Differentially-private Distributed Principal Component Analysis

Hafiz Imtiaz & Anand D. Sarwate

Department of Electrical and Computer Engineering
Rutgers, the State University of New Jersey

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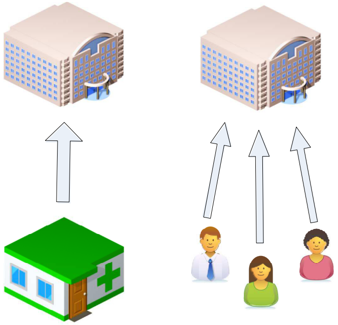
Outline

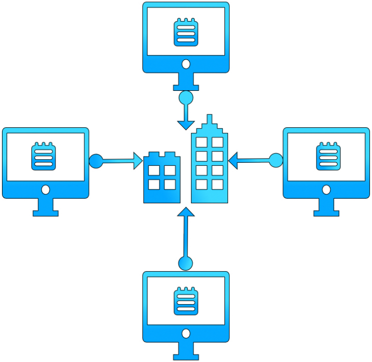
1. Motivation
2. Problem Formulation
3. Differential Privacy
4. Proposed Algorithm
5. Experimental Results
6. Conclusion

Why learn from private data?



- Much of private/sensitive data is being digitized
- Want to learn about population – using/reusing data
- Free and open sharing – ethical, legal, and technological obstacles

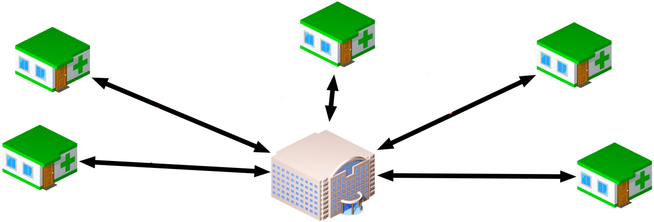
Why learn in distributed setting?



Good feature learning requires large sample sizes.

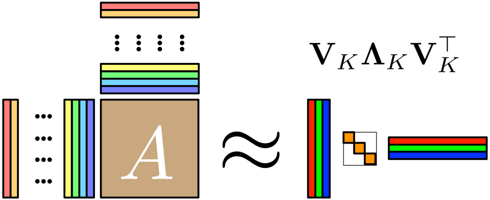
- Data at a single site may not be sufficient for statistical learning
- Pooling data in one location may not be possible

An example in neuroimaging



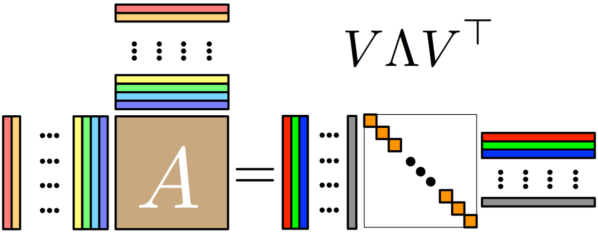
- Multiple fMRI collection centers
- Each has a moderate number of samples, at best
- **Goal:** find a way to reduce the sample dimension

We can perform principal component analysis (PCA)



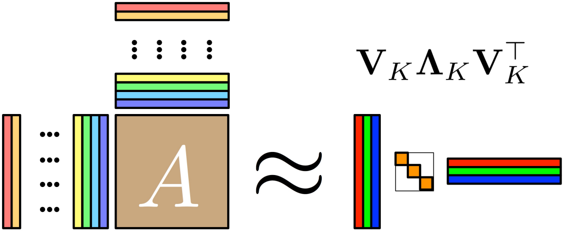
Principal Component Analysis

The PCA problem: pooled case



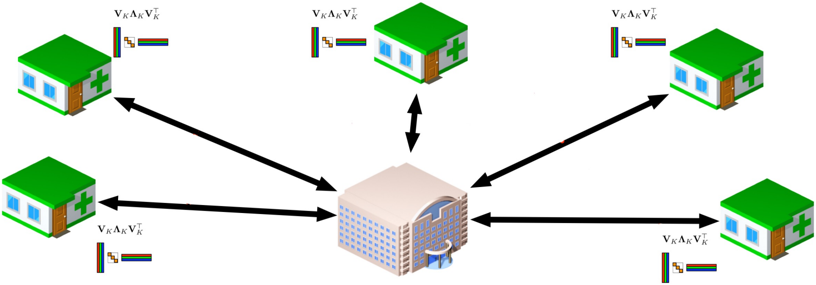
- Data matrix: $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \ldots \ \mathbf{x}_N] \in \mathbb{R}^{D \times N}$
- Second-moment matrix: $\mathbf{A} = \frac{1}{N}\mathbf{X}\mathbf{X}^\top$
- We can decompose $\mathbf{A}$ as: $\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^\top$
- Here, $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_D)$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_D \geq 0$

The PCA problem: pooled case



- The best rank-K approximation of $\mathbf{A}$: $\mathbf{A}_K = \mathbf{V}_K \mathbf{\Lambda}_K \mathbf{V}_K^\top$
- The top-K PCA subspace is the span of the corresponding columns of $\mathbf{V}$: $\mathbf{V}_K(\mathbf{A})$

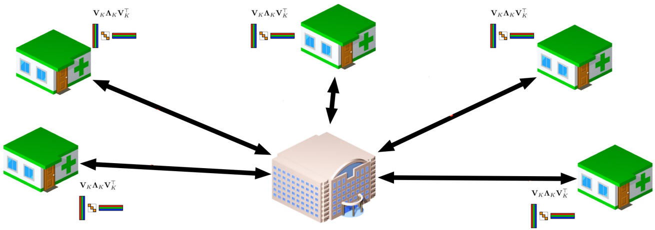
The PCA problem: distributed case



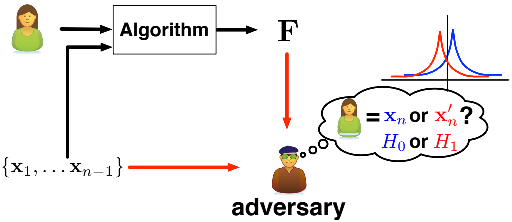
- One aggregator, S different sites with disjoint datasets
- Local data matrix: $\mathbf{X}_s = [\mathbf{x}_{s,1} \ldots \mathbf{x}_{s,N_s}] \in \mathbb{R}^{D \times N_s}$
- Local second-moment matrix: $\mathbf{A}_s = \frac{1}{N_s} \mathbf{X}_s \mathbf{X}_s^\top$
- All parties: "nice but curious"

How can we compute a **global** $\mathbf{V}_K$?

What are our options?

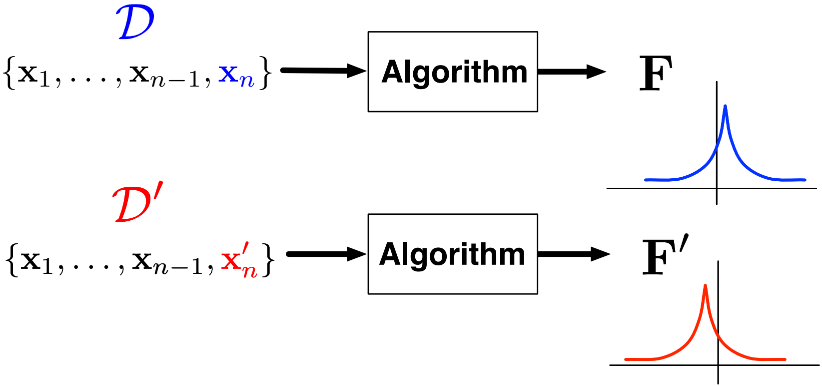


- Send $\mathbf{X}_s$ to aggregator
  - $\rightarrow$ huge communication cost
  - $\rightarrow$ privacy violation
- Compute $\mathbf{V}_K$ using local data
  - $\rightarrow$ poor quality of the subspace



Differential Privacy

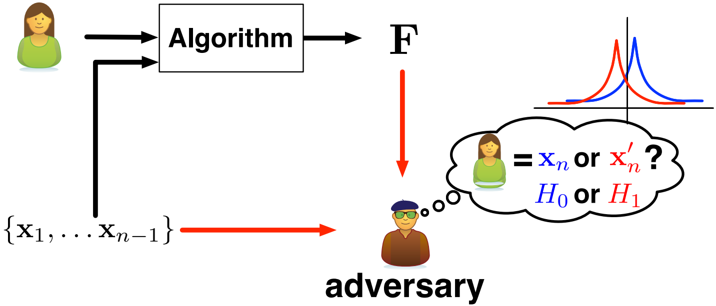
Differential privacy: a definition



[Dwork et al. 2006] An algorithm $\mathcal{A}$ is (ε, δ)-differentially private if for any set of outputs $\mathcal{F}$, and all $(\mathcal{D}, \mathcal{D}')$ differing in a single point,

$$\mathbb{P}\left(\mathcal{A}(\mathcal{D}) \in \mathcal{F}\right) \leq \exp(\varepsilon) \cdot \mathbb{P}\left(\mathcal{A}(\mathcal{D}') \in \mathcal{F}\right) + \delta$$

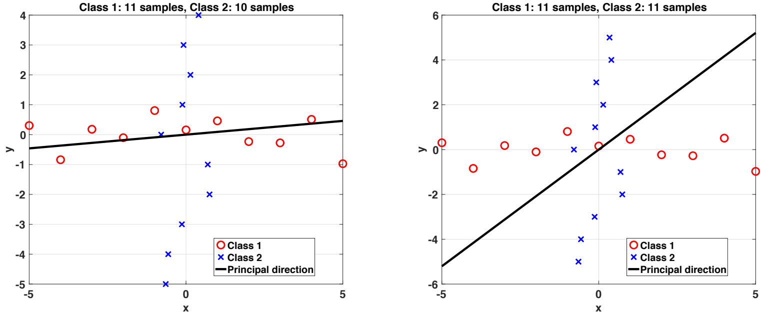
Differential privacy: hypothesis testing



$$\log \frac{\mathbb{P}\left(\mathcal{A}(\mathcal{D}) \in \mathcal{F}\right)}{\mathbb{P}\left(\mathcal{A}(\mathcal{D}') \in \mathcal{F}\right)} \leq \varepsilon$$

We want to design algorithms that satisfy differential privacy

Why we need privacy in PCA?



Changing one sample can significantly change the principal direction

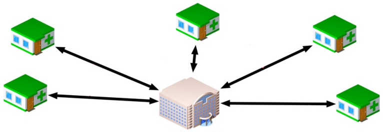
What are we trying to address?

Goal: compute an accurate $\mathbf{V}_K$

- want to exploit all samples across all sites
- want a lower communication cost
- want to preserve a formal privacy definition

Idea: send the differentially private partial square root of $\mathbf{A}_s$

- Compute $\mathbf{A}_s \leftarrow \frac{1}{N_s}\mathbf{X}_s\mathbf{X}_s^\top$
- Generate $D \times D$ symmetric matrix $\mathbf{E}$
- Compute $\hat{\mathbf{A}}_s \leftarrow \mathbf{A}_s + \mathbf{E}$
- Perform SVD $\hat{\mathbf{A}}_s = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^\top$
- Compute $\mathbf{P}_s \leftarrow \mathbf{U}_R\mathbf{\Sigma}_R^{\frac{1}{2}}$
- Send $\mathbf{P}_s$ to aggregator



- Compute $\mathbf{A}_c \leftarrow \frac{1}{S}\sum_{s=1}^{S}\mathbf{P}_s\mathbf{P}_s^\top$
- Perform SVD $\mathbf{A}_c = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top$

Output: Differentially-private rank-K subspace $\mathbf{V}_K$

Input: Data matrix $\mathbf{X}_s$ for $s \in [S]$; privacy parameters ϵ, δ; intermediate dimension R; reduced dimension K

Proposed Algorithm

Differentially-private Distributed PCA (DPdisPCA)

Input: Data matrix $\mathbf{X}_s$ for $s \in [S]$; privacy parameters ϵ, δ; intermediate dimension R; reduced dimension K

→ for $s = 1, 2, \ldots, S$ do :

- Compute $\mathbf{A}_s \leftarrow \frac{1}{N_s}\mathbf{X}_s\mathbf{X}_s^\top$
- Generate $D \times D$ symmetric matrix $\mathbf{E}$ where $\{\mathbf{E}_{ij} : i \in [D], j \leq i\}$ drawn i.i.d. $\sim \mathcal{N}(0, \Delta_{\epsilon,\delta}^2)$ and $\Delta_{\epsilon,\delta} = \frac{1}{N_s\epsilon}\sqrt{2\log(\frac{1.25}{\delta})}$
- Compute $\hat{\mathbf{A}}_s \leftarrow \mathbf{A}_s + \mathbf{E}$
- Perform SVD $\hat{\mathbf{A}}_s = \mathbf{U\Sigma U}^\top$
- Compute $\mathbf{P}_s \leftarrow \mathbf{U}_R\mathbf{\Sigma}_R^{\frac{1}{2}}$; send $\mathbf{P}_s$ to aggregator

→ Compute $\mathbf{A}_c \leftarrow \frac{1}{S}\sum_{s=1}^{S}\mathbf{P}_s\mathbf{P}_s^\top$

→ Perform SVD $\mathbf{A}_c = \mathbf{V\Lambda V}^\top$

Output: Differentially-private rank-K subspace $\mathbf{V}_K$

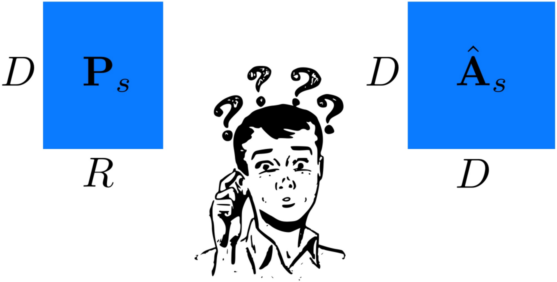
Privacy guarantee of DPdisPCA

Theorem (Privacy of DPdisPCA Algorithm)

DPdisPCA *computes an* (ϵ, δ) *differentially private approximation to the optimal subspace* $\mathbf{V}_K(\mathbf{A})$.

- $\mathcal{L}_2$ sensitivity of $\mathbf{A}_s$ is $\frac{1}{N_s}$
- By AG [Dwork et al. 2014] algorithm: computation of $\hat{\mathbf{A}}_s$ is (ϵ, δ) differentially private
- Differential-privacy is invariant to post-processing: computation of $\mathbf{V}_K$ also satisfies (ϵ, δ) differential privacy

Some comments on DPdisPCA



- $\mathbf{P}_s$ is $D \times R$: communication cost is proportional to $S \times D \times R$
- If we send $\hat{\mathbf{A}}_s$, the cost would be proportional to $S \times D^2$. Typically, $K < R < D$
- Sending $\mathbf{P}_s$ instead of $\hat{\mathbf{A}}_s$ does introduce some errors – cost of cheaper communication

Experimental Results

Datasets

- *Synthetic* dataset ($D = 200,\ K = 50$) generated with zero mean and a pre-determined covariance matrix
- *MNIST* dataset ($D = 784,\ K = 50$) (MNIST)
- *Covertype* dataset ($D = 54,\ K = 10$) (COVTYPE)

The trade-offs

We are interested to find out:

- how performance varies with "privacy risk" ϵ
- how performance varies with sample size N_s

Performance measures

Table: Notation of performance measures

Algorithm / Setting	Performance Index
Pooled Data	q_{pooled}
DPdisPCA	q_{DPdisPCA}
Local Data	q_{local}
Sending $\hat{\mathbf{A}}_s$	q_{full}

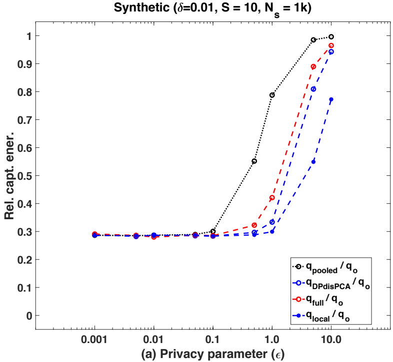
- Quality of a subspace $\mathbf{V}$: *captured energy* of $\mathbf{A}$

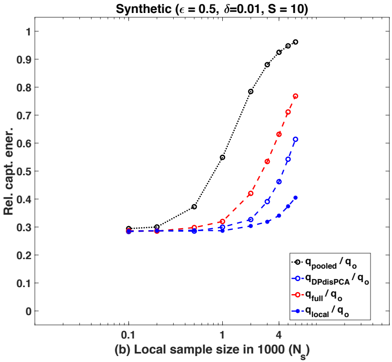
$$q(\mathbf{V}) = \operatorname{tr}(\mathbf{V}^\top \mathbf{A} \mathbf{V})$$

- We plot the ratio of these quantities with respect to the true captured energy q_o

Performance variation

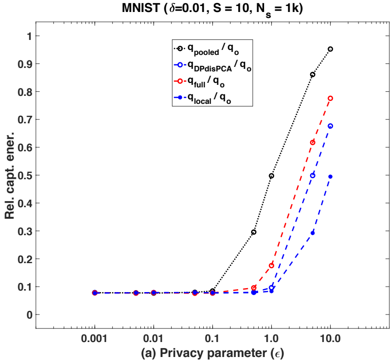
For synthetic data

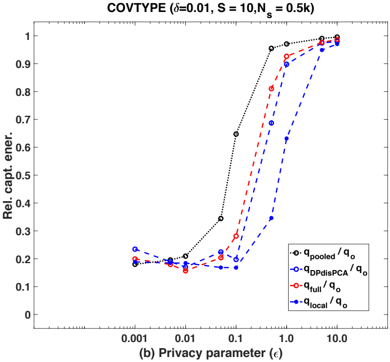




Performance variation

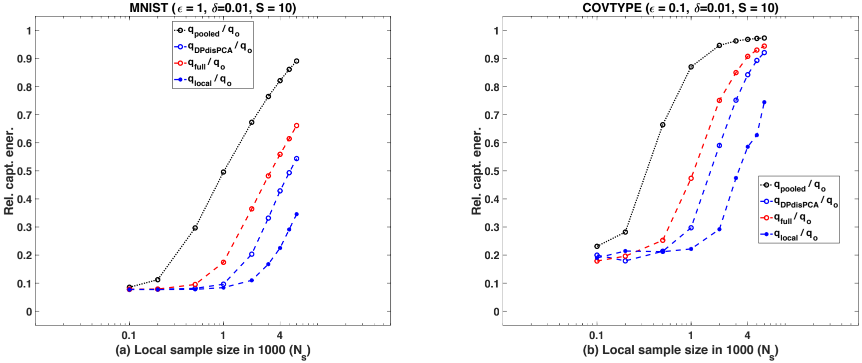
with ϵ





Performance variation

with N_s



Concluding Remarks

Concluding remarks

- The distributed algorithm clearly outperforms the local PCA algorithm
- Increasing ϵ improves performance at the cost of lower privacy
- Datasets with lower D allows smaller ϵ for achieving the same utility
- Increasing N_s improves performance for a fixed privacy level
- The cost of sending $\mathbf{P}_s$ instead of $\hat{\mathbf{A}}_s$ is noticeable in all datasets

Questions

Thank you