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The Department of Intermedia Art and Science

ENVELOPE ESTIMATION BY TANGENTIALLY CONSTRAINED SPLINE

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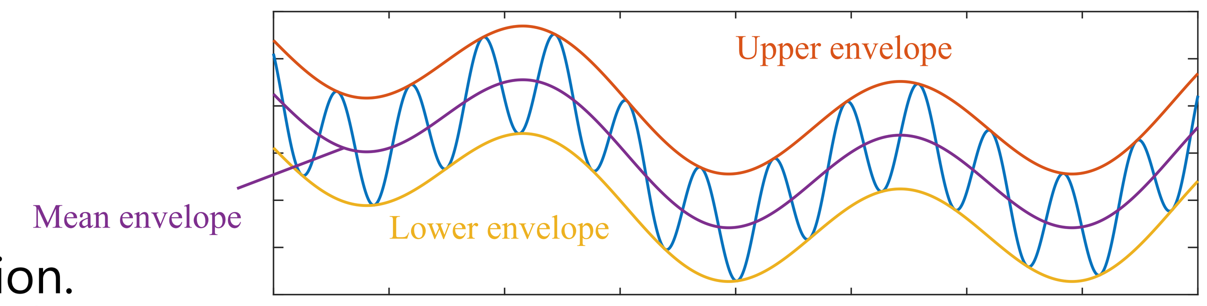
Introduction

- An envelope of a signal is a smooth wrapping curve containing much information about the signal.
  →it is widely considered in signal processing. (e.g., empirical mode decomposition (EMD)).
- The accuracy of EMD is determined by the accuracy of envelope estimation because subtraction of the mean of upper and lower envelopes is the central step in the process of EMD.
- Conventional envelope estimation method is interpolating all extrema using the cubic C^2-spline interpolation. It can obtain smooth envelopes, but often causes the undershoot problems. → **leads to erroneous results in EMD.**
- In this paper, tangentially constrained spline together with a gradient-based tangential points optimization method is proposed.

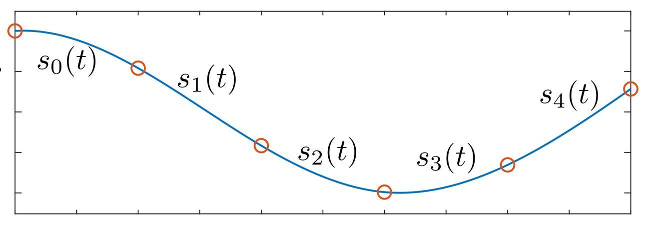


Spline functions and cubic C^2-spline interpolation

Spline functions

- A **spline is a continuous function defined by piecewise polynomials** that has been widely used in signal processing owing to its flexibility and optimality.
- These polynomials are different, but each polynomial is **connected continuously**.

$s_k(t)$: Polynomials.



Cubic C^2-spline interpolations

- Finding a cubic C^2-spline function passing through all given points (x_k, y_k) is characterized by

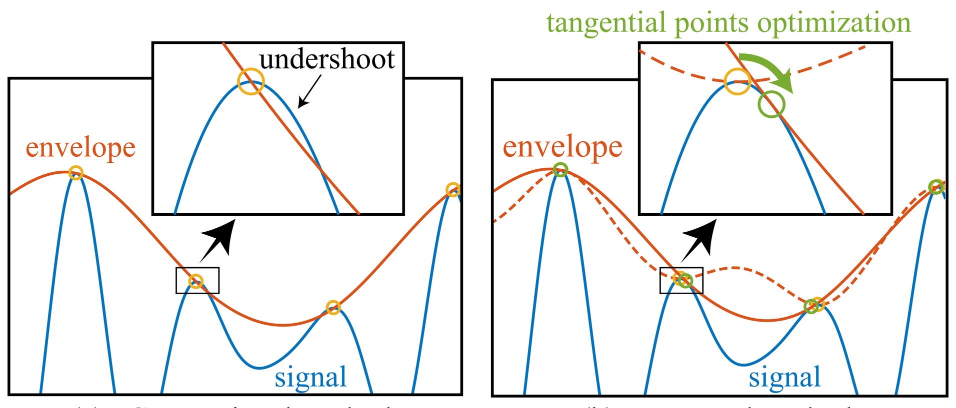
$$\underset{s\in S_3^2(\mathbf{x})}{\text{minimize}}\ \frac{1}{2}\int_{x_0}^{x_n}|s''(t)|^2dt \quad \text{subject to} \quad s(x_k)=y_k \ \text{ for all } k.$$

$S_3^2(\mathbf{x})$: Set of all spline functions of degree 3 and C^2-continuous.

Proposed method

- In order to estimate the envelope without undershoot problem while maintaining smoothness, a **tangentially constrained spline** with **tangential points optmization** is proposed.

- **Tangentially constrained spline** → An estimated envelope is **strictly tangent at interpolation points**.
- **Tangential points optmization** → An estimated envelope is **optimally smooth**.



(a) Conventional method (b) Proposed method

Tangentially constrained spline

- Tangentially constrainde spline s_{TC} :
  A quartic C^2-spline function constrained with first derivatives

$$\underset{s_{\mathrm{TC}}\in S_4^2(\boldsymbol{\tau})}{\text{minimize}}\ \frac{1}{2}\int_{\tau_0}^{\tau_n}|s''_{\mathrm{TC}}(t)|^2dt \quad \text{subject to} \quad s_{\mathrm{TC}}(\tau_k)=u(\tau_k),\ s'_{\mathrm{TC}}(\tau_k)=u'(\tau_k) \ \text{ for all } k\,.$$

$u(t)$: Signal.
$S_4^2(\boldsymbol{\tau})$: Set of all spline functions of degree 4 and C^2-continuous.
$\boldsymbol{\tau}=[\tau_0,\tau_1,\dots,\tau_n]^T$: Tangential points.

$\mathbf{s}=[\mathbf{z}^T,\mathbf{p}^T,\mathbf{P}^T]^T$, $\mathbf{z}=[z_0,\dots,z_n]^T$, $\mathbf{p}=[p_0,\dots,p_n]^T$, $\mathbf{P}=[P_0,\dots,P_n]^T$, $z_k=s_{\mathrm{TC}}(\tau_k)$, $p_k=s'_{\mathrm{TC}}(\tau_k)$, $P_k=s''_{\mathrm{TC}}(\tau_k)$, $h_k=\tau_{k+1}-\tau_k$

- $\displaystyle\int_{\tau_k}^{\tau_{k+1}}|s''_{\mathrm{TC}}(t)|^2dt=\frac{6}{5h_k}\left(p_k-p_{k+1}+\frac{h_k}{12}(P_k+P_{k+1})\right)^2+\frac{h_k}{24}(3P_k^2-2P_kP_{k+1}+3P_{k+1}^2),$
  $=\mathbf{s}_k^T\mathbf{A}_k\mathbf{s}_k$
- $s_{\mathrm{TC}}\in S_4^2(\boldsymbol{\tau})$
  $\rightarrow 12(z_k-z_{k+1})+6h_k(p_k+p_{k+1})+h_k^2(P_k-P_{k+1})=0$
- $s_{\mathrm{TC}}(\tau_k)=u(\tau_k),\ s'_{\mathrm{TC}}(\tau_k)=u'(\tau_k)$

$\Leftrightarrow\ \mathbf{Es}=\mathbf{b}$

- This problem can be descritized as

$$\underset{\mathbf{s}\in\mathbb{R}^{3n+3}}{\text{minimize}}\ \frac{1}{2}\mathbf{s}^T\mathbf{A}\,\mathbf{s} \quad \text{subject to} \quad \mathbf{Es}=\mathbf{b}\,. \qquad \cdots(*)$$

- Its solution is obtained by solving Karush–Kuhn–Tucker system,

$$\mathbf{K}\boldsymbol{\xi}-\tilde{\mathbf{b}}=\mathbf{0}. \qquad \boldsymbol{\xi}=\begin{bmatrix}\mathbf{s}\\ \boldsymbol{\nu}\end{bmatrix},\quad \mathbf{K}=\begin{bmatrix}\mathbf{A} & \mathbf{E}^T\\ \mathbf{E} & \mathbf{O}\end{bmatrix},\quad \tilde{\mathbf{b}}=\begin{bmatrix}\mathbf{0}\\ \mathbf{b}\end{bmatrix}$$

Tangential points optimization

- Smoothness of tangentially constrained spline depends on the choice of $\boldsymbol{\tau}$.
  →**The tangentially constrained spline is smooth only when appropriate tangential points are chosen as the interpolation points.**
- Optimization problem of tangential points is formulated as

$$\underset{\boldsymbol{\tau}\in\mathbb{R}^{n+1}}{\text{minimize}}\ I(\boldsymbol{\tau})=\frac{1}{2}\mathbf{s}^\star(\boldsymbol{\tau})^T\mathbf{A}(\boldsymbol{\tau})\,\mathbf{s}^\star(\boldsymbol{\tau}).$$

$\mathbf{s}^\star$: The solution of Eq. $(*)$.

- It can be rewritten as

$$\underset{\boldsymbol{\tau}\in\mathbb{R}^{n+1}}{\text{minimize}}\ \frac{1}{2}\boldsymbol{\xi}^T\tilde{\mathbf{A}}(\boldsymbol{\tau})\,\boldsymbol{\xi} \quad \text{subject to} \quad \mathbf{K}(\boldsymbol{\tau})\boldsymbol{\xi}-\tilde{\mathbf{b}}(\boldsymbol{\tau})=\mathbf{0}.$$

$$\tilde{\mathbf{A}}=\begin{bmatrix}\mathbf{A} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{bmatrix},\quad \boldsymbol{\xi}^T\tilde{\mathbf{A}}(\boldsymbol{\tau})\,\boldsymbol{\xi}=\begin{bmatrix}\mathbf{s}\\ \boldsymbol{\nu}\end{bmatrix}^T\begin{bmatrix}\mathbf{A} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{s}\\ \boldsymbol{\nu}\end{bmatrix}=\mathbf{s}^T\mathbf{A}(\boldsymbol{\tau})\,\mathbf{s}$$

- In order to solve the problem using a gradient-based optimization method, gradient $\nabla_{\boldsymbol{\tau}}I(\boldsymbol{\tau})$ is calculated by the following steps through the adjoint-state method:

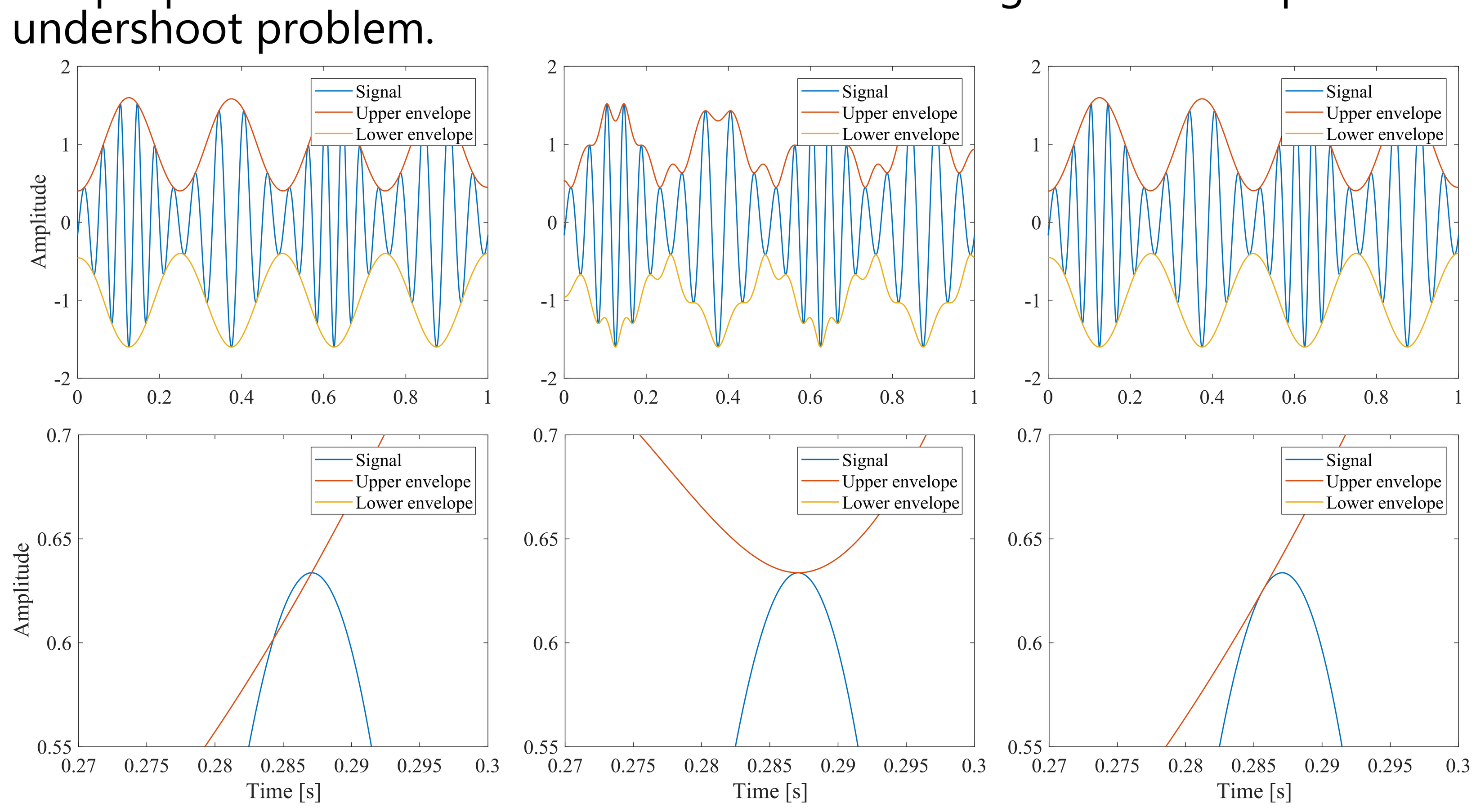
Step 1. Solve $\mathbf{K}\boldsymbol{\xi}^\star=\tilde{\mathbf{b}}$.
Step 2. Solve $\mathbf{K}\boldsymbol{\lambda}^\star=\tilde{\mathbf{A}}\boldsymbol{\xi}^\star$.
Step 3. Calculate the gradient by

$$\nabla_{\boldsymbol{\tau}}I(\boldsymbol{\tau})=\frac{1}{2}\mathbf{D}_{\mathbf{A}}^T\boldsymbol{\xi}^\star-\left(\mathbf{D}_{\mathbf{K}}-\frac{\partial\tilde{\mathbf{b}}}{\partial\boldsymbol{\tau}}\right)^T\boldsymbol{\lambda}^\star.\quad \mathbf{D}_{\mathbf{A}}=\left[\frac{\partial\tilde{\mathbf{A}}}{\partial\tau_0}\boldsymbol{\xi},\dots,\frac{\partial\tilde{\mathbf{A}}}{\partial\tau_n}\boldsymbol{\xi}\right],\quad \mathbf{D}_{\mathbf{K}}=\left[\frac{\partial\mathbf{K}}{\partial\tau_0}\boldsymbol{\xi},\dots,\frac{\partial\mathbf{K}}{\partial\tau_n}\boldsymbol{\xi}\right]$$

Numerical experiment

Comparison of estimated envelopes

- An envelope estimation problem of a simulated signal was considered.
- The proposed method can estimate a smooth tangential envelope without undershoot problem.

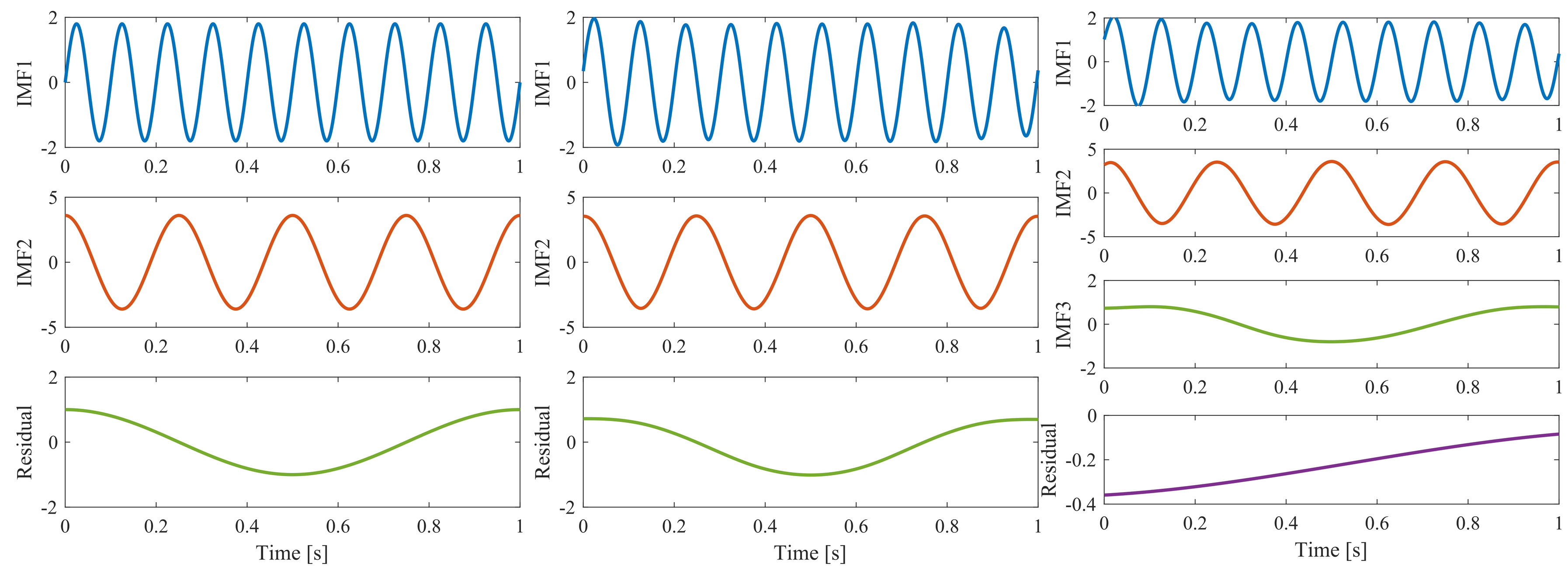


Cubic spline interpolation | Proposed method without tangential points optmization | Proposed method

Estimated envelopes based on the spline interpolation methods.

Application of the EMD

- EMD was applied to a signal consisting of three components.
- EMD with the proposed method correctly recovered the original components because undershoot is avoided in the proposal.
  → **The envelopes estimated by the proposed method may have better characteristics than the conventional ones.**



Components of the original signal | The result of EMD based on the proposed method | The result of EMD based on the conventional method

Results of EMD with different envelope estimation methods.

Conclusion

- A tangentially constrained spline, which is a quartic C^2-spline constrained by first derivatives at the tangential points is proposed for estimating envelopes without undershoot problem.
- A tangential points optimization method is also proposed so that an optimally smooth envelope among the proposed splines is obtained.
- Future works include considerations of appropriate boundary conditions.