Circularity Preserving DFT

The quest for invariance

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Second-order statistics in $\mathbb{C}$

Second-order statistics [1]

For a random variable $x \in \mathbb{C}$:

Hermitian variance:

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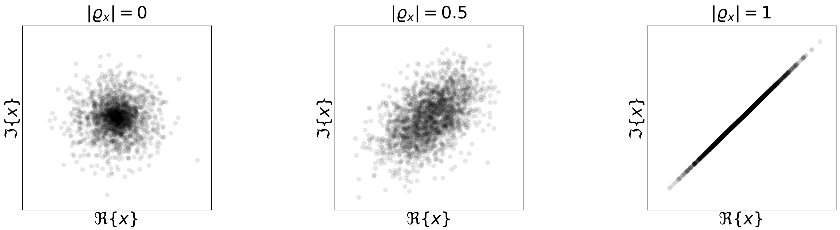
Both σ_x^2 and τ_x are required for **full** description of second-order statistics [2]

Circular vs noncircular distributions

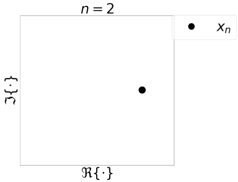
Metric for second-order noncircularity [3]

$$|\varrho_x| = \begin{cases} 0, & \text{circular} \\ 1, & \text{noncircular} \\ (0,1) & \text{otherwise} \end{cases}$$

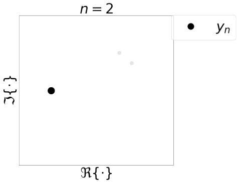


Distributions can be ambiguous

Consider the following evolving distributions:



Scatter plot of periodic deterministic signal



Scatter plot of uniformly distributed signal

The statistics are equivalent, even though one is deterministic (left panel) and the other is random (right panel)!

$$E\left\{|x|^2\right\} = E\left\{|y|^2\right\} = \sigma^2 \qquad \text{(variance)}$$

$$|E\left\{x^2\right\}| = |E\left\{y^2\right\}| = 0 \qquad \text{(abs. pseudo-variance)}$$

$$\frac{|E\left\{x^2\right\}|}{E\left\{|x|^2\right\}} = \frac{|E\left\{y^2\right\}|}{E\left\{|y|^2\right\}} = 0 \qquad \text{(circularity coeff.)}$$

How can we distinguish x from y? ↬ we must abandon conventional statistics.

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A dynamical systems perspective on signals

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$\Rightarrow$ we have a time-invariant measure of x_n, since at any time instant, n, x_n can be expressed in terms of the initial value, x_0.

Periodic deterministic systems

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A time-invariant measure

We can now unfold:

$$\begin{aligned} x_n &= e^{\jmath\omega} x_{n-1} \\ &= e^{\jmath 2\omega} x_{n-2} \\ &\vdots \\ &= e^{\jmath\omega n} x_0 \end{aligned}$$

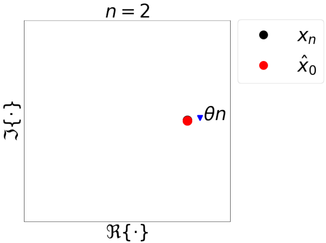
or, equivalently,

$$\boxed{x_0 = e^{-\jmath\omega n} x_n}$$

Invariance of periodic deterministic systems

We can estimate $\hat{x}_0 = e^{-\jmath\omega n} x_n$ and $\hat{y}_0 = e^{-\jmath\omega n} y_n$ at every time instant n.

Observe that $\hat{x}_0$ is constant for the deterministic x, however $\hat{y}_0$ is random!

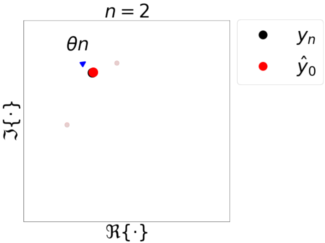


Scatter plot of periodic deterministic signal original (black) and unfolded (red)

$$E\left\{|\hat{x}_0|^2\right\} = \sigma^2$$

$$|E\left\{\hat{x}_0^2\right\}| = \sigma^2$$

$$\frac{|E\left\{\hat{x}_0^2\right\}|}{E\left\{|\hat{x}_0|^2\right\}} = 1$$



Scatter plot of uniformly distributed signal original (black) and unfolded (red)

$$E\left\{|\hat{y}_0|^2\right\} = \sigma^2$$

$$|E\left\{\hat{y}_0^2\right\}| = 0$$

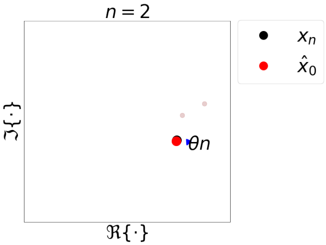
$$\frac{|E\left\{\hat{y}_0^2\right\}|}{E\left\{|\hat{y}_0|^2\right\}} = 0$$

Periodic systems in Gaussian environments

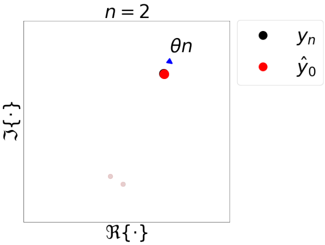
Let's test whether deterministic systems with additive circular Gaussian noise also has time-invariant measures

$$x_n = e^{\jmath\omega} x_{n-1} + w_n$$

with $w_n \sim \mathcal{N}\left(0, \sigma_w^2\right)$



Scatter plot of noisy periodic deterministic signal original (black) and unfolded (red)



Scatter plot of noisy uniformly distributed signal original (black) and unfolded (red)

We can still distinguish between the distributions!

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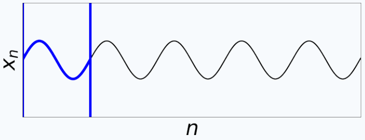
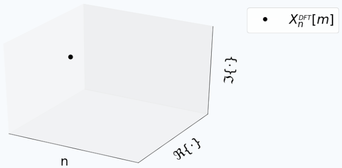
The sliding DFT [4, 5]

$$X_n^{\mathrm{DFT}}[m] = \sum_{k=0}^{N-1} x_{n+k} e^{-\jmath \omega_m k}$$

Consider the signal given by $x_n = \sin(\omega_m n)$, with $\omega_m = 2\pi m/N$.

The evolution of $X_n^{\mathrm{DFT}}[m]$ shows:





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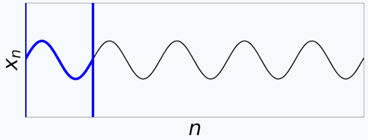

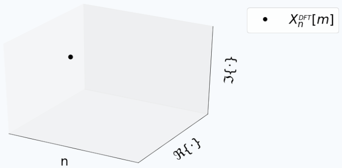

The "rotation" arises due to a change in reference frame for the phase at each time step increment.

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Circularity Preserving DFT

$$X_n[m] = \sum_{k=0}^{N-1} x_{n+k} e^{-\jmath\omega_m(n+k)}$$

The CPDFT modifies the frame of reference of the phase such that it becomes the "initial phase".

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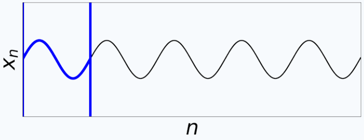

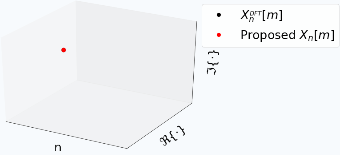

Circularity-Preserving DFT

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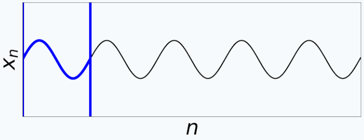
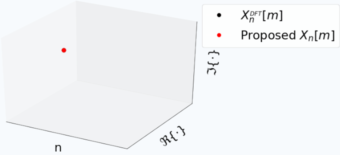
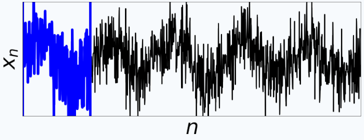


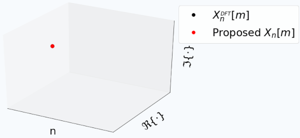


The phase of the CPDFT is stationary! $\implies$ we can confidently exploit statistics in $\mathbb{C}$

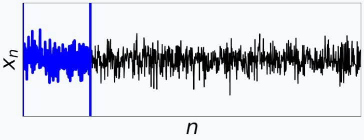
CPDFT: more examples

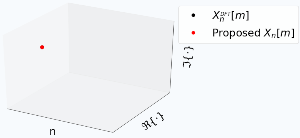
Sinusoid in additive Gaussian noise





Gaussian noise





Full spectral description

Full second-order statistical description in the frequency domain

As with complex-valued random variables in general, the full second-order statistical description of the CPDFT coefficents, $X[m]$, requires both quantities:

Hermitian variance $\leftrightarrow$ Power Spectrum: $R[m] = E\left\{|X[m]|^2\right\}$

Pseudo-variance $\leftrightarrow$ Panorama: $P[m] = E\left\{X^2[m]\right\}$

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Using both the power spectrum and panorama [6, 7], we can distinguish between deterministic and random frequency bins using the spectral circularity:

$$\varrho[m] = \frac{E\left\{X^2[m]\right\}}{E\left\{|X[m]|^2\right\}} = \frac{P[m]}{R[m]} \tag{1}$$

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For a specific frequency bin m:

Deterministic $\implies$ noncircular ($|\varrho[m]| = 1$)

Random $\implies$ circular ($|\varrho[m]| = 0$)

CPDFT in practice

Example: 3 sinusoids in correlated noise

Consider the harmonic signal in noise, given by

$$x_n = \cos(0.15(2\pi n)) + 0.25\cos(0.25(2\pi n)) + 0.1\cos(0.4(2\pi n)) + \eta_n$$

where η_n is generated by filtering a zero-mean uncorrelated Gaussian random process with a digital filter with system function given by

$$H(z) = \frac{1}{1 - 1.6\cos(0.2(2\pi))z^{-1} + 0.64z^{-2}}$$

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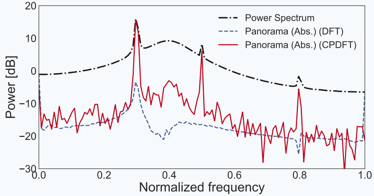

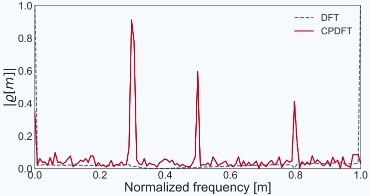

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