ESTIMATION OF SCATTER MATRIX WITH CONVEX STRUCTURE UNDER T-DISTRIBUTION

Bruno Mériaux[1], Chengfang Ren[1], Mohammed Nabil El Korso[2], Arnaud Breloy[2] and Philippe Forster[3]

[1]SONDRA, CentraleSupélec, France [2]Université Paris-Nanterre/LEME, France [3]SATIE, ENS Paris-Saclay, France

Objectives

Covariance Matrix (CM):

- plays a central role in adaptive signal processing ⇒ CM estimation
- generally exhibits a specific structure (e.g. Toeplitz for ULA)

Structured CM estimation:

- Gaussian context: COvariance Matching Estimation Technique [1]
- t-distribution framework: used as heavy-tailed model
  - normalizing the data: RCOMET [2], COCA [3], Constrained Tyler [4]
  - taking into account the texture → still an open problem

The purposes of this work consist in:

- proposing a new estimation procedure, for t-distributed data with a convexly structured CM matrix.
- studying the asymptotic performance: consistency, normality and efficiency.

Problem setup

N i.i.d. t-distributed data, $\mathbf{y}_n \sim \mathbb{C}t_{m,d}(\mathbf{0}, \mathbf{R})$, $n = 1, \ldots, N$ [5]:

- $\mathbf{y}_n \in \mathbb{C}^m$ with $N > m$
- d degrees of freedom assumed known

Scatter matrix $\mathbf{R}$

- belongs to $\mathcal{S}$, a convex subset of Hermitian positive-definite matrices
- there exists a one-to-one differentiable mapping $\boldsymbol{\mu} \mapsto \mathbf{R}(\boldsymbol{\mu})$ from $\mathbb{R}^P$ to $\mathcal{S}$

Unknown interest parameter: $\boldsymbol{\mu} \in \mathbb{R}^p$, with exact value $\boldsymbol{\mu}_e$

Maximum Likelihood Estimator (MLE) of $\boldsymbol{\mu}$

$$\hat{\boldsymbol{\mu}}_{\mathrm{ML}} = \arg\max_{\boldsymbol{\mu}} \; -(d+m)\sum_{n=1}^{N} \log\left(1 + \frac{\mathbf{y}_n^H \mathbf{R}(\boldsymbol{\mu})^{-1}\mathbf{y}_n}{d}\right) - N \log|\mathbf{R}(\boldsymbol{\mu})|$$

Fisher information matrix

Let be $\mathbf{y} \sim \mathbb{C}t_{m,d}(\mathbf{0}, \mathbf{R}(\boldsymbol{\mu}_e))$, with $\boldsymbol{\mu}_e \in \mathbb{R}^P$. The FIM is expressed by [6]

$$\mathbf{F}(\boldsymbol{\mu}_e) = \left.\frac{\partial \mathbf{r}(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}}\right|_{\boldsymbol{\mu}_e}^H \mathbf{Y}_e \left.\frac{\partial \mathbf{r}(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}}\right|_{\boldsymbol{\mu}_e} \qquad (1)$$

where $\frac{\partial \mathbf{r}(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}}$ refers to the Jacobian matrix of $\mathbf{r}(\boldsymbol{\mu}) = \mathrm{vec}(\mathbf{R}(\boldsymbol{\mu}))$,

$\mathbf{W}_e = \mathbf{R}_e^T \otimes \mathbf{R}_e$ and $\mathbf{Y}_e = \dfrac{(d+m)\mathbf{W}_e^{-1} - \mathrm{vec}\left(\mathbf{R}_e^{-1}\right)\mathrm{vec}\left(\mathbf{R}_e^{-1}\right)^H}{d+m+1}$.

Main results

Proposed algorithm

- Step 1: unstructured MLE of $\mathbf{R}$

The unstructured MLE, $\widehat{\mathbf{R}}$, is the solution of the fixed point equation:

$$\widehat{\mathbf{R}} = \frac{d+m}{N}\sum_{n=1}^{N} \frac{\mathbf{y}_n \mathbf{y}_n^H}{d + \mathbf{y}_n^H \widehat{\mathbf{R}}^{-1}\mathbf{y}_n} \triangleq \mathcal{H}_N(\widehat{\mathbf{R}}) \qquad (2)$$

Existence and uniqueness of this solution, convergence of the iterative algorithm $\mathbf{R}_{k+1} = \mathcal{H}_N(\mathbf{R}_k)$ to $\widehat{\mathbf{R}}$ for any initialization and consistency of $\widehat{\mathbf{R}}$ are ensured [5].

- Step 2: Estimation of $\boldsymbol{\mu}$

The minimization of the following criterion w.r.t $\boldsymbol{\mu}$

$$\hat{\boldsymbol{\mu}} = \arg\min_{\boldsymbol{\mu}} \; (\hat{\mathbf{r}} - \mathbf{r}(\boldsymbol{\mu}))^H \, \widehat{\mathbf{Y}} \, (\hat{\mathbf{r}} - \mathbf{r}(\boldsymbol{\mu})) \qquad (3)$$

with $\widehat{\mathbf{Y}} = (d+m)\widehat{\mathbf{W}}^{-1} - \mathrm{vec}\left(\widehat{\mathbf{R}}^{-1}\right)\mathrm{vec}\left(\widehat{\mathbf{R}}^{-1}\right)^H$ and $\widehat{\mathbf{W}} = \widehat{\mathbf{R}}^T \otimes \widehat{\mathbf{R}}$ yields a unique solution $\hat{\boldsymbol{\mu}}$ for $\boldsymbol{\mu}$.

Asymptotic analysis

$\hat{\boldsymbol{\mu}}$, obtained by (3), is consistent, asymptotically Gaussian and efficient:

$$\sqrt{N}\left(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_e\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}, \mathbf{F}(\boldsymbol{\mu}_e)^{-1}\right)$$

Numerical results

Simulation settings:

- $m = 4$, $d = 5$, 5000 sets of N i.i.d. $\mathbf{y}_n \sim \mathbb{C}t_{m,d}(\mathbf{0}, \mathbf{R}_e)$, $n = 1, \ldots, N$.
- $\mathbf{R}_e \triangleq \mathbf{R}(\boldsymbol{\mu}_e)$ is Hermitian Toeplitz, $\boldsymbol{\mu}_e$ is a real-valued vector containing the real and imaginary parts of the first row of $\mathbf{R}_e$.

Comparison of the performance with the state of the art:

- Proposed algorithm with unstructured ML, $\widehat{\mathbf{R}}$ as step 1
- Proposed algorithm with joint estimation of d and $\mathbf{R}$ [7] as step 1
  ⇒ to deal with the possibility of unknown parameter d
- RCOMET [2] and COCA [3] based on $\mathbf{z}_n = \mathbf{y}_n / \|\mathbf{y}_n\|$
- Projection onto the Toeplitz set by averaging the diagonals of $\widehat{\mathbf{R}}$

Conclusion

In this paper, we addressed structured covariance estimation for convex structures. A consistent, asymptotically unbiased and efficient estimator is proposed for t-distribution. A generalization for any Complex Elliptically Symmetric distributions is studied in [8]. Numerical simulations confirm the theoretical analysis and the practical interest of this approach.

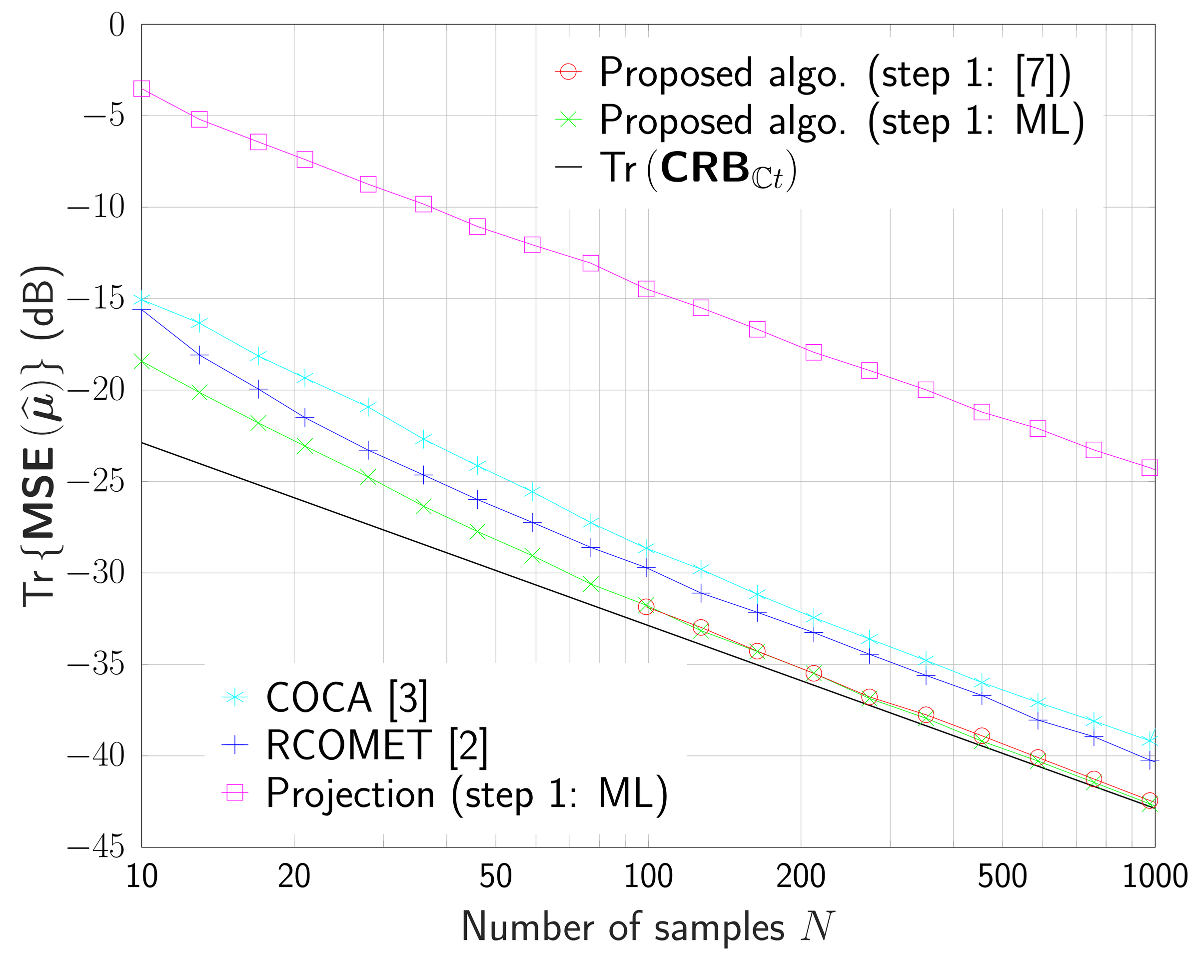

Figure: Efficiency simulation

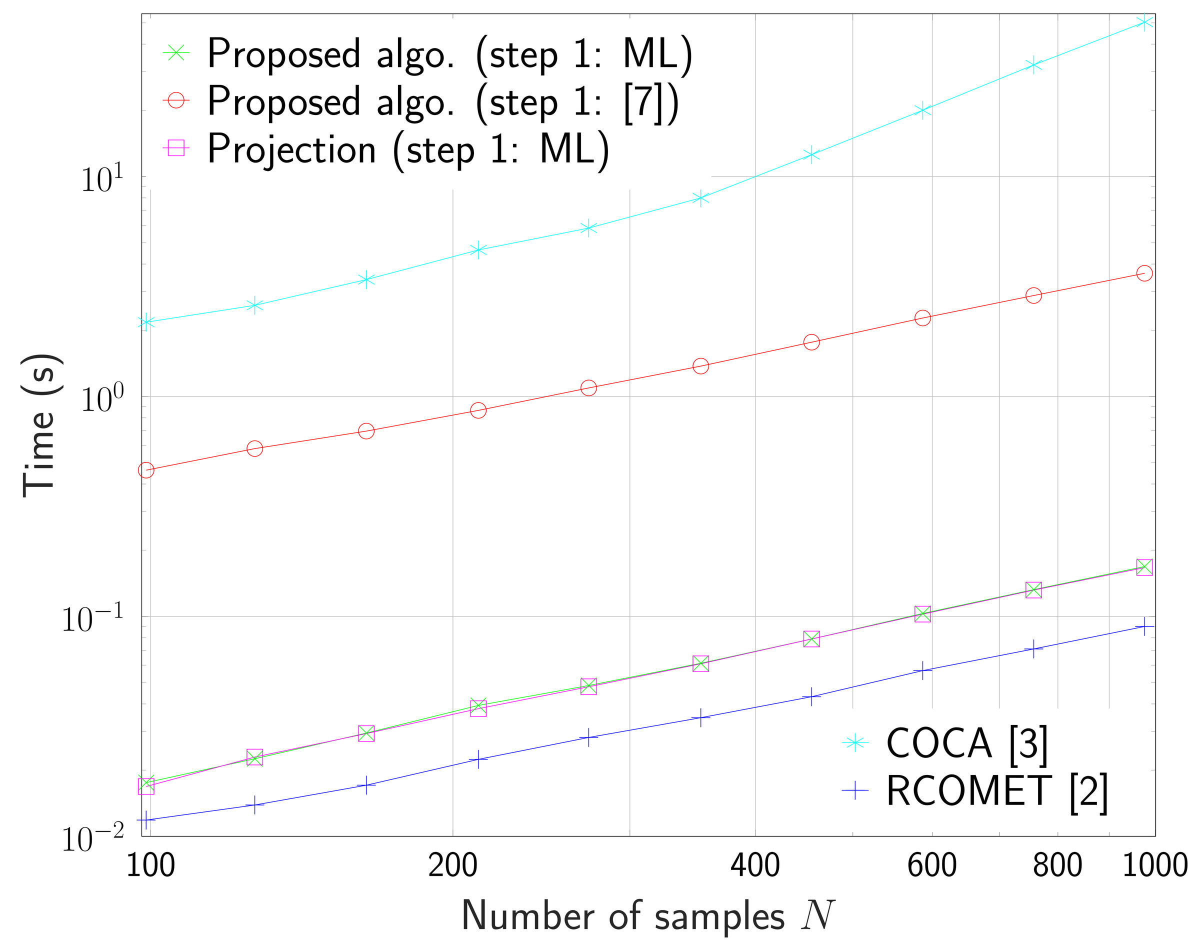

Figure: Average computing time

References

[1] B. Ottersten, P. Stoica, and R. Roy, "Covariance matching estimation techniques for array signal processing applications," *ELSEVIER DSP*, vol. 8, no. 3, pp. 185–210, 1998.

[2] B. Mériaux, C. Ren, M. N. El Korso, A. Breloy, and P. Forster, "Robust-COMET for covariance estimation in convex structures: algorithm and statistical properties," in *Proc. of IEEE CAMSAP*, 2017.

[3] I. Soloveychik and A. Wiesel, "Tyler's covariance matrix estimator in elliptical models with convex structure," *IEEE TSP*, vol. 62, no. 20, pp. 5251–5259, Oct. 2014.

[4] Y. Sun, P. Babu, and D. P. Palomar, "Robust estimation of structured covariance matrix for heavy-tailed elliptical distributions," *IEEE TSP*, vol. 14, no. 64, pp. 3576–3590, Jul. 2016.

[5] E. Ollila, D. E. Tyler, V. Koivunen, and H. V. Poor, "Complex elliptically symmetric distributions: Survey, new results and applications," *IEEE TSP*, vol. 60, no. 11, pp. 5597–5625, Nov. 2012.

[6] O. Besson and Y. I. Abramovich, "On the fisher information matrix for multivariate elliptically contoured distributions," *IEEE SPL*, vol. 20, no. 11, pp. 1130–1133, Nov. 2013.

[7] S. Fortunati, F. Gini, and M. S. Greco, "Matched, mismatched, and robust scatter matrix estimation and hypothesis testing in complex t-distributed data," *EURASIP Journal on Advances in Signal Processing*, vol. 2016, no. 1, p. 123, Nov. 2016.

[8] B. Mériaux, C. Ren, M. N. El Korso, A. Breloy, and P. Forster, "Efficient structured scatter matrix estimation for complex elliptically symmetric distributions," submitted to *IEEE TSP*.