Grid-Free Direction-of-Arrival Estimation with Compressed Sensing and Arbitrary Antenna Arrays

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Direction of Arrival Estimation

Reconstruction Schemes

Main Results

Simulations

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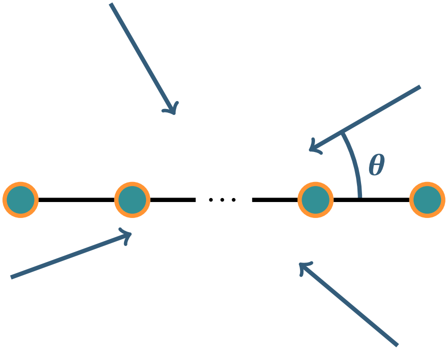
Preliminaries & Motivation

- Given an array of M antennas
- Planar (narrowband) waves in the far field impinging from unknown directions
- We take noisy measurements
- **Goal:** Estimate directions of arrival (DOA) for each wave
- **Motivation:** direction finding [1], massive MIMO and 5G [2] and radar [3]



[1] Valaee, Champagne, and Kabal, "Parametric localization of distributed sources", *1995*.
[2] Chen, Yao, and Hudson, "Source localization and beamforming", *2002*.
[3] Blair and Brandt-Pearce, "Monopulse DOA estimation of two unresolved Rayleigh targets", *2001*.

State of the Art

- Grid-free subspace based DOA (MUSIC, ESPRIT)
- Grid-bound Sparse Recovery based DOA [4]
- Grid-free sparsity based line spectral estimation [5]
- Grid-free Sparse Recovery based DOA for (randomly subselected) ULAs [6]
- Grid-free compressed sensing based line spectral estimation [7]

[4] Malioutov, Cetin, and Willsky, "A sparse signal reconstruction perspective for source localization with sensor arrays", *2005*.
[5] Bhaskar, Tang, and Recht, "Atomic Norm Denoising With Applications to Line Spectral Estimation", *2013*.
[6] Xenaki and Gerstoft, "Grid-free compressive beamforming", *2015*.
[7] Heckel and Soltanolkotabi, "Generalized Line Spectral Estimation via Convex Optimization", *2017*.

Main Contributions

- Grid-free sparsity based DOA estimation via *ANM*
- Arbitary antenna arrays via *Effective Aperture Distribution Function* (EADF)
- Spatial compression with an analog combining network using generalized line spectral estimation

Beampatterns

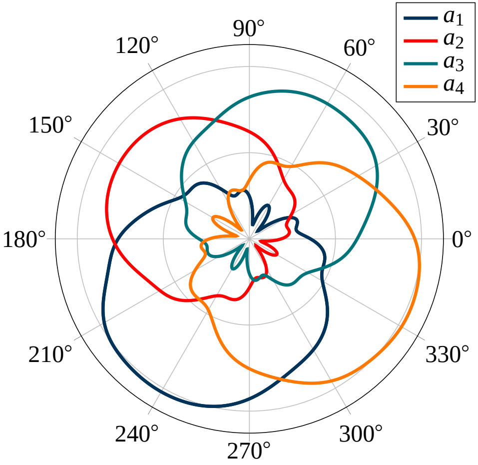


Figure. Beampatterns of $M = 4$ elements.

- Let $\mathbf{a}(\theta) : [0, 2\pi) \to \mathbb{C}^M$ model response of antenna array comprising of M antennas for a planar wave impinging from azimuth angle θ
- Can be measured in practice
- Can model **arbitrary**, thus more **realistic** antennas
- Formal model for beam patterns?

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Fourier Series to the Rescue

- Antenna patterns $\mathbf{a}(\theta)$ are periodic in θ and smooth
- Good approximation via truncated Fourier series element wise

$$a_m(\theta) \approx \sum_{\ell=-\frac{L-1}{2}}^{\frac{L-1}{2}} g_{m,\ell} \mathrm{e}^{\jmath\theta\ell}$$

- For $\mathbf{G} \in \mathbb{C}^{M \times L}$ containing the $g_{m,\ell}$ and $\mathbf{f}(\theta) = \left[\mathrm{e}^{-\jmath\theta(L-1)/2}, \ldots, \mathrm{e}^{\jmath\theta(L-1)/2}\right]$ we can summarize

$$\mathbf{a}(\theta) \approx \mathbf{G} \cdot \mathbf{f}(\theta)$$

- Concise and efficient description of the whole antenna behaviour [8]
- We observe measurements of S waves impinging with unknown amplitudes c_s

$$\mathbf{x} = \sum_{s=1}^{S} c_s \cdot \mathbf{a}(\theta_s) = \mathbf{G} \cdot \sum_{s=1}^{S} c_s \cdot \mathbf{f}(\theta_s)$$

[8] Landmann and Galdo, “Efficient antenna description for MIMO channel modelling and estimation”, *2004*.

Spice things up with Compression

- **Motivation:** reduction of hardware complexity
- We currently have $\mathbf{G} \in \mathbb{C}^{M \times L}$ and

$$\mathbf{x} = \mathbf{G} \cdot \sum_{s=1}^{S} c_s \cdot \mathbf{f}(\theta_s)$$

- Pick a matrix $\mathbf{\Phi} \in \mathbb{C}^{K \times M}$ for some $K \in \mathbb{N}$ and we get

$$\mathbf{y} = \mathbf{\Phi} \cdot \mathbf{x} = \mathbf{\Phi} \cdot \mathbf{G} \cdot \sum_{s=1}^{S} c_s \cdot \mathbf{f}(\theta_s) + \mathbf{n}$$

- Action of $\mathbf{\Phi}$ is realized via analog combining

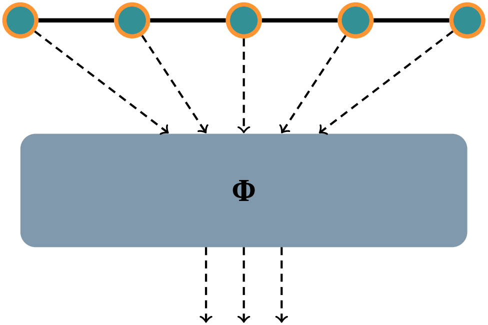

Figure. Spatial compression scheme from 5 antenna ports down to 3 outputs

Atomic Norm Minimization for Line Spectral Estimation

- For the much simpler model

$$\mathbf{z} = \sum_{s=1}^{S} c_s \mathbf{f}(\theta_s) + \mathbf{n}$$

  with $\mathbf{f}(\theta) = \left[\mathrm{e}^{-\jmath\theta(L-1)/2}, \ldots, \mathrm{e}^{\jmath\theta(L-1)/2}\right]$

- Define an atomic set

$$\mathscr{A} = \{\mathbf{f}(\theta) \in \mathbb{C}^M \mid \theta \in [0, 2\pi)\}$$

- Define the atomic norm

$$\mathbf{x} \mapsto \|\mathbf{x}\|_{\mathscr{A}} = \inf\{t > 0 \mid \mathbf{x} \in t \cdot \mathrm{conv}(\mathscr{A})\}$$

- Pose following convex optimization problem for some suitably chosen ϵ

$$\min_{\mathbf{x}} \|\mathbf{x}\|_{\mathscr{A}} \text{ subject to } \|\mathbf{z} - \mathbf{x}\|_2 \leq \epsilon$$

$\rightsquigarrow$ Atomic Norm Minimization (ANM)

The Semidefinite Reformulation

- **Problem:** How to get a solution to the atomic norm minimization?
- **Solution:**
  - Reformulate it as a semidefinite program according to [9] using duality:

$$\begin{aligned} \min_{(\mathbf{x},\mathbf{u},t)\in\mathbb{C}^L\times\mathbb{C}^L\times\mathbb{R}} \quad & \frac{1}{2n}\operatorname{tr}\operatorname{Toep}(\mathbf{u}) + \frac{1}{2}t \\ \text{subject to} \quad & \begin{pmatrix} \operatorname{Toep}(\mathbf{u}) & \mathbf{x} \\ \mathbf{x}^H & t \end{pmatrix} \succeq 0, \\ & \|\mathbf{z}-\mathbf{x}\|_2 \leqslant \epsilon. \end{aligned} \tag{1}$$

  - Extract the frequencies, i.e. DOAs $\theta_1,\ldots,\theta_S$, from $\operatorname{Toep}(\mathbf{u})$ using covariance based spectral estimation methods, like MUSIC or ESPRIT.

[9] Megretski, “Positivity of trigonometric polynomials”, *2003*.

Atomic Norm Minimization for DOA

- Incorporate the model for compression and EADF in optimization
- Modify atomic norm minimization to

$$\min_{\mathbf{x}} \ \|\mathbf{x}\|_{\mathscr{A}} \ \text{subject to} \ \|\mathbf{z} - \boldsymbol{\Phi} \cdot \mathbf{G} \cdot \mathbf{x}\|_2 \leq \epsilon$$

- Modify the semidefinite program to

$$\begin{aligned} \min_{(\mathbf{x},\mathbf{u},t) \in \mathbb{C}^L \times \mathbb{C}^L \times \mathbb{R}} \quad & \frac{1}{2n} \operatorname{tr} \operatorname{Toep}(\mathbf{u}) + \frac{1}{2} t \\ \text{subject to} \quad & \begin{pmatrix} \operatorname{Toep}(\mathbf{u}) & \mathbf{x} \\ \mathbf{x}^H & t \end{pmatrix} \succeq 0, \\ & \|\mathbf{z} - \boldsymbol{\Phi} \cdot \mathbf{G} \cdot \mathbf{x}\|_2 \leqslant \epsilon. \end{aligned} \tag{2}$$

- Still we can use Standard ESPRIT to estimate the $\theta_1, \ldots, \theta_S$
- Conditions on the $\theta_1, \ldots, \theta_S$, $\boldsymbol{\Phi}$ and $\mathbf{G}$ such that ANM has a unique solution?

Atomic Norm Minimization for DOA

- Requirements on $\mathbf{\Phi}$
  - Good reconstruction properties
  - Fairly easy to implement with phase shifters
- Draw $\mathbf{\Phi}$ randomly

Definition (sub-Gaussian)

A real valued random variable X is called sub-Gaussian with variance factor c, if

$$\mathbb{E}\exp(\lambda X) \leqslant \lambda^2 \frac{c}{2}$$

for all $\lambda \in \mathbb{R}$.

- Examples: Gaussian, Binomial, *Rademacher*
- Conditions on the $\theta_1, \ldots, \theta_S$, $\mathbf{\Phi}$ and $\mathbf{G}$ such that ANM has a unique solution *with high probability*?
- Matrices can be sub-Gaussian as well
- $\rightsquigarrow$ Rademacher

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- Combine results in [10] with our model and measurement setup

Theorem (S. S.)

If

$$|\theta_i - \theta_j| > \frac{1}{L}$$

for all $i \neq j$*, then using the Rademacher measurement setup, the matrix* $\mathbf{\Phi} \cdot \mathbf{G}$ *is* b*-sub-Gaussian with* $b^{-1} = \max_{1 \leqslant \ell \leqslant L} \|\mathbf{g}_\ell\|_2^2$ *and* $\Sigma = \mathbf{G}^{\mathrm{H}}\mathbf{G}$*. Thus the ANM has a unique solution with high probability, if the number of measurements* K *obeys*

$$K \geqslant \hat{c} \cdot S \cdot \log(M) \cdot \max_{1 \leqslant \ell \leqslant L} \|\mathbf{g}_\ell\|_2^2 \cdot \sigma(\mathbf{G}^{\mathrm{H}}\mathbf{G}),$$

where $\sigma(\mathbf{G}^{\mathrm{H}}\mathbf{G})$ *denotes the condition number of* $\mathbf{G}^{\mathrm{H}}\mathbf{G}$*.*

[10] Heckel and Soltanolkotabi, "Generalized Line Spectral Estimation via Convex Optimization", *2017*.

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Simulations using genuine antenna measurements

Figure. The simulated SPUCPA setup

- Stacked polarimetric uniform circular patch array (SPUCPA) with 58 ports.
- Used only the ports corresponding to the two stacked circular arrays with 12 elements per ring
- ANM to estimate covariance Toep($\mathbf{u}$) done via cvx using the SDPT3 solver
- DOA estimation based on covariance via Standard ESPRIT

Figure. Estimation error vs. SNR for $S = 2$ sources, $M = 24$ elements, $L = 25$ Fourier coefficients and $K = 12$ measurements

Conclusion & Future Extensions

Conclusion

- Arbitrary Antennas
- Compression
- Grid-free

Outlook

- Estimate multidimensional frequencies / directions
- Bistatic Tx / Rx setups for channel sounding / radar

Thank You!

Questions?

BACKUP SLIDES

Figure. Estimation error vs. SNR for $S = 3$ sources, $M = 24$ elements, $L = 25$ Fourier coefficitent and $K = 15$ measurements

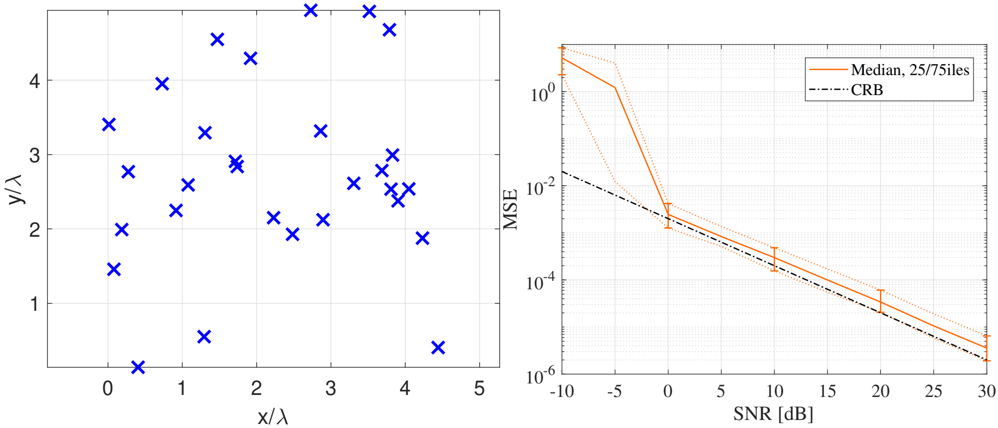

Figure. Randomly drawn virtual (toy) array geometry ($M = 29$)

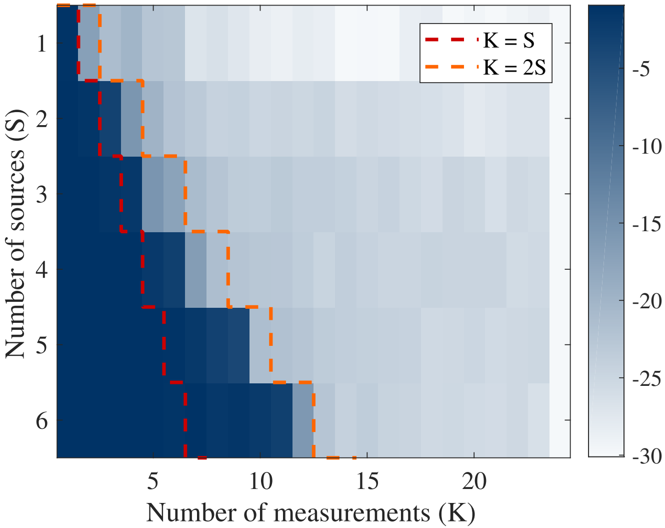

Figure. Estimation error (logarithmic scale) vs. *K*, *S* for the noise-free case and Rademacher distributed compression matrices.

$$C(\theta) = \frac{\sigma^2}{2}\,\mathbf{tr}\left(\left[\Re(\mathbf{D}^H \Pi_{\mathbf{A}}^{\perp} \mathbf{D} \odot (\mathbf{c}\mathbf{c}^H)^T)\right]^{-1}\right), \qquad (3)$$

where $\mathbf{A} = \mathbf{G}\mathbf{F}(\theta)$, $\quad \mathbf{D} = \jmath \mathbf{G} diag(\mu) \mathbf{F}(\theta)$ and $\Pi_{\mathbf{A}}^{\perp} = \mathbf{I} - \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$. Here $\mathbf{F}(\theta) = [\mathbf{f}(\theta_1), \ldots, \mathbf{f}(\theta_S)]$, the vector $\mu = -(L-1)/2, \ldots, +(L-1)/2$ and $\Re$ denotes the real part of a complex number. If we incorporate compression, it changes to

$$C(\theta) = \frac{\sigma^2}{2}\,\mathbf{tr}\left(\left[\Re(\bar{\mathbf{D}}^H \Pi_{\bar{\mathbf{A}}}^{\perp} \bar{\mathbf{D}} \odot (\mathbf{c}\mathbf{c}^H)^T)\right]^{-1}\right), \qquad (4)$$

where $\bar{\mathbf{A}} = \Phi\mathbf{A} = \Gamma\mathbf{F}(\theta)$ and $\bar{\mathbf{D}} = \Phi\mathbf{D} = \jmath\Gamma diag(\mu)\mathbf{F}(\theta)$.