TV-SVM: Support Vector Machine with Total Variational Regularization

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Abstract

To leverage the spatial relationship of lattice data, such as images, we introduce total variational (TV) regularization into support vector machines (SVM), called TV-SVM. TV-SVM encourages local smoothness and sparsity in gradient domain of the learned parameters. TV-SVM is optimized via the alternating direction method of multipliers (ADMM) algorithm and is significantly better than (Linear) SVM for image classifications.

The Problem of SVM

Typically, SVM contains a hinge loss function and an L_2 regularization. Given a set of training data $\{x_i, y_i\}_{i=1}^n$, where $x_i \in \mathbf{R}^d$ and $y_i \in \{-1, 1\}$, SVM is formulated as:

$$\min_{w,b} \frac{1}{2}w^T w + C\sum_{i=1}^{n}[1 - y_i(w^T x_i + b)]_+ \quad (1)$$

Notice that each input x_i must be represented as a vector. However, some kinds of real-world data are naturally represented as matrices (images) or even tensors (videos).

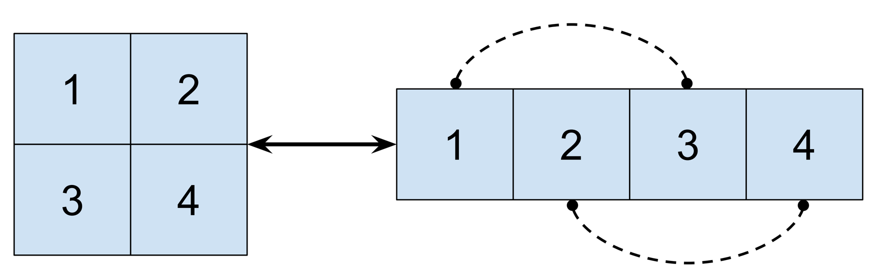


Figure: Reshape a matrix into a vector.

When matrices or tensors are reshaped into vectors, their spatial relationships are partially destroyed. Thus the spatial information is not successfully considered in SVM.

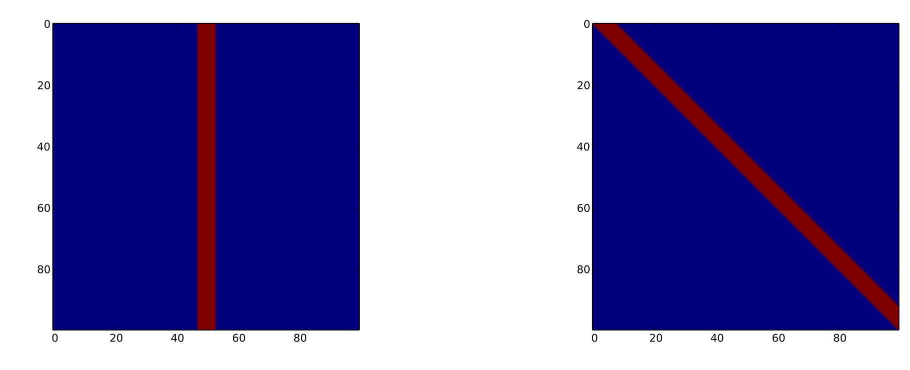
Low-rank Solutions

To deal with the above issue, several methods have been proposed

- rank-k SVM → $\mathbf{W}$ is the sum of k rank-1 matrices
- bilinear SVM → $\mathbf{W}$ is factorized into two low-rank matrices
- SMM → adds the nuclear norm of $\mathbf{W}$ as a regularization term

Nuclear norm introduced by Support Matrix Machine (SMM) is an approximation of matrix rank. Thus, all those methods are based on the low-rank assumption of $\mathbf{W}$, i.e. the rows or columns are highly correlated.

For natural image classification, the low-rank assumption may suffer problems. Rank is sensitive to rotations, but an image classification algorithm is required to be insensitive to rotations.



(a) Low-rank (b) High-rank

Figure: Rank is sesitive to rotations.

Total Variation

In this work, the TV-norm for 2D discrete signal is defined as

$$L_{TV}(X) = \sum_{i,j}|X_{i+1,j} - X_{i,j}| + |X_{i,j+1} - X_{i,j}| \quad (2)$$

where $\mathbf{X} \in \mathbf{R}^{m\times n}$ is the input matrix. L_{TV} can be also represented as linear operators combined with standard L_1 norm.

$$L_{TV}(X) = \|D_x x\|_1 + \|D_y x\|_1 \quad (3)$$

where x is the vector representation of $\mathbf{X}$. $\mathbf{D}_x$ and $\mathbf{D}_y$ are the so-called 2D forward differentiation operators in x-direction and y-direction, respectively. For simplicity, let $D = [D_x; D_y]$, then (3) is rewritten as:

$$L_{TV}(X) = \|Dx\|_1 \quad (4)$$

Then the spatial relationships of images are carried on $\mathbf{D}$.

TV-SVM

To take the advantage of the local smooth assumption and restrain the regression matrix $\mathbf{W}$ to be sparse in the gradient domain, we introduce TV regularization of $\mathbf{W}$ into SVM, instead of low-rank regularization.

$$\min_{\mathbf{W},b} \frac{1}{2}tr(\mathbf{W}^T\mathbf{W}) + \tau L_{TV}(\mathbf{W}) + C\sum_{i=1}^{n}[1 - y_i(tr(\mathbf{W}^T\mathbf{X}_i) + b)]_+ \quad (5)$$

Denote x and w are the vector representations of $\mathbf{X}$ and $\mathbf{W}$. Then, TV-SVM can also be represented as vectorized form:

$$\min_{w,b} \frac{1}{2}w^T w + \tau|Dw|_1 + C\sum_{i=1}^{n}[1 - y_i(w^T x_i + b)]_+ \quad (6)$$

It is equivalent to the following problem:

$$\arg\min_{w,b,z} f(w,b) + g(z) \quad s.t.\ z - w = 0 \quad (7)$$

$$f(w,b) = \frac{1}{2}w^T w + C\sum_{i=1}^{n}[1 - y_i(w^T x_i + b)]_+ \quad (8)$$

$$g(z) = \tau|Dz|_1 \quad (9)$$

Because both f and g are convex, the whole problem is optimized by fast ADAM algorithm. In each iteration, (w_k, b_k) is solved in a similar way as SVM and z_k is solved via an optimized taut-string method.

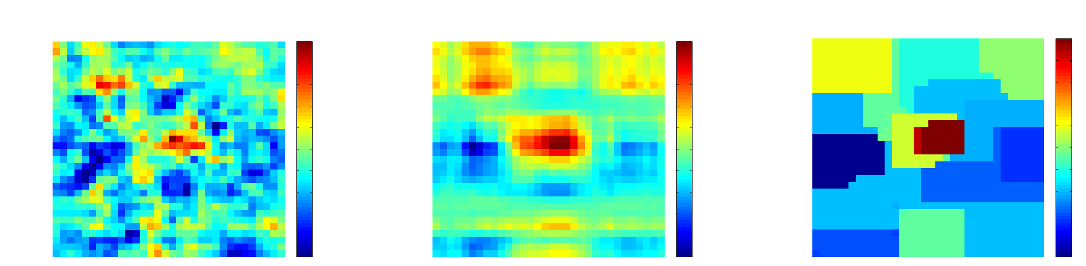
Fast ADMM for TV-SVM

Initialize $z^{-1} = \hat{z}^0$, $u^{-1} = \hat{u}^0$, $\rho = 1$, $t^1 = 1$, $c^0 = 0$ and $\eta \in (0,1)$
for $k = 1, 2, 3...$ **do**
  $(w^k, b^k) = \arg\min_{w,k} f(w,b) + \frac{\rho}{2}\|\hat{z}^k - w + \hat{u}^k\|_2^2$
  $z^k = \arg\min_z g(z) + \frac{\rho}{2}\|z - w^k + \hat{u}^k\|_2^2$
  $u^k = \hat{u}^k + \rho(z^k - w^k)$
  $c^k = \frac{1}{\rho}\|u^k - \hat{u}^k\|_2^2 + \rho\|z^k - \hat{z}^k\|_2^2$
  if $c^k < \eta c^{k-1}$ **then**
    $t^{k+1} = \frac{1 + \sqrt{1 + 4(t^k)^2}}{2}$
    $\hat{z}^{k+1} = z^k + \frac{t^k - 1}{t^{k+1}}(z^k - z^{k-1})$
    $\hat{u}^{k+1} = u^k + \frac{t^k - 1}{t^{k+1}}(u^k - u^{k-1})$
  else
    $t^{k+1} = 1$
    $\hat{z}^{k+1} = z^{k-1}$, $\hat{u}^{k+1} = u^{k-1}$
    $c^k = \frac{c^{k-1}}{\eta}$
  end if
end for

Experiments

Method	INRIA person		CIFAR-10		IG02	
	Time	Accuracy	Time	Accuracy	Time	Accurcy
L-SVM	-	80.12	-	67.60	-	72.91
SMM	**42.21**	81.98	15.40	**68.62**	**11.62**	73.68
TV-SVM	42.87	**82.99**	**14.79**	68.23	11.80	**73.86**

Table: Training time (second) and classification accuracy (%). There 2114 samples (160×90) on INRIA person, 2037 samples (32×32) on CIFAR-10, and 785 samples (90×120) on IG02.



(a) L-SVM (b) SMM (c) TV-SVM

Figure: Normalized $\mathbf{W}$ learned from CIFAR-10.

References

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[2] *Fast alternating direction optimization methods.* Siam Journal on Imaging Sciences, 2014.

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