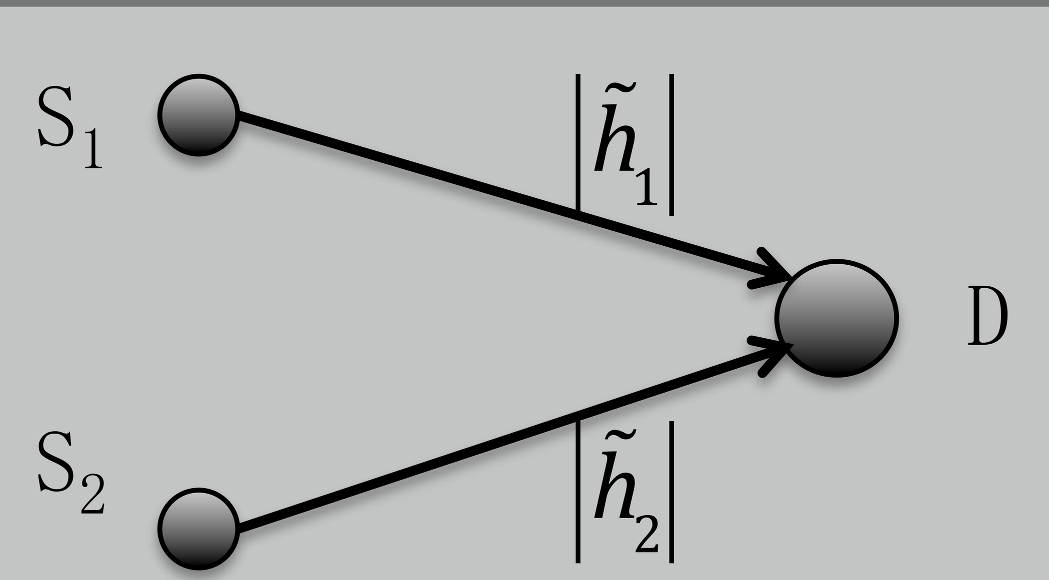


Figure: Two-User Real Gaussian Multiple Access Channel

- The received signal at the access point D can be written as:

$$y = |\tilde{h}_1|\tilde{w}_1 s_1 + |\tilde{h}_2|\tilde{w}_2 s_2 + n,$$

where $s_k \in \{\pm(2\ell-1)\}_{\ell=1}^{M_k/2}$, $k=1,2$ are drawn from a standard PAM constellation with equal probability, $0<\tilde{w}_1\le 1$ and $0<\tilde{w}_2\le 1$ are the weighting coefficients;
- A coherent maximum-likelihood (ML) detector is used by the access point D to estimate the transmitted signals in a symbol-by-symbol fashion. Mathematically, the estimated signals can be expressed as $(\hat{s}_1,\hat{s}_2) = \arg\min_{(s_1,s_2)} |y - (|\tilde{h}_1|\tilde{w}_1 s_1 + |\tilde{h}_2|\tilde{w}_2 s_2)|$;
- The minimum Euclidean distance between two constellations are given by

$$d(m,n) = \frac{1}{2}|y(s_1,s_2) - y(\tilde{s}_1,\tilde{s}_2)| = ||\tilde{h}_1|\tilde{w}_1 n - |\tilde{h}_2|\tilde{w}_2 m|,$$
$$(m,n)\in \mathbb{Z}^2_{(M_1-1,M_2-1)}\setminus\{(0,0)\}.$$

3. The Design Problem for the Finite-Alphabet NOMA

The Weighting Coefficients Design Problem:
Problem 1: Find the optimal $(\tilde{w}_1^*,\tilde{w}_2^*)$ subject to the individual average power constraint such that the minimum Euclidean distance d^* of the received constellation points is maximized, i.e.,

$$(\tilde{w}_1^*,\tilde{w}_2^*) = \arg\max_{(\tilde{w}_1,\tilde{w}_2)} \min_{(m,n)\in\mathbb{Z}^2_{(M_1-1,M_2-1)}\setminus\{(0,0)\}} d(m,n)$$
$$\text{s.t. } 0<\tilde{w}_1\le 1 \text{ and } 0<\tilde{w}_2\le 1.$$

To that end, we should solve the following optimization problem first:
Problem 2: Find the optimal $(\tilde{w}_1^*(k),\tilde{w}_2^*(k))$ such that

$$g\Big(\frac{b_k}{a_k},\frac{b_{k+1}}{a_{k+1}}\Big) = \max_{(\tilde{w}_1,\tilde{w}_2)} \min_{(m,n)\in\mathbb{F}^2_{(M_1-1,M_2-1)}} d(m,n)$$
$$\text{s.t. } \frac{b_k}{a_k} < \frac{|\tilde{h}_2|\tilde{w}_2}{|\tilde{h}_1|\tilde{w}_1} \le \frac{b_{k+1}}{a_{k+1}}, 0<\tilde{w}_1\le 1 \text{ and } 0<\tilde{w}_2\le 1,$$

where the punched Farey sequence given by $\mathfrak{P}_{M_2-1}^{M_1-1} = \big(\frac{b_1}{a_1},\frac{b_2}{a_2},\cdots,\frac{b_C}{a_C}\big)$ whose definition will be elaborated in the following part.

4. Punched Farey Sequence

We now propose a new definition in number theory called **Punched Farey sequence** which characterizes the relationship between two positive integers:
Definition: The punched Farey sequence $\mathfrak{P}_K^L$ is the ascending sequence of irreducible fractions whose denominators are no greater than K and numerators are no greater than L.
Example: $\mathfrak{P}_5^2$ is the ordered sequence $\big(\frac{0}{1},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{2}{3},\frac{1}{1},\frac{2}{1},\frac{1}{0}\big)$.
Properties:

- If $\frac{n_1}{m_1}$ and $\frac{n_2}{m_2}$ are two adjacent terms in $\mathfrak{P}_K^L$ $(\min\{K,L\}\ge 2)$ such that $\frac{n_1}{m_1}<\frac{n_2}{m_2}$, then, 1) $\frac{n_1+n_2}{m_1+m_2}\in\big(\frac{n_1}{m_1},\frac{n_2}{m_2}\big)$, $\frac{m_1+m_2}{n_1+n_2}\in\big(\frac{m_2}{n_2},\frac{m_1}{n_1}\big)$; 2) $m_1n_2-m_2n_1=1$; 3) If $n_1+n_2\le L$, then $m_1+m_2>K$ and if $m_1+m_2\le K$, then $n_1+n_2>L$; 4) $n_1+n_2\ge 1$ where the equality is attained if and only if $\frac{n_1}{m_1}=\frac{0}{1}$ and $\frac{n_2}{m_2}=\frac{1}{K}$. Likewise, $m_1+m_2\ge 1$ where the equality is attained if and only if $\frac{n_1}{m_1}=\frac{L}{1}$ and $\frac{n_2}{m_2}=\frac{1}{0}$.
- If $\frac{n_1}{m_1}$, $\frac{n_2}{m_2}$ and $\frac{n_3}{m_3}$ are three consecutive terms in $\mathfrak{P}_K^L$ with $\min\{K,L\}\ge 2$ such that $\frac{n_1}{m_1}<\frac{n_2}{m_2}<\frac{n_3}{m_3}$, then $\frac{n_2}{m_2}=\frac{n_1+n_3}{m_1+m_3}$.
- Let $\frac{n_1}{m_1},\frac{n_2}{m_2},\frac{n_3}{m_3},\frac{n_4}{m_4}\in\mathfrak{P}_K^L$ with $\min\{K,L\}\ge 3$. If $\frac{n_1}{m_1}<\frac{n_2}{m_2}<\frac{n_3}{m_3}<\frac{n_4}{m_4}$, where $\frac{n_2}{m_2},\frac{n_3}{m_3}$ are successive in $\mathfrak{P}_K^L$, then $\frac{n_1+n_3}{m_1+m_3}\le\frac{n_2}{m_2}$ and $\frac{n_3}{m_3}\le\frac{n_2+n_4}{m_2+m_4}$.

5.1 The Solution to Problem 2

We now can solve **Problem 2**, i.e., restricting $\frac{|\tilde{h}_2|\tilde{w}_2}{|\tilde{h}_1|\tilde{w}_1}$ into a certain punched Farey interval determined by the corresponding Farey pair where a closed-form solution is attainable,

Lemma 1: The optimal solution to Problem 2 is given as follows:

- If $\frac{|\tilde{h}_2|}{|\tilde{h}_1|}\le\frac{b_k+b_{k+1}}{a_k+a_{k+1}}$, then $g\big(\frac{b_k}{a_k},\frac{b_{k+1}}{a_{k+1}}\big)=\frac{|\tilde{h}_2|}{b_k+b_{k+1}}$ and $(\tilde{w}_1^*(k),\tilde{w}_2^*(k))=\big(\frac{|\tilde{h}_2|(a_k+a_{k+1})}{|\tilde{h}_1|(b_k+b_{k+1})},1\big)$;
- If $\frac{|\tilde{h}_2|}{|\tilde{h}_1|}>\frac{b_k+b_{k+1}}{a_k+a_{k+1}}$, then $g\big(\frac{b_k}{a_k},\frac{b_{k+1}}{a_{k+1}}\big)=\frac{|\tilde{h}_1|}{a_k+a_{k+1}}$ and $(\tilde{w}_1^*(k),\tilde{w}_2^*(k))=\big(1,\frac{|\tilde{h}_1|(b_k+b_{k+1})}{|\tilde{h}_2|(a_k+a_{k+1})}\big)$.

5.2 The Solution to Problem 1

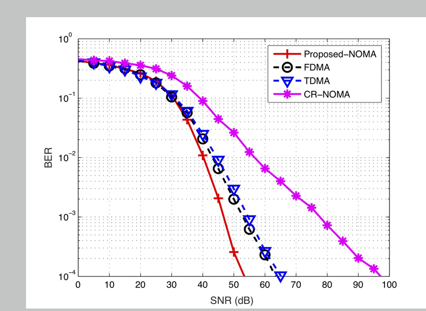
Theorem: Closed-form optimal weighting coefficients: The optimal solution to Problem 1 in terms of (w_1^*,w_2^*) is given by:

- If $\frac{|h_2|}{|h_1|}\le\sqrt{\frac{P_1(M_2^2-1)}{P_2M_2^2(M_1^2-1)}}$, then $(w_1^*,w_2^*)=\Big(\sqrt{\frac{3P_2M_2^2}{2(M_2^2-1)}\frac{|h_2|}{|h_1|}},\sqrt{\frac{3P_2}{2(M_2^2-1)}}\Big)$, $d_{\text{noma}}=\sqrt{\frac{3P_2}{2(M_2^2-1)}}|h_2|$;
- If $\sqrt{\frac{P_1(M_2^2-1)}{P_2M_2^2(M_1^2-1)}}<\frac{|h_2|}{|h_1|}\le\sqrt{\frac{P_1M_1^2(M_2^2-1)}{P_2M_2^2(M_1^2-1)}}$, then $(w_1^*,w_2^*)=\Big(\sqrt{\frac{3P_1}{2(M_1^2-1)}},\sqrt{\frac{3P_1}{2M_2^2(M_1^2-1)}\frac{|h_1|}{|h_2|}}\Big)$, $d_{\text{noma}}=\sqrt{\frac{3P_1}{2M_2^2(M_1^2-1)}}|h_1|$;
- If $\sqrt{\frac{P_1M_1^2(M_2^2-1)}{P_2M_2^2(M_1^2-1)}}<\frac{|h_2|}{|h_1|}\le\sqrt{\frac{P_1M_1^2(M_2^2-1)}{P_2(M_1^2-1)}}$, then $(w_1^*,w_2^*)=\Big(\sqrt{\frac{3P_2}{2M_1^2(M_2^2-1)}\frac{|h_2|}{|h_1|}},\sqrt{\frac{3P_2}{2(M_2^2-1)}}\Big)$, $d_{\text{noma}}=\sqrt{\frac{3P_2}{2M_1^2(M_2^2-1)}}|h_2|$;
- If $\sqrt{\frac{P_1M_1^2(M_2^2-1)}{P_2(M_1^2-1)}}<\frac{|h_2|}{|h_1|}$, then $(w_1^*,w_2^*)=\Big(\sqrt{\frac{3P_1}{2(M_1^2-1)}},\sqrt{\frac{3P_1M_1^2}{2(M_1^2-1)}\frac{|h_1|}{|h_2|}}\Big)$, $d_{\text{noma}}=\sqrt{\frac{3P_1}{2(M_1^2-1)}}|h_1|$.

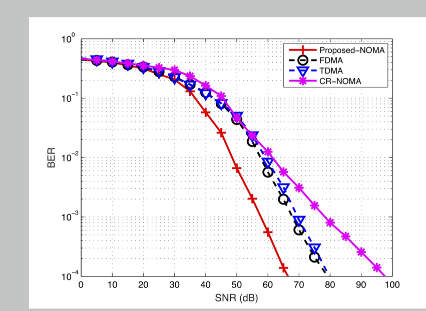
Corollary: The sum-constellation at the receiver is a *standard* $M_1^2M_2^2$-QAM constellation with the minimum Euclidean distance d_{noma}, where a quantization receiver can be employed to implement ML detection [R1].

Example: If $\frac{|h_2|}{|h_1|}\le\sqrt{\frac{P_1(M_2^2-1)}{P_2M_2^2(M_1^2-1)}}$, then $|h_1|w_1^*s_1+|h_2|w_2^*s_2=\sqrt{\frac{3P_2M_2^2}{2(M_2^2-1)}\frac{|h_2|}{|h_1|}}|h_1|s_1+\sqrt{\frac{3P_2}{2(M_2^2-1)}}|h_2|s_2=\sqrt{\frac{3P_2}{2(M_2^2-1)}}|h_2|(M_2s_1+s_2)$.
Recall that $s_1\in\{\pm(2k-1)\}_{k=1}^{M_1/2}$, $s_2\in\{\pm(2k-1)\}_{k=1}^{M_2/2}$, and therefore $M_2s_1+s_2\in\{\pm(2k-1)\}_{k=1}^{M_1M_2/2}$.

6. Numerical Results



(a)



(b)

Figure: Comparison between the Proposed-NOMA, CR-NOMA, TDMA and FDMA methods in Rayleigh fading with variances of the channel coefficients of δ_1^2 and δ_2^2 of User-1 and User-2, respectively: (a) $(\delta_1^2,\delta_2^2)=(1,1)$, (b) $(\delta_1^2,\delta_2^2)=(1,1/64)$.

- Each user adopts 64-QAM for the proposed NOMA design and 64-PSK is used by each user in CR-NOMA. Meanwhile, for TDMA and FDMA methods, each user uses 4096-QAM.
- From Fig.(1a) and Fig. (1b), the proposed NOMA design outperforms all the benchmark designs in moderate and high SNR regimes. The FDMA method has a better error performance than the TDMA scheme as expected. The CR-NOMA has the highest BER since the PSK constellation has a smaller Euclidean distance under the same power constraint compared with QAM constellation.
- The BER performance gap between the proposed NOMA and OMA methods are enlarged with "near-far" channel realizations.

7. Reference

[R1] Z. Dong, Y. Y. Zhang, J. K. Zhang and X. C. Gao, "Quadrature Amplitude Modulation Division for Multiuser MISO Broadcast Channels," IEEE J. Sel. Topics Signal Process., 2016.