Permissible Support Patterns for Identifying the Spreading Function of Time-Varying Channels

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Time-Varying Channels & Operators

- ▷ A time-varying communication channel is modeled by the input-output relation

$$y(t) = (\mathrm{H}x)(t) = \int_{\mathbb{R}} h_{\mathrm{H}}(\tau,t)\, x(t-\tau)\, \mathrm{d}\tau\ , \qquad t \in \mathbb{R}$$

with the time-varying impulse response h_{H}.

- ▷ Similarly, taking the Fourier transform of $h_{\mathrm{H}}(\tau,\cdot)$, one obtains

$$y(t) = (\mathrm{H}x)(t) = \iint_{\mathbb{R}\times\mathbb{R}} \eta_{\mathrm{H}}(\tau,\nu)\, \mathrm{e}^{\mathrm{i}2\pi\nu(t-\tau)} x(t-\tau)\, \mathrm{d}\nu\, \mathrm{d}\tau$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} \eta_{\mathrm{H}}(\tau,\nu)\, \big(\mathrm{M}_\nu \mathrm{T}_\tau x\big)(t)\, \mathrm{d}\nu\, \mathrm{d}\tau \qquad (1)$$

with the spreading functions $\eta_{\mathrm{H}}(\tau,\nu) = \big(\mathcal{F}h_{\mathrm{H}}(\tau,\cdot)\big)(\nu)$, with the translation operator T_τ, and with the modulation operator M_ν, given by

$$(\mathrm{T}_\tau x)(t) = x(t-\tau) \qquad \text{and} \qquad (\mathrm{M}_\nu x)(t) = x(t)\, \mathrm{e}^{\mathrm{i}2\pi\nu t}\ .$$

- ▷ Every bounded linear operator $\mathrm{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ can be represented in the form (1).

Identification of Stochastic Operators

Stochastic channels: The spreading function $\eta(\tau,\nu)$ may be considered as a two-dimensional stochastic process with covariance function

$$R_{\mathrm{H}}(\tau,\tau',t,t') = \mathbb{E}\big[\eta(\tau,t)\, \overline{\eta(\tau',t')}\big] \qquad (2)$$

Problem: Assuming a sounding signal of the form

$$x_{\mathrm{s}}(t) = \sum_{n\in\mathbb{Z}} c_n\, \delta(t - nT)$$

with an N-periodic sequence $\{c_n\}_{n\in\mathbb{Z}}$. Determine the covariance function (2) of the operator H from the covariance

$$R_y(t,t') = \mathbb{E}\big[y(t)\overline{y(t')}\big] = \mathbb{E}\big[(\mathrm{H}x_{\mathrm{s}})(t)\overline{(\mathrm{H}x_{\mathrm{s}})(t')}\big]$$

of the channel output $y(t) = (\mathrm{H}x_{\mathrm{s}})(t)$.

Reformulation in Finite Dimensions

Stochastic operator estimation

Determine the covariance $\mathbf{X} = \mathbb{E}[\boldsymbol{\eta}\boldsymbol{\eta}^*] \in \mathbb{C}^{N^2\times N^2}$ of a random spreading vector $\boldsymbol{\eta} \in \mathbb{C}^{N^2}$ from the covariance $\mathbf{Y} = \mathbb{E}[\boldsymbol{y}\boldsymbol{y}^*] \in \mathbb{C}^{N\times N}$ of the channel output

$$\boldsymbol{y} = H\boldsymbol{c} = \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} \eta(k,\ell)\big(\mathrm{M}^\ell \mathrm{T}^k \boldsymbol{c}\big) = \mathbf{G}_{\boldsymbol{c}}\boldsymbol{\eta}\ ,$$

where $\mathbf{G}_{\boldsymbol{c}} = [\mathrm{M}^\ell \mathrm{T}^k \boldsymbol{c}]_{k,\ell=0}^{N-1} \in \mathbb{C}^{N\times N^2}$ is the Gabor matrix generated by $\boldsymbol{c} \in \mathbb{C}^N$.

- ▷ Columns of measurement matrix $\mathbf{G}_{\boldsymbol{c}} \in \mathbb{C}^{N\times N^2}$ are time-frequency shifts of $\boldsymbol{c}$.
- ▷ To recover $\mathbf{X}$ from $\mathbf{Y}$, one needs to solve the undetermined linear system

$$\vec{\boldsymbol{y}} = (\overline{\mathbf{G}_{\boldsymbol{c}}} \otimes \mathbf{G}_{\boldsymbol{c}})\vec{\boldsymbol{x}} \qquad \text{with} \qquad \vec{\boldsymbol{y}} = \mathrm{vec}(\mathbf{Y}) \in \mathbb{C}^{N^2} \text{ and } \vec{\boldsymbol{x}} = \mathrm{vec}(\mathbf{X}) \in \mathbb{C}^{N^4}.$$

- ▷ $\vec{\boldsymbol{x}}$ needs to be sparse to get a unique solution.

Problem: Assume the support pattern Γ of $\vec{\boldsymbol{x}}$ (i.e. of $\mathbf{X}$) is known. Find an identifier $\boldsymbol{c} \in \mathbb{C}^N$ such that the matrix $\overline{\mathbf{G}_{\boldsymbol{c}}} \otimes \mathbf{G}_{\boldsymbol{c}}|_\Gamma$ is invertible.

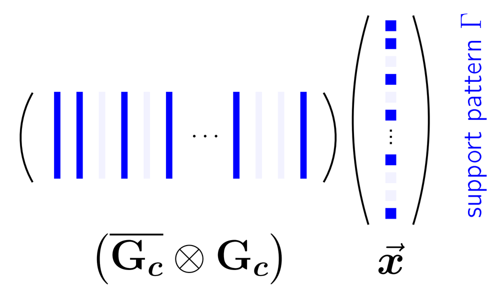
Permissible & Defective Support Pattern

We consider (covariance) matrices $\mathbf{X}$ of size $N^2 \times N^2$. The support pattern of $\mathbf{X}$ is a set $\Lambda \subset (\mathbb{Z}_N\times\mathbb{Z}_N)\times(\mathbb{Z}_N\times\mathbb{Z}_N)$ such that

$$\Lambda = \big\{(\lambda,\lambda') \in \Lambda\ :\ \mathbf{X}(\lambda,\lambda') \neq 0\big\} \qquad \text{where} \qquad \lambda = (k,l) \text{ with } k,l \in \mathbb{Z}_N\ .$$

We say that Λ is a positive semi-definite (psd) pattern if

$$(\lambda,\lambda') \in \Gamma \ \Rightarrow\ (\lambda,\lambda),\ (\lambda',\lambda),\ (\lambda',\lambda') \in \Gamma. \qquad (3)$$



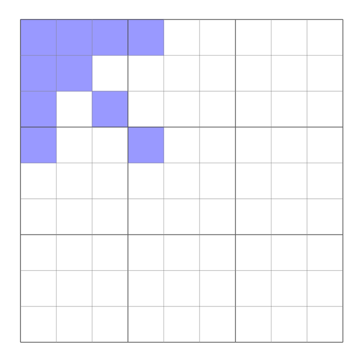
$(\overline{\mathbf{G}_{\boldsymbol{c}}} \otimes \mathbf{G}_{\boldsymbol{c}})$ $\vec{\boldsymbol{x}}$

Given a support pattern Γ. Is it possible to find a $\boldsymbol{c} \in \mathbb{C}^N$ such that $\overline{\mathbf{G}_{\boldsymbol{c}}} \otimes \mathbf{G}_{\boldsymbol{c}}|_\Gamma$ is injective?

- ▷ Yes! $\Rightarrow$ Γ is permissible.
- ▷ No! $\Rightarrow$ Γ is defective.

If Γ is permissible: How to choose $\boldsymbol{c} \in \mathbb{C}^N$?

Examples of Defective Patterns

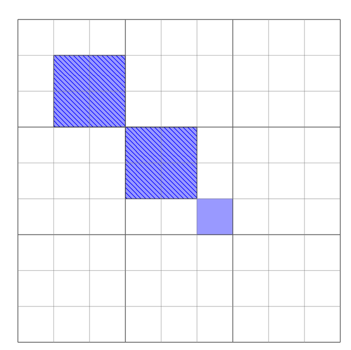


⇐ Arrowhead pattern

$\Gamma_{\mathrm{L}} = \big(\{\lambda\}\times\Lambda\big) \cup \big(\Lambda\times\{\lambda\}\big) \cup \mathrm{diag}(\Lambda)$ with $\lambda \in \Lambda$ and $|\Lambda| \geq N+1$.

Rank-two defective pattern ⇒

$\Gamma_{\mathrm{R}} = (\Lambda_1 \times \Lambda_1) \cup (\Lambda_2 \times \Lambda_2)$



I. Permissible Pattern of the First Kind

Theorem: Let $\mathbf{A} \in \mathbb{C}^{m\times n}$ with $m \leq n$, and let $\Lambda \subseteq \{0,1,\ldots,n-1\}$ with $|\Lambda| \geq 2$. Then the following statements are equivalent

(a) $\mathbf{A}|_\Lambda \in \mathbb{C}^{m\times|\Lambda|}$ is injective. (b) $\overline{\mathbf{A}} \otimes \mathbf{A}|_{\Lambda\times\Lambda} \in \mathbb{C}^{m^2\times|\Lambda|^2}$ is injective.

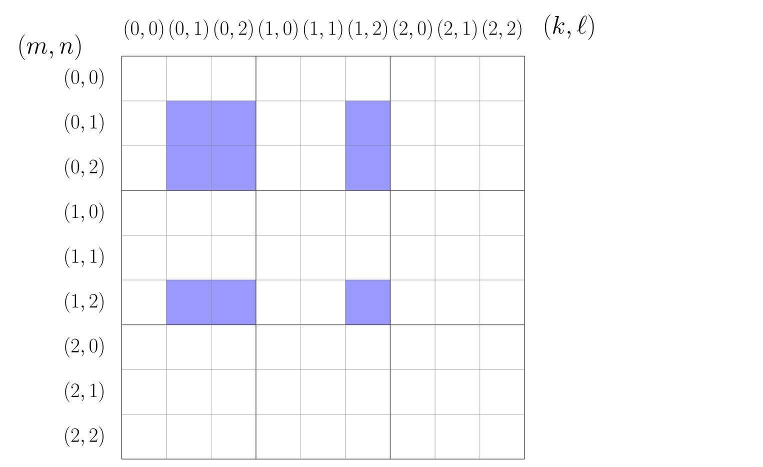
(c) There exist nonempty disjoint subsets $\Lambda_1, \Lambda_2 \subset \Lambda$ with $\Lambda_1 \cup \Lambda_2 = \Lambda$ such that $\overline{\mathbf{A}} \otimes \mathbf{A}|_{(\Lambda_1\times\Lambda_1)\cup(\Lambda_2\times\Lambda_2)}$ is injective.

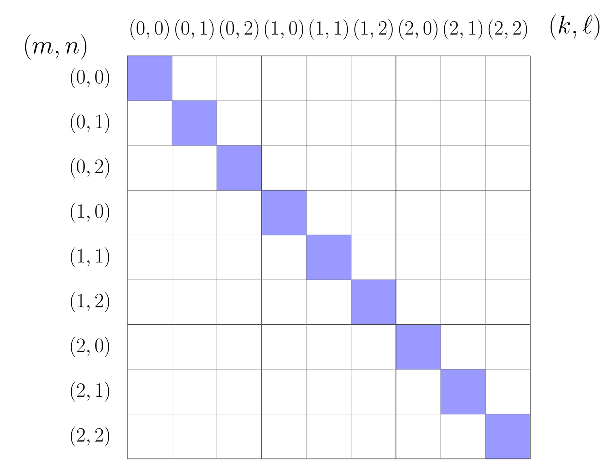
(d) There exist nonempty disjoint subsets $\Lambda_1, \Lambda_2 \subset \Lambda$ with $\Lambda_1 \cup \Lambda_2 = \Lambda$ such that $\overline{\mathbf{A}} \otimes \mathbf{A}|_{(\Lambda_1\times\Lambda_2)\cup(\Lambda_2\times\Lambda_1)\cup\mathrm{diag}(\Lambda)}$ is injective.

(e) There exists $\lambda \in \Lambda$ for which $\overline{\mathbf{A}} \otimes \mathbf{A}|_{(\{\lambda\}\times\Lambda)\cup(\Lambda\times\{\lambda\})\cup\mathrm{diag}(\Lambda)}$ is injective.

and $|\Lambda| \leq \mathrm{rank}\,\mathbf{A}$ $(\leq m)$ and $\overline{\mathbf{A}}\otimes\mathbf{A}|_\Gamma$ is injective for every subpattern $\Gamma \subset \Lambda\times\Lambda$.

- ▷ Choose $\boldsymbol{c} \in \mathbb{C}^N$ randomly then $\mathbf{G}_{\boldsymbol{c}} \in \mathbb{C}^{N\times N^2}$ is injective (with probability 1) $\Rightarrow$ (b)–(e) yields permissible patterns if $|\Lambda| \leq N$.

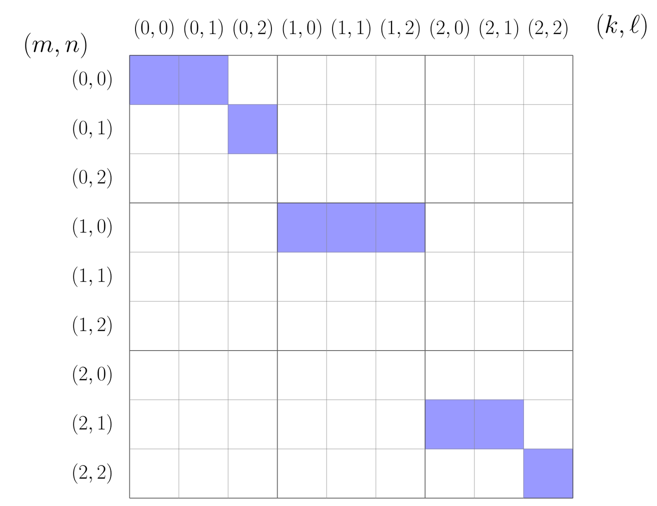


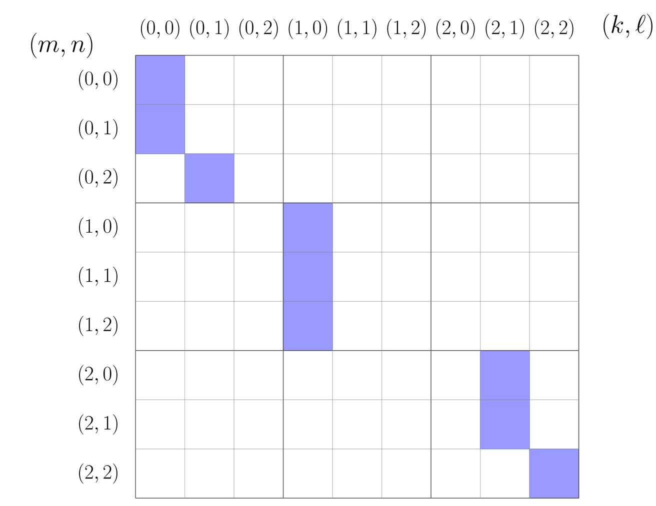


II. Generalized Diagonal Pattern

As a generalization of the diagonal pattern $\Lambda_{\mathrm{diag}} = \{(k,\ell,k,\ell) : k,\ell \in \mathbb{Z}_N\}$, we consider patterns of the form

$$\Gamma = \cup_{k=0}^{L-1}\big\{(k,\ell,k,n_{k,\ell}) : \ell \in \mathbb{Z}_N\big\} \qquad \Gamma = \cup_{k=0}^{L-1}\big\{(k,\ell_{k,n},k,n) : n \in \mathbb{Z}_N\big\}$$





where $n_{k,\ell} \in \mathbb{Z}_N$ and $\ell_{k,n} \in \mathbb{Z}_N$ are arbitrary sequences in k,ℓ and k,n.

Theorem: Let $N \geq 2$ be any integer and let Γ be any pattern of the above form. The matrix $\overline{\mathbf{G}_{\boldsymbol{c}}} \otimes \mathbf{G}_{\boldsymbol{c}}|_\Gamma = [\overline{\pi(\lambda)\boldsymbol{c}} \otimes \pi(\lambda')\boldsymbol{c}]_{(\lambda,\lambda')\in\Gamma} \in \mathbb{C}^{N^2\times N^2}$ is invertible for all $\boldsymbol{c}$ in a dense open subset of $\mathbb{C}^N$ with full measure.

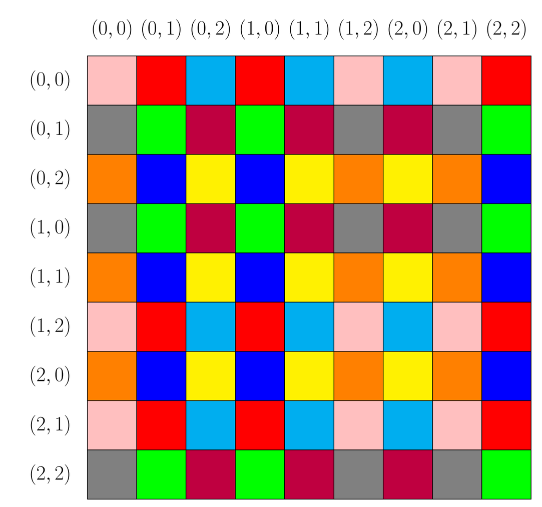
- ▷ A generalized pattern is a psd-pattern only when it is a diagonal pattern.
- ▷ Choosing $\boldsymbol{c} \in \mathbb{C}^N$ randomly $\Rightarrow$ $\overline{\mathbf{G}_{\boldsymbol{c}}} \otimes \mathbf{G}_{\boldsymbol{c}}|_\Gamma$ is invertible (with prob. 1).

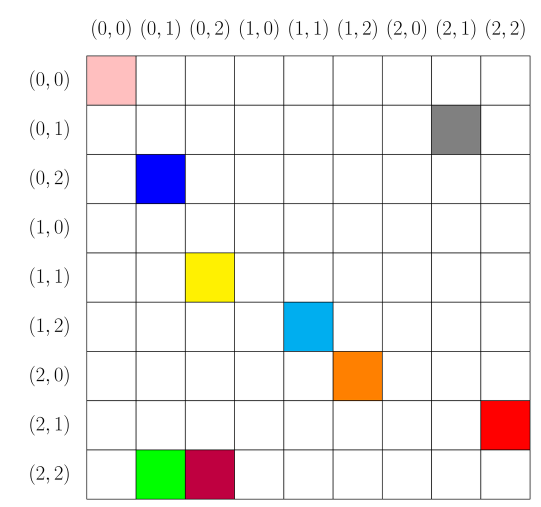
III. Scattered Patterns

- ▷ Divide $\mathbb{Z}_N \times \mathbb{Z}_N$ into $N+1$ additive subgroups V_p of cardinality N.
- ▷ Consider the cosets of these subgroups (left figure).
- ▷ A pattern Γ is obtained by choosing one element from each coset (right figure).

$$\Gamma_p = \big\{(\lambda_{q,q'}, \widetilde{\lambda}_{q,q'})\big\}_{q,q'=0}^{N-1} \subset (\mathbb{Z}_N\times\mathbb{Z}_N)\times(\mathbb{Z}_N\times\mathbb{Z}_N), \qquad (4)$$

with $\lambda_{q,q'} \in V_p + (0,q)$ and $\widetilde{\lambda}_{q,q'} \in V_p + (0,q')$





Theorem: Let $N \geq 2$ be a prime and let $\Gamma \subset (\mathbb{Z}_N\times\mathbb{Z}_N)\times(\mathbb{Z}_N\times\mathbb{Z}_N)$ be any pattern of the form (4). Then

1. $\overline{\mathbf{G}_{\boldsymbol{c}}} \otimes \mathbf{G}_{\boldsymbol{c}}|_\Gamma$ is invertible for all $\boldsymbol{c}$ in a dense open subset of $\mathbb{C}^N$ of full measure.
2. there exist explicit vectors $\boldsymbol{c} \in \mathbb{C}^N$ for which $\overline{\mathbf{G}_{\boldsymbol{c}}} \otimes \mathbf{G}_{\boldsymbol{c}}|_\Gamma$ is unitary.

- ▷ There are psd-pattern of this structure (upper-left 3×3 corner in example).
- ▷ Explicit construction of $\boldsymbol{c}$ $\Rightarrow$ unitary $\overline{\mathbf{G}_{\boldsymbol{c}}} \otimes \mathbf{G}_{\boldsymbol{c}}|_\Gamma$ $\Rightarrow$ stable recovery.