PhaseSplit: A Variable Splitting Framework for Phase Retrieval

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1. Phase Retrieval

- Objective: To estimate a signal $\boldsymbol{x}^* \in \mathbb{R}^n$ from noisy quadratic measurements of the form $y_i = \left|\boldsymbol{a}_i^\top \boldsymbol{x}^*\right|^2 + \xi_i,\ i = 1, 2, \cdots, m$.
- Ill-posed in general, can be solved using oversampling ($m > n$), or a signal prior such as sparsity, or both.
- Applications: X-ray crystallography, microscopy, astronomy, frequency-domain optical coherence tomography (FDOCT), etc.

2. The Proposed Variable Splitting Approach

- Maximum-likelihood estimation: $\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \frac{1}{2}\sum_{i=1}^m \left(y_i - \left|\boldsymbol{a}_i^\top \boldsymbol{x}\right|^2\right)^2$.
  - Nonconvex in $\boldsymbol{x}$, no closed-form solution.
- Variable splitting: Write $\left|\boldsymbol{a}_i^\top \boldsymbol{x}\right|^2 = \boldsymbol{x}^\top \boldsymbol{a}_i \boldsymbol{a}_i^\top \boldsymbol{x}$ as $\boldsymbol{u}^\top \boldsymbol{A}_i \boldsymbol{v}$ subject to $\boldsymbol{u} = \boldsymbol{v}$.

$$\text{PhaseSplit} : (\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}) = \arg\min_{\boldsymbol{u},\boldsymbol{v}} \frac{1}{2}\sum_{i=1}^m \left(y_i - \boldsymbol{u}^\top \boldsymbol{A}_i \boldsymbol{v}\right)^2 \text{ subject to } \boldsymbol{u} = \boldsymbol{v}.$$

- Enforcing the sparsity prior:

$$(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}) = \arg\min_{\boldsymbol{u},\boldsymbol{v}} \frac{1}{2}\sum_{i=1}^m \left(y_i - \boldsymbol{u}^\top \boldsymbol{A}_i \boldsymbol{v}\right)^2 \text{ s.t. } \boldsymbol{u} = \boldsymbol{v} \text{ and } \|\boldsymbol{u}\|_0 \leqslant s.$$

3. PhaseSplit with Alternating Minimization

- Solve $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}) = \arg\min_{\boldsymbol{u},\boldsymbol{v}} \frac{1}{2}\sum_{i=1}^m \left(y_i - \boldsymbol{u}^\top \boldsymbol{A}_i \boldsymbol{v}\right)^2 + \frac{\lambda}{2}\|\boldsymbol{u} - \boldsymbol{v}\|_2^2$.
- The cost, when viewed as a function of $\boldsymbol{u}$ for a fixed $\boldsymbol{v}$, can be expressed as

$$q(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u}^\top \boldsymbol{C}_{\boldsymbol{v}} \boldsymbol{u} - \boldsymbol{d}_{\boldsymbol{v}}^\top \boldsymbol{u} + (\text{terms independent of } \boldsymbol{u}), \text{ where}$$

$$\boldsymbol{C}_{\boldsymbol{v}} = \lambda \boldsymbol{I} + \sum_{i=1}^m \left(\boldsymbol{a}_i^\top \boldsymbol{v}\right)^2 \boldsymbol{A}_i \text{ and } \boldsymbol{d}_{\boldsymbol{v}} = \lambda \boldsymbol{v} + \sum_{i=1}^m \left(\boldsymbol{a}_i^\top \boldsymbol{v}\right) y_i \boldsymbol{a}_i. \quad (1)$$

- $q(\boldsymbol{u})$ is a convex quadratic in $\boldsymbol{u}$, and can be minimized in closed-form for a fixed $\boldsymbol{v}$. Similarly, the cost can be minimized over $\boldsymbol{v}$ for a fixed $\boldsymbol{u}$.

PS-AM Algorithm

- **Input**: Measurements $\{y_i\}_{i=1}^m$, the sampling vectors $\{\boldsymbol{a}_i\}_{i=1}^m$, maximum number of iterations N_{iter}, and λ.
- **Spectral initialization:** Set $t = 0$ and $\boldsymbol{v}^t = \boldsymbol{v}_{\max}$, the eigenvector corresponding to the largest eigenvalue of $\boldsymbol{S} = \sum_{i=1}^m y_i \boldsymbol{a}_i \boldsymbol{a}_i^\top$.
- **For** $t = 1 : N_{\text{iter}}$ **do**:
  1. $\boldsymbol{u}^t = \boldsymbol{C}_{\boldsymbol{v}^{t-1}}^{-1} \boldsymbol{d}_{\boldsymbol{v}^{t-1}}$, where $\boldsymbol{C}_{\boldsymbol{v}}$ and $\boldsymbol{d}_{\boldsymbol{v}}$ are as in (1),
  2. $\boldsymbol{v}^t = \boldsymbol{C}_{\boldsymbol{u}^t}^{-1} \boldsymbol{d}_{\boldsymbol{u}^t}$, and
- **Output:** the current estimates $\boldsymbol{u}^t$ or $\boldsymbol{v}^t$.

A Fixed-Point Interpretation of PS-AM

- Define $\boldsymbol{v}^t = \boldsymbol{x}^{2t}$ and $\boldsymbol{u}^{t+1} = \boldsymbol{x}^{2t+1}$, for $t = 0, 1, 2, \cdots$.
- The update rule can be expressed succinctly as $\boldsymbol{x}^{t+1} = \boldsymbol{C}_{\boldsymbol{x}^t}^{-1} \boldsymbol{d}_{\boldsymbol{x}^t}$.
- In the absence of measurement noise, $y_i = \left|\boldsymbol{a}_i^\top \boldsymbol{x}^*\right|^2 \implies \boldsymbol{d}_{\boldsymbol{u}} = \left(\lambda \boldsymbol{I} + \sum_{i=1}^m y_i \boldsymbol{A}_i\right)\boldsymbol{u} = \boldsymbol{C}_{\boldsymbol{x}^*}\boldsymbol{u}$.
- The update rule becomes $\boldsymbol{x}^{t+1} = \boldsymbol{C}_{\boldsymbol{x}^t}^{-1} \boldsymbol{C}_{\boldsymbol{x}^*} \boldsymbol{x}^t = h(\boldsymbol{x}^t)$: Fixed-point algorithm.
- $h(\boldsymbol{x}^*) = \boldsymbol{x}^*$: The ground-truth $\boldsymbol{x}^*$ is a fixed-point.

4. Alternating Direction Method of Multipliers

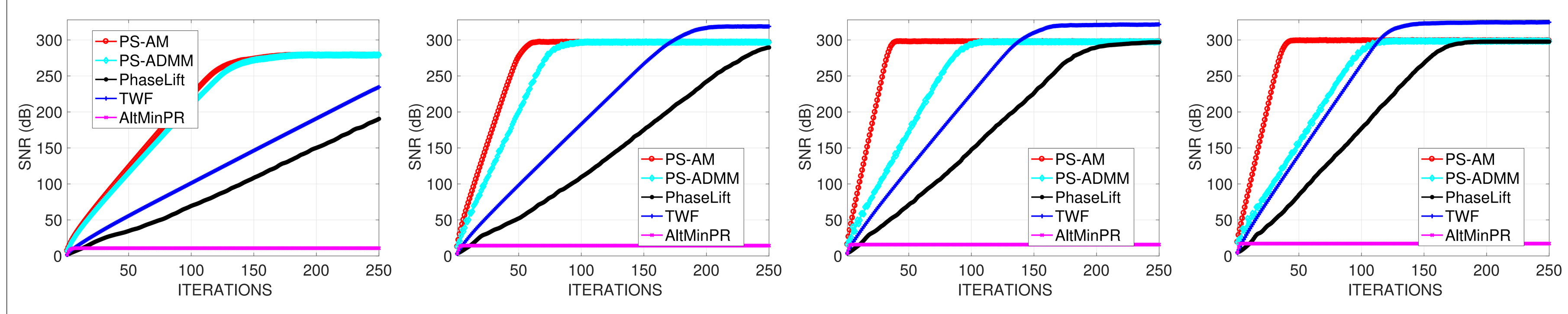
- Augmented Lagrangian function:
  $\mathcal{L}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\mu}) = \frac{1}{2}\sum_{i=1}^m \left(y_i - \boldsymbol{u}^\top \boldsymbol{A}_i \boldsymbol{v}\right)^2 + \frac{\lambda}{2}\|\boldsymbol{u} - \boldsymbol{v}\|_2^2 + \boldsymbol{\mu}^\top (\boldsymbol{u} - \boldsymbol{v})$.

PS-ADMM Algorithm

- **Input:** Measurements $\{y_i\}_{i=1}^m$, $\{\boldsymbol{a}_i\}_{i=1}^m$, N_{iter}, and λ.
- **Spectral initialization:** Set $t = 0$, initialize $\boldsymbol{v}^t$ and $\boldsymbol{\mu}^t$.
- **For** $t = 1 : N_{\text{iter}}$ **do**
  1. $\boldsymbol{u}^{t+1} = \arg\min_{\boldsymbol{u}} \mathcal{L}(\boldsymbol{u}, \boldsymbol{v}^t, \boldsymbol{\mu}^t) = \boldsymbol{C}_{\boldsymbol{v}^t}^{-1}\left(\boldsymbol{d}_{\boldsymbol{v}^t} - \boldsymbol{\mu}^t\right)$,
  2. $\boldsymbol{v}^{t+1} = \arg\min_{\boldsymbol{v}} \mathcal{L}(\boldsymbol{u}^{t+1}, \boldsymbol{v}, \boldsymbol{\mu}^t) = \boldsymbol{C}_{\boldsymbol{u}^{t+1}}^{-1}\left(\boldsymbol{d}_{\boldsymbol{u}^{t+1}} + \boldsymbol{\mu}^t\right)$, and
  3. $\boldsymbol{\mu}^{t+1} = \boldsymbol{\mu}^t + \lambda\left(\boldsymbol{u}^{t+1} - \boldsymbol{v}^{t+1}\right)$.
- **Output:** the current estimate $\boldsymbol{u}^t$ or $\boldsymbol{v}^t$.
- The sparse counterparts of PS-AM and PS-ADMM are obtained by inserting the additional step of obtaining the best s-sparse approximation of the updates in each iteration: $\boldsymbol{v}^t \leftarrow \mathcal{P}_s\left(\boldsymbol{v}^t\right)$.

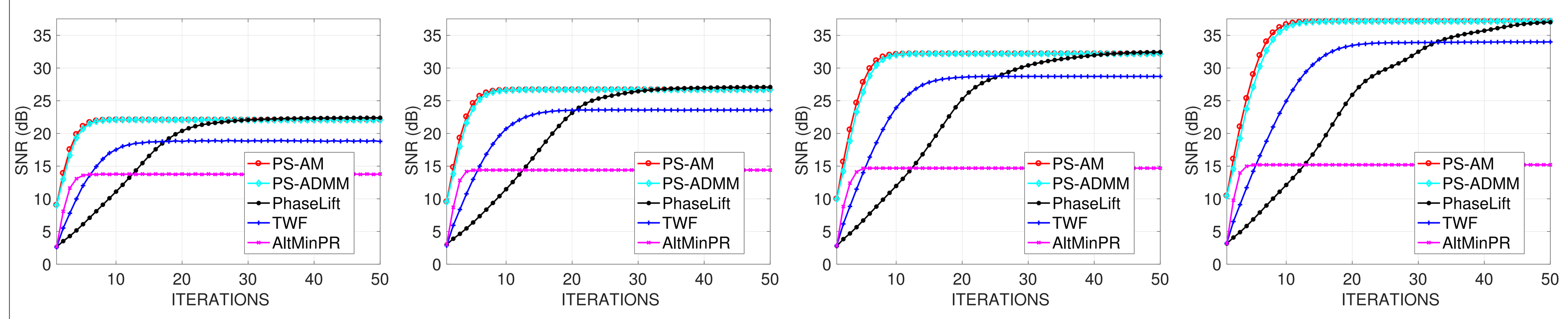
5. Simulation Results

- Clean measurements: Reconstruction SNR vs. the oversampling factor $\frac{m}{n}$.



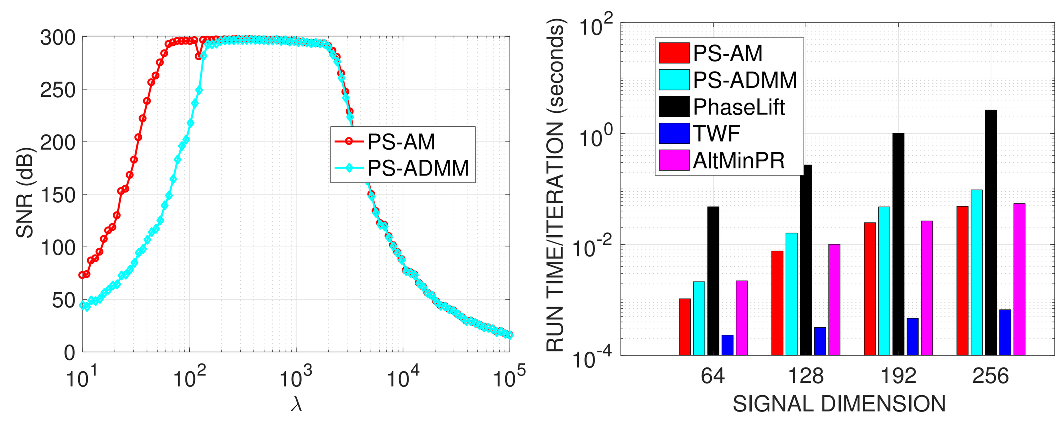
(a) $\frac{m}{n} = 4$ (b) $\frac{m}{n} = 6$ (c) $\frac{m}{n} = 8$ (d) $\frac{m}{n} = 10$

- Noisy measurements: $\frac{m}{n} = 6$, output SNR values averaged over 20 trials.



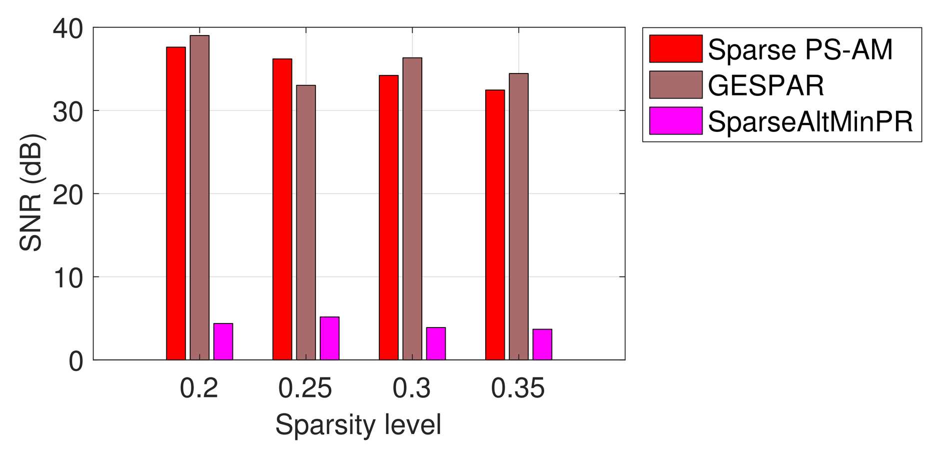
(e) $\text{SNR}_{\text{in}} = 15$ dB (f) $\text{SNR}_{\text{in}} = 20$ dB (g) $\text{SNR}_{\text{in}} = 25$ dB (h) $\text{SNR}_{\text{in}} = 30$ dB

- Parameter selection and run-time comparison
  - A large range of λ values lead to accurate reconstruction.
  - Same per-iteration complexity as PhaseLift and AltMinPR ($\mathcal{O}(n^3)$).



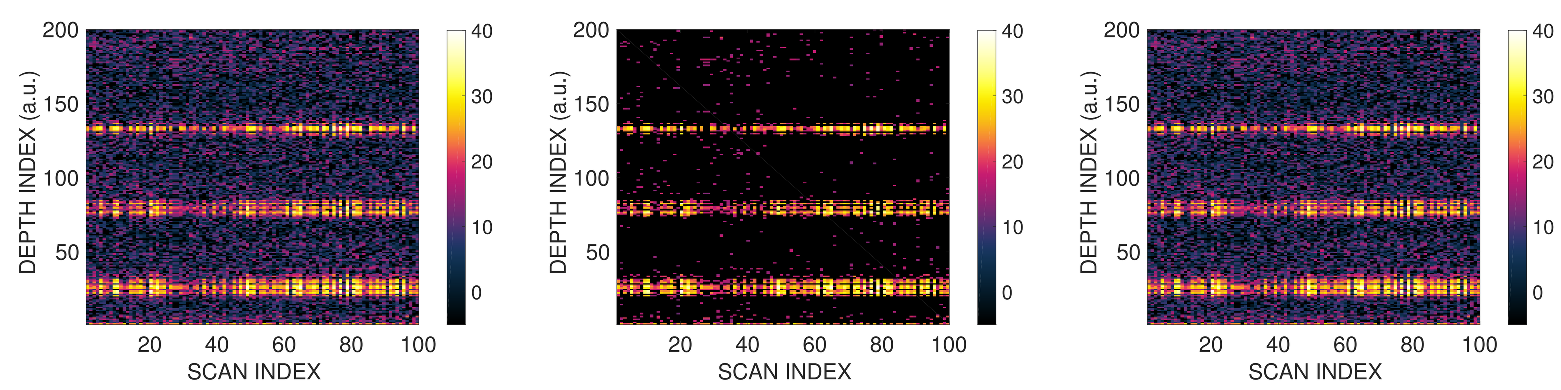
(i) Reconstruction SNR versus λ (j) Run-time versus n

- PhaseSplit with sparsity prior



6. Application to FDOCT

- Objective: To reconstruct the backscattered wave from the Fourier intensity of its interference with the reference wave.
  - Exhibits a strong peak when there is a change in refractive index in the specimen.
  - The backscattered wave is sparse and contains structural information about the specimen.



PS-AM Sparse PS-AM Direct inversion

Figure 1: Reconstruction of glass specimen.

7. Summary

- Proposed a variable-splitting approach for phase retrieval and developed two algorithms based on alternating minimization and ADMM.
- Sparsity prior can be enforced via a hard-thresholding step.
- Superior convergence, robust to parameter selection, and same per-iteration complexity as PhaseLift.
- Demonstrated an application to image reconstruction in FDOCT. Imposing the sparsity prior helps eliminate background noise.

References

- P. Netrapalli, P. Jain, and S. Sanghavi, "Phase retrieval using alternating minimization," *IEEE TSP*, vol. 63, no. 18, pp. 4814–4826, Sep. 2015.
- Y. Wang, W. Yin, and J. Zeng, "Global convergence of ADMM in nonconvex nonsmooth optimization," *arXiv:1511.06324v4*, Nov. 2016.

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