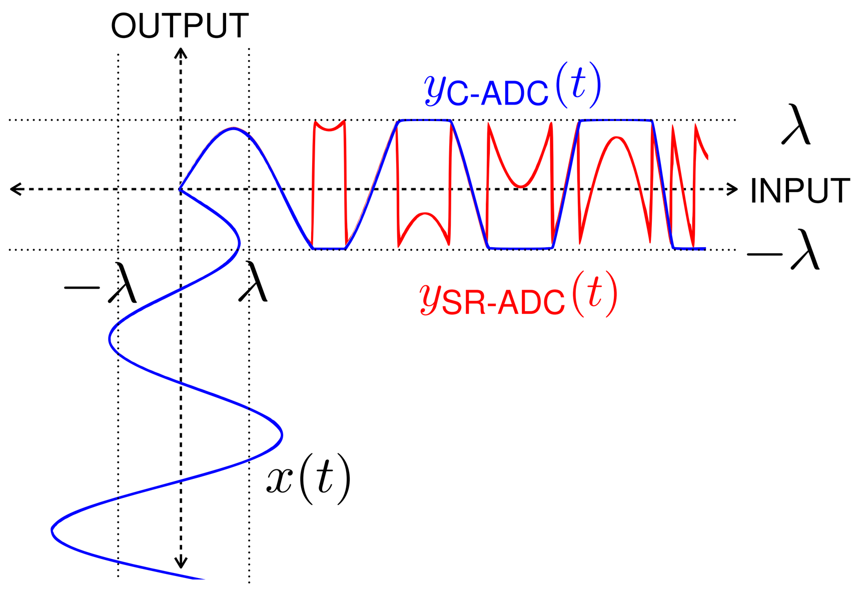
WAVELET-BASED RECONSTRUCTION FOR UNLIMITED SAMPLING

Sunil Rudresh, Aniruddha Adiga, Basty Ajay Shenoy, and Chandra Sekhar Seelamantula
Department of Electrical Engineering, Indian Institute of Science, Bangalore - 560012, India
Email: {sunilr, aaniruddha, bastys}@iisc.ac.in, chandra.sekhar@ieee.org

SPECTRUM LAB

1. Overview

Self-reset analog-to-digital converter (SR-ADC):

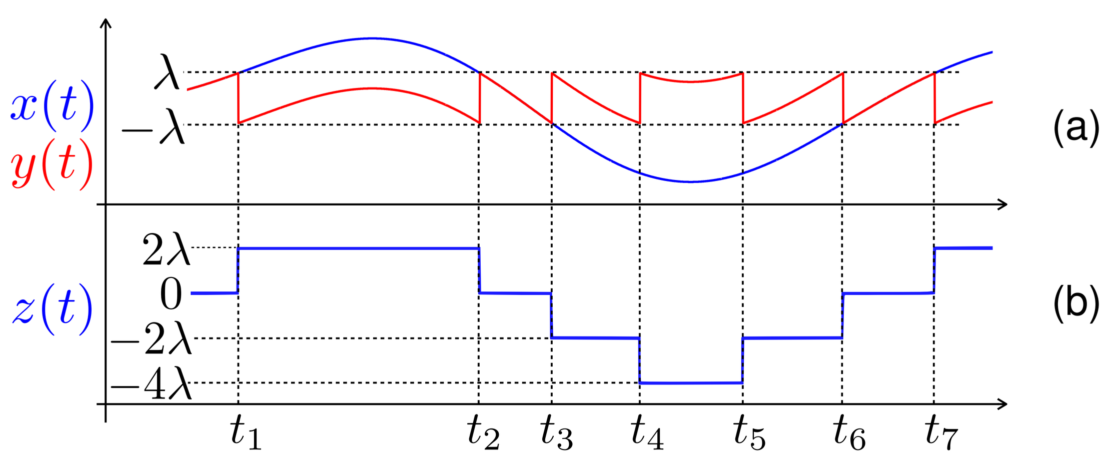


An illustration of the transfer characteristics of a clipping-ADC and a self-reset-ADC with the thresholds at $\pm\lambda$.

Output:

$y(t) = \mathcal{M}_\lambda\{x(t)\} :=$ mod $(x(t)+\lambda, 2\lambda) - \lambda,$
$= x(t) - z(t),$

where $z(t) = \sum_k \alpha_k \mathbf{1}_{[t_k, t_{k+1}]}(t)$ is a piecewise-constant signal.



Modulo operation: (a) Input signal $x(t)$ and its modulo version $y(t)$; and (b) piecewise-constant signal $z(t)$.

Discrete version:

$y(nT) = \mathcal{M}_\lambda\{x(nT)\} = x(nT) - z(nT),$

where $z(nT) = \sum_k \alpha_k \mathbf{1}_{[\![n_k, n_{k+1}]\!]}(nT)$.

2. Problem Formulation

1. Given $y(t) = \mathcal{M}_\lambda\{x(t)\}$, reconstruct $x(t)$.
2. Given $y(nT) = \mathcal{M}_\lambda\{x(nT)\}$, reconstruct $x(nT)$.

3. Continuous-Domain Analysis

Definition 1 (Vanishing moments)
A wavelet ψ has $p+1$ vanishing moments if $\int t^k \psi(t)\,\mathrm{d}t = 0$, for $k \in [\![0,p]\!]$.

Key idea: Use wavelets to annihilate the smooth parts of $y(t)$.

Lemma 1 *Let $x(t)$ be a polynomial of degree p, $\psi(t)$ be a wavelet with $p+1$ vanishing moments, and $y(t) = \mathcal{M}_\lambda\{x(t)\}$. Then, $y_\psi(t) := (y * \psi)(t) = \sum_k (\alpha_k - \alpha_{k-1})\xi(t-t_k)$, where $\xi(t) = -\int_{-\infty}^{t} \psi(\tau)\,\mathrm{d}\tau$.*

- Estimation of α_ks: by matched filtering.
- Sufficient condition for exact reconstruction: $(t_k - t_{k-1}) > (2p-1) \implies \frac{2\lambda}{L} < (2p-1)$.

4. Discrete-Domain Analysis

$$y(nT) = \mathcal{M}_\lambda\{x(nT)\} = x(nT) - z(nT)$$

Annihilation of a sampled polynomial of degree p: $\sum_{n\in\mathbb{Z}} n^k g[n] = 0$, for $k \in [\![0,p]\!]$.

Compactly supported $g[n]$: Daubechies filter of order p with support $[\![0, 2p-1]\!]$.

Lemma 2 *Let $x[n]$ be the samples of a polynomial of degree p, and let $g[n]$ be a discrete wavelet filter with $p+1$ vanishing moments. Then, $y_g[n] := (y * g)[n] = \sum_k \beta_k\, m[n-n_k]$ with $\beta_k = (\alpha_k - \alpha_{k-1})$ and $m[n] := -\sum_{k=-\infty}^{n} g[k]$.*

Computing $\{n_k\}$:

LASSO: $\arg\min_{\mathbf{h}} = \|\mathbf{y}_g - \mathbf{A}\mathbf{h}\|_2^2 + \gamma\|\mathbf{h}\|_1$

$\mathbf{A}$ – convolutional dictionary of time-shifted versions of $m[n]$

$\mathbf{h}$ – sparse vector with $\mathbf{h}[n_k] = \beta_k$ and supp$\{\mathbf{h}\}$=cardinality of $\{n_k\}$

Selection of sampling interval (T):

- Let $T_f := \min_k (n_k - n_{k-1})$ be the minimum sampling interval.
- For no overlap: $T_f >$ supp$\{m[n]\}$.
- Support of Daubechies wavelet filter of order p is $2p$.
- A sufficient condition on T: $T < \frac{T_f}{2p}$.

References

[1] A. Bhandari, F. Krahmer, and R. Raskar, "On unlimited sampling," in *Proceedings of International Conference on Sampling Theory and Applications (SampTA)*, July 2017, pp. 31– 35.

[2] K. Sasagawa, T. Yamaguchi, M. Haruta, Y. Sunaga, H. Takehara, T. Noda, T. Tokuda, and J. Ohta, "An implantable CMOS image sensor with self-reset pixels for functional brain imaging," *IEEE Transactions on Electron Devices*, vol. 63, no. 1, pp. 215–222, Jan 2016.

[3] I. Daubechies, "Ten Lectures on Wavelets," *SIAM*, 1992.

[4] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, 2008.

[5] R. Tibshirani, "Regression shrinkage and selection via the lasso," *Journal of the Royal Statistical Society*, Series B, vol. 58, pp. 267–288, 1994.

Acknowledgements

- The work has been funded by the Science and Engineering Research Board (SERB), Government of India through the project "Sub-Nyquist Sampling."
- Conference travel funded by Indian Institute of Science, Bangalore and SPCOM 2018 travel grant.

5. Main Result: WAVE-BUS

WAVE-BUS: *WAVElet-Based Unlimited Sampling*

Lemma 3 *Let $y(t) = \mathcal{M}_\lambda\{x(t)\}$, where $x(t)$ is a Lipschitz-continuous signal that satisfies $|x(a) - x(b)| \le L|b-a|$. Then, we have $\tilde{T}_f := \frac{2\lambda}{L} \le T_f$.*

Proof: From Lipschitz continuity, $|x(t+\tilde{T}_f) - x(t)| \le L\tilde{T}_f < 2\lambda, \quad \forall t.$
Hence, at any t_k, no folding happens in the interval $(t_k, t_k + \tilde{T}_f)$. Thus, $\tilde{T}_f \le T_f$.

- Sufficient condition for exact reconstruction: Sampling interval $T \le \frac{T_f}{2p} \le \frac{2\lambda}{L(2p)}$.

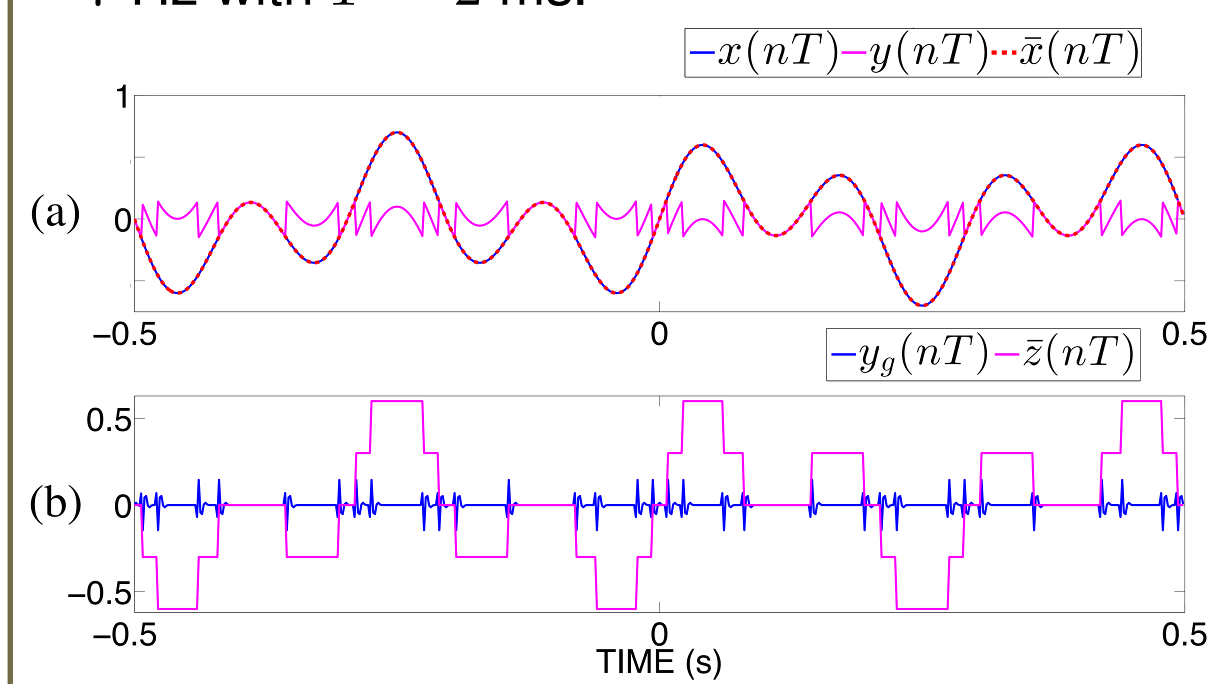
6. Reconstruction Algorithm

Algorithm 1 : WAVE-BUS.

- Input: $y(nT) = \mathcal{M}_\lambda\{x(nT)\}, L, \lambda, p, T$
- Output: $\bar{x}(nT)$
- Method:
  1. Wavelet filtering: $(y_g)[n] = (y * g)[n]$
  2. LASSO (compute n_ks and α_ks): $\arg\min_{\mathbf{h}} \|\mathbf{y}_g - \mathbf{A}\mathbf{h}\|_2^2 + \gamma\|\mathbf{h}\|_1$
  3. Compute $\bar{z}[n] : \sum_k \alpha_k \mathbf{1}_{[\![n_k, n_{k+1}]\!]}$
  4. Reconstruct $x[n] : \bar{x}[n] = y[n] + \bar{z}[n]$

7. Simulation Results

- Sum of sinusoids of frequency 4 Hz and 7 Hz with $T = 2$ ms.



WAVE-BUS with $\lambda = 0.15$: (a) Signal $x(nT)$, its modulo samples $y(nT)$, and the reconstruction $\bar{x}(nT)$; (b) wavelet filter output $y_g(nT)$, and the estimate $\bar{z}(nT)$. The reconstruction error was computed to be -330 dB.

- Signal-to-reconstruction-noise ratio (SRNR) vs input SNR for WAVE-BUS and repeated finite-difference (RFD)[1] method.