Optimal algorithms and CRB for reciprocity calibration in Massive MIMO

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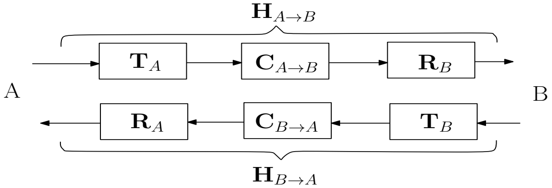
Outline

TDD Massive MIMO and Channel Reciprocity

- A Massive number of Transmit (Tx) Antennas at the base station (BS) talking to a small number of user equipment (UE) in Time Division Duplexing (TDD) mode.
- Channel observed in the digital domain from BS to UE (in general, from antennas A to antennas B) not reciprocal.



- $\mathbf{C}_{A\to B}$ and $\mathbf{C}_{B\to A}$ model the reciprocal propagation channels.
- Diagonal Matrices $\mathbf{T}_A$, $\mathbf{R}_A$, $\mathbf{T}_B$, $\mathbf{R}_B$ model the response of the transmit and receive RF front-ends

TDD Massive MIMO and Channel Reciprocity (Cont'd)

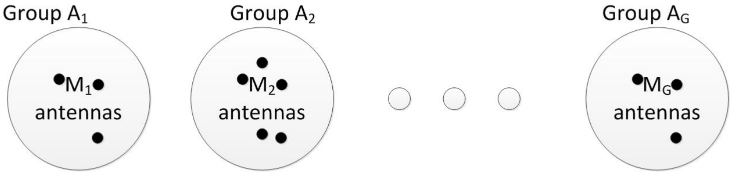
- $\mathbf{H}_{A\to B} = \mathbf{R}_B \mathbf{C}_{A\to B} \mathbf{T}_A, \qquad \mathbf{H}_{B\to A} = \mathbf{R}_A \mathbf{C}_{B\to A} \mathbf{T}_B,$
- DL (Downlink) channel is derived from the UL (Uplink) channel for Tx beamforming.
- As $\mathbf{C}_{A\to B} = \mathbf{C}_{B\to A}^T$,

$$\mathbf{H}_{A\to B} = \underbrace{\mathbf{R}_B \mathbf{T}_B^{-T}}_{\mathbf{F}_B^{-T}} \mathbf{H}_{B\to A}^T \underbrace{\mathbf{R}_A^{-T} \mathbf{T}_A}_{\mathbf{F}_A} = \mathbf{F}_B^{-T} \mathbf{H}_{B\to A}^T \mathbf{F}_A. \tag{1}$$

Relative calibration factor.

- It has been shown that calibration at the BS is more crucial for Multi-user MIMO performance.
- Hence, in what follows, calibration will be done internally at the BS without involving the UE.

General Framework and antenna grouping at BS



- Partition the M antennas into G groups with M_i antennas each.
- Proposed in Jiang et al., "A Framework for Over-the-air Reciprocity Calibration for TDD Massive MIMO Systems".
- Each group A_i transmits pilots $\mathbf{P}_i$ for L_i channel uses.

$$\text{Bidirectional Tx} \begin{cases} \mathbf{Y}_{i\to j} = \mathbf{R}_j \mathbf{C}_{i\to j} \mathbf{T}_i \mathbf{P}_i + \mathbf{N}_{i\to j}, \\ \mathbf{Y}_{j\to i} = \mathbf{R}_i \mathbf{C}_{j\to i} \mathbf{T}_j \mathbf{P}_j + \mathbf{N}_{j\to i}, \end{cases} \quad (2)$$

General Framework

- Eliminating the propagation channel $\mathbf{C}$ and after some matrix manipulations, we get

$$\mathcal{Y}(\mathbf{P})\mathbf{f} = \widetilde{\mathbf{n}}, \qquad \mathbf{F} = \text{diag}\{\mathbf{f}\} = \mathbf{R}^{-T}\mathbf{T}.$$

$$\mathcal{Y}(\mathbf{P}) = \underbrace{\begin{bmatrix} (\mathbf{Y}_{2\to1}^T * \mathbf{P}_1^T) & -(\mathbf{P}_2^T * \mathbf{Y}_{1\to2}^T) & 0 & \cdots \\ (\mathbf{Y}_{3\to1}^T * \mathbf{P}_1^T) & 0 & -(\mathbf{P}_3^T * \mathbf{Y}_{1\to3}^T) & \cdots \\ 0 & (\mathbf{Y}_{3\to2}^T * \mathbf{P}_2^T) & -(\mathbf{P}_3^T * \mathbf{Y}_{2\to3}^T) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{(\sum_{j=2}^{G}\sum_{i=1}^{j-1} L_i L_j)\times M}.$$

where $*$ denotes the Khatri–Rao product (or column-wise Kronecker product[1]). $\widetilde{\mathbf{n}}$ is colored noise.

[1] $\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_M\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_M\end{bmatrix}$ where $\mathbf{a}_i$ and $\mathbf{b}_i$ are column vectors for $i \in 1\dots M$, then, $\mathbf{A} * \mathbf{B} = \begin{bmatrix}\mathbf{a}_1\otimes\mathbf{b}_1 & \mathbf{a}_2\otimes\mathbf{b}_2 & \dots & \mathbf{a}_M\otimes\mathbf{b}_M\end{bmatrix}$.

General Framework (Cont'd)

$$\mathcal{Y}(\mathbf{P})\mathbf{f} = \widetilde{\mathbf{n}}$$

- Least squares (LS) estimate of $\mathbf{f}$

$$\begin{aligned}\hat{\mathbf{f}} &= \arg\min_{\mathbf{f}} \|\mathcal{Y}(\mathbf{P})\,\mathbf{f}\|^2 \\ &= \arg\min_{\mathbf{f}} \sum_{i<j} \|(\mathbf{Y}_{j\to i}^T * \mathbf{P}_i^T)\mathbf{f}_i - (\mathbf{P}_j^T * \mathbf{Y}_{i\to j}^T)\mathbf{f}_j\|^2 \end{aligned}, \quad (3)$$

- To exclude the trivial solution, $\hat{\mathbf{f}} = \mathbf{0}$, this needs to be augmented with a constraint

$$\mathcal{C}(\hat{\mathbf{f}}, \mathbf{f}) = 0.$$

General Framework (Cont'd)

- Typical choices for the constraint are
  1) Norm plus phase constraint (NPC):

$$\text{norm: } \mathrm{Re}\{\mathcal{C}(\hat{\mathbf{f}}, \mathbf{f})\} = ||\hat{\mathbf{f}}||^2 - c\,, \; c = ||\mathbf{f}||^2, \tag{4}$$

$$\text{phase: } \mathrm{Im}\{\mathcal{C}(\hat{\mathbf{f}}, \mathbf{f})\} = \mathrm{Im}\{\hat{\mathbf{f}}^H \mathbf{f}\} = 0. \tag{5}$$

  2) Linear constraint:

$$\mathcal{C}(\hat{\mathbf{f}}, \mathbf{f}) = \hat{\mathbf{f}}^H \mathbf{g} - c = 0\,. \tag{6}$$

  The most popular linear constraint is the First Coefficient Constraint (FCC), which is (6) with $\mathbf{g} = \mathbf{e}_1$, $c = 1$.

- We now proceed to investigate some optimal estimation methods and performance limits.

Cramer Rao Bound (CRB) - Defining an auxiliary channel

- Re-parameterize in terms of the relative calibration factors and an auxiliary internal channel.
- As $\mathbf{F}_i = \mathbf{R}_i^{-T}\mathbf{T}_i$,

$$\mathbf{Y}_{i\to j} = \mathbf{R}_j\mathbf{C}_{i\to j}\mathbf{T}_i\mathbf{P}_i + \mathbf{N}_{i\to j} = \underbrace{\mathbf{R}_j\mathbf{C}_{i\to j}\mathbf{R}_i^T}_{\mathcal{H}_{i\to j}}\mathbf{F}_i\mathbf{P}_i + \mathbf{N}_{i\to j}. \quad (7)$$

Auxiliary internal channel

- $\mathcal{H}_{i\to j}$ is a nuisance parameter and does not correspond to any physically measurable quantity.
- $\mathcal{H}_{i\to j}$ is reciprocal: $\mathcal{H}_{i\to j} = \mathcal{H}_{j\to i}^T$.

Cramer Rao Bound

- Stacking these observations into a vector

$$\begin{aligned}\mathbf{y} &= \left[\mathrm{vec}(\mathbf{Y}_{1\to 2})^T\,\mathrm{vec}(\mathbf{Y}_{2\to 1}^T)^T\,\mathrm{vec}(\mathbf{Y}_{1\to 3})^T \ldots\right]^T \\ &= \mathcal{H}(\mathbf{h},\mathbf{P})\mathbf{f} + \mathbf{n} = \mathcal{F}(\mathbf{f},\mathbf{P})\mathbf{h} + \mathbf{n},\end{aligned} \tag{8}$$

where $\mathbf{h} = \left[\mathrm{vec}(\mathcal{H}_{1\to 2})^T\,\mathrm{vec}(\mathcal{H}_{1\to 3})^T\,\mathrm{vec}(\mathcal{H}_{2\to 3})^T \ldots\right]^T$, $\mathbf{f} = [\mathbf{f}_1,\, \mathbf{f}_2,\, \ldots,\, \mathbf{f}_M]$ (Relative calibration factors for M antennas) and $\mathbf{n}$ is the corresponding noise vector.

Cramer Rao Bound (Cont'd)

- The composite matrices $\mathcal{H}$ and $\mathcal{F}$ are given by,

$$\mathcal{H}(\mathbf{h},\mathbf{P}) = \begin{bmatrix} \mathbf{P}_1^T * \mathcal{H}_{1\to 2} & 0 & 0 & \dots \\ 0 & \mathcal{H}_{1\to 2}^T * \mathbf{P}_2^T & 0 & \dots \\ \mathbf{P}_1^T * \mathcal{H}_{1\to 3} & 0 & 0 & \dots \\ 0 & 0 & \mathcal{H}_{1\to 3}^T * \mathbf{P}_3^T & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\mathcal{F}(\mathbf{f},\mathbf{P}) = \begin{bmatrix} \mathbf{P}_1^T\mathbf{F}_1 \otimes \mathbf{I} & 0 & 0 & 0 & \dots \\ \mathbf{I} \otimes \mathbf{P}_2^T\mathbf{F}_2 & 0 & 0 & 0 & \dots \\ 0 & \mathbf{P}_1^T\mathbf{F}_1 \otimes \mathbf{I} & 0 & 0 & \dots \\ 0 & \mathbf{I} \otimes \mathbf{P}_3^T\mathbf{F}_3 & 0 & 0 & \dots \\ 0 & 0 & \mathbf{P}_2^T\mathbf{F}_2 \otimes \mathbf{I} & 0 & \dots \\ 0 & 0 & \mathbf{I} \otimes \mathbf{P}_3^T\mathbf{F}_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (9)$$

Cramer Rao Bound (Cont'd)

$$\mathbf{y} = \mathcal{H}(\mathbf{h}, \mathbf{P})\mathbf{f} + \mathbf{n} = \mathcal{F}(\mathbf{f}, \mathbf{P})\mathbf{h} + \mathbf{n},$$

- The scenario is now identical to that encountered in some blind channel estimation scenarios and hence we can take advantage of some existing tools[2] [3].
- Treating $\mathbf{h}$ and $\mathbf{f}$ as deterministic unknown parameters, $\mathbf{n} \sim \mathcal{CN}(0, \sigma^2\mathbf{I})$, the Fisher Information Matrix (FIM) $\mathbf{J}$ for jointly estimating $\mathbf{f}$ and $\mathbf{h}$ is,

$$\mathbf{J} = \frac{1}{\sigma^2} \begin{bmatrix} \mathcal{H}^H \\ \mathcal{F}^H \end{bmatrix} \begin{bmatrix} \mathcal{H} & \mathcal{F} \end{bmatrix}. \tag{10}$$

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[2] Carvalho and Slock, "Semi–Blind Methods for FIR Multichannel Estimation".

[3] Carvalho, Omar, and Slock, "Performance and complexity analysis of blind FIR channel identification algorithms based on deterministic maximum likelihood in SIMO systems".

Cramer Rao Bound (Cont'd)

- Real interest is in deriving the CRB for $\mathbf{f}$, in the presence of the nuisance parameters $\mathbf{h}$.
- Incorporating the effect of the constraint $\mathcal{C}$ on $\mathbf{f}$, we can derive from[4], the following constrained CRB for $\mathbf{f}$

$$\text{CRB}_{\mathbf{f}} = \sigma^2 \mathcal{V}_{\mathbf{f}} \left(\mathcal{V}_{\mathbf{f}}^H \mathcal{H}^H \mathcal{P}_{\mathcal{F}}^{\perp} \mathcal{H} \mathcal{V}_{\mathbf{f}} \right)^{-1} \mathcal{V}_{\mathbf{f}}^H \tag{11}$$

where $\mathcal{P}_{\mathcal{F}} = \mathcal{F}(\mathcal{F}^H \mathcal{F})^{\dagger} \mathcal{F}^H$ is the projection operator on the column space of matrix $\mathcal{F}$ and $\mathcal{P}_{\mathcal{F}}^{\perp} = \mathbf{I} - \mathcal{P}_{\mathcal{F}}$.

- $\dagger$ corresponds to the Moore-Penrose pseudo inverse.
- The $M \times (M-1)$ matrix $\mathcal{V}_{\mathbf{f}}$ is such that its column space spans the orthogonal complement of that of $\dfrac{\partial \mathcal{C}(f)}{\partial \mathbf{f}^*}$, i.e., $\mathcal{P}_{\mathcal{V}_{\mathbf{f}}} = \mathcal{P}^{\perp}_{\frac{\partial \mathcal{C}}{\partial \mathbf{f}^*}}$.

[4] Carvalho and Slock, *Cramér-Rao bounds for blind multichannel estimation*.

Maximum Likelihood(ML) & LS

- The log likelihood for the ML estimator may be written as $\frac{1}{\sigma^2}\left\|\mathbf{y}-\mathcal{F}(\mathbf{f},\mathbf{P})\mathbf{h}\right\|^2$.
- Optimizing w.r.t. $\mathbf{h}$ leads to $\mathbf{h}=(\mathcal{F}^H\mathcal{F})^{\dagger}\mathcal{F}^H\mathbf{y}$.
- Substituting back this estimate yields an ML estimator for $\hat{\mathbf{f}}$ minimizing

$$\mathbf{y}^H\mathcal{F}^{\perp}(\mathcal{F}^{\perp H}\mathcal{F}^{\perp})^{\dagger}\mathcal{F}^{\perp H}\mathbf{y}, \tag{12}$$

- Now, it can be shown that,

$$\mathcal{Y}(\mathbf{P})\mathbf{f}=\mathcal{F}^{\perp H}\mathbf{y}=\widetilde{\mathbf{n}}, \tag{13}$$

- Hence, LS estimator minimizes $\left\|\mathcal{Y}(\mathbf{P})\mathbf{f}\right\|^2=\mathbf{y}^H\mathcal{F}^{\perp}\mathcal{F}^{\perp H}\mathbf{y}$.

Maximum Likelihood(ML) & LS Cont'd

ML Metric

$$\mathbf{y}^H \mathcal{F}^{\perp} (\mathcal{F}^{\perp H} \mathcal{F}^{\perp})^{\dagger} \mathcal{F}^{\perp H} \mathbf{y}$$

LS Metric

$$\mathbf{y}^H \mathcal{F}^{\perp} \mathcal{F}^{\perp H} \mathbf{y}$$

- Thus, ML estimator is LS with $\mathbf{F}$ dependent weighting.
- In the case of single antenna grouping, $(\mathcal{F}^{\perp H} \mathcal{F}^{\perp})$ has a diagonal structure and is a multiple of the Identity matrix if all calibration values have equal magnitude.
- This implies that the ML techniques are more useful when the calibration factors vary over a significant range.

Variational Bayes (VB) Estimation

- In VB, a Bayesian estimate is obtained by computing an approximation (variational distribution) to the posterior distribution of the parameters $\mathbf{h}, \mathbf{f}$
- $\mathbf{h}, \mathbf{f}$ priors : $\mathbf{f} \sim \mathcal{CN}(0, \alpha^{-1}\mathbf{I_M})$, $\mathbf{h} \sim \mathcal{CN}(0, \beta^{-1}\mathbf{I_{N_h}})$.
- α, β are assumed to have themselves a uniform prior. N_h is the number of elements in $\mathbf{h}$.
- The variational distribution, is chosen to minimize the Kullback-Leibler distance between the true posterior distribution $p(\mathbf{h}, \mathbf{f}, \alpha, \beta|\mathbf{y})$ and a factored variational distribution

$$p(\mathbf{h}, \mathbf{f}, \alpha, \beta|\mathbf{y}) \approx q_{\mathbf{h}}(\mathbf{h})\, q_{\mathbf{f}}(\mathbf{f})\, q_{\alpha}(\alpha)\, q_{\beta}(\beta).$$

Variational Bayes Estimation (Cont'd)

- The factors can be obtained in an alternating fashion as,

$$\ln(q_{\psi_i}(\psi_i)) = < \ln p(\mathbf{y}, \mathbf{h}, \mathbf{f}, \alpha, \beta) >_{k \neq i} + c_i, \tag{14}$$

where ψ_i refers to the i^{th} block of $\psi = [\mathbf{h}, \mathbf{f}, \alpha, \beta]$ and $<>_{k \neq i}$ represents the expectation operator over the distributions q_{ψ_k} for all $k \neq i$. c_i is a normalizing constant.

- The log likelihood,

$$\begin{aligned} \ln p(\mathbf{y}, \mathbf{h}, \mathbf{f}, \alpha, \beta) &= \ln p(\mathbf{y}|\mathbf{h}, \mathbf{f}, \alpha, \beta) + \ln p(\mathbf{f}|\alpha) + \ln p(\mathbf{h}|\beta) \\ &= -N_y \ln \sigma^2 - \frac{1}{\sigma^2} \|\mathbf{y} - \mathcal{H}\mathbf{f}\|^2 + M \ln \alpha - \alpha \|\mathbf{f}\|^2 \\ &+ N_h \ln \beta - \beta \|\mathbf{h}\|^2 + c. \end{aligned} \tag{15}$$

Here, N_y refers to the number of elements in $\mathbf{y}$ and c is a constant.

Variational Bayes Estimation - Overall algorithm

- $\mathbf{f} \sim \mathcal{CN}(\hat{\mathbf{f}}, \mathbf{C}_{\tilde{f}\tilde{f}})$ and $\mathbf{h} \sim \mathcal{CN}(\widehat{\mathbf{h}}, \mathbf{C}_{\tilde{h}\tilde{h}})$.

Initialization: Initialize $\hat{\mathbf{f}}$ using existing calibration methods. Use this to determine $\widehat{\mathbf{h}}, <\alpha>, <\beta>$.

repeat

$$<\mathcal{H}^H\mathcal{H}>= \mathcal{H}^H(\widehat{\mathbf{h}})\mathcal{H})(\widehat{\mathbf{h}}) + <\mathcal{H}^H(\tilde{\mathbf{h}})\mathcal{H}(\tilde{\mathbf{h}})>$$

$$\hat{\mathbf{f}} = (<\mathcal{H}^H\mathcal{H}> + <\alpha> \mathbf{I})^{-1}\mathcal{H}^{\mathbf{H}}\mathbf{y}$$

$$\mathbf{C}_{\tilde{f}\tilde{f}} = (<\mathcal{H}^H\mathcal{H}> + <\alpha> \mathbf{I})^{-1}$$

$$<\mathcal{F}^H\mathcal{F}>= \mathcal{F}^H(\hat{\mathbf{f}})\mathcal{F}(\hat{\mathbf{f}}) + <\mathcal{F}^H(\tilde{\mathbf{f}})\mathcal{F}(\tilde{\mathbf{f}})>$$

$$\widehat{\mathbf{h}} = (<\mathcal{F}^H\mathcal{F}> + <\beta> \mathbf{I})^{-1}\mathcal{F}^{\mathbf{H}}\mathbf{y}$$

$$\mathbf{C}_{\tilde{h}\tilde{h}} = (<\mathcal{F}^H\mathcal{F}> + <\beta> \mathbf{I})^{-1}$$

$$<\alpha>= \frac{M}{<\|\mathbf{f}\|^2>}, \; <\|\mathbf{f}\|^2>= \hat{\mathbf{f}}^H\hat{\mathbf{f}} + tr\{\mathbf{C}_{\tilde{f}\tilde{f}}\} \; .$$

$$<\beta>= \frac{N_h}{<\|\mathbf{h}\|^2>}, \; <\|\mathbf{h}\|^2>= \widehat{\mathbf{h}}^H\widehat{\mathbf{h}} + tr\{\mathbf{C}_{\tilde{h}\tilde{h}}\}.$$

until convergence.

Variational Bayes Estimation (Cont'd)

- The log likelihood,

$$\ln p(\mathbf{y}, \mathbf{h}, \mathbf{f}, \alpha, \beta) = -N_y \ln \sigma^2 - \frac{1}{\sigma^2} \left\| \mathbf{y} - \mathcal{H}\mathbf{f} \right\|^2 + M \ln \alpha - \alpha \left\| \mathbf{f} \right\|^2 + N_h \ln \beta - \beta \left\| \mathbf{h} \right\|^2 + c. \tag{16}$$

- The parameters of the prior distributions of $\mathbf{f}$ and $\mathbf{h}$ act as regularization parameters in their estimation.
- The advantage of using VB is that it automatically computes the optimal values for these regularization parameters too!!

Expectation Consistent Variational Bayes Estimation

- For single antenna grouping $(G = M)$, $\mathbf{C}_{\tilde{f}\tilde{f}}$ and $\mathbf{C}_{\tilde{h}\tilde{h}}$ are diagonal and $<\mathcal{F}^H(\tilde{\mathbf{f}})\mathcal{F}(\tilde{\mathbf{f}})>$, $<\mathcal{H}^H(\tilde{\mathbf{h}})\mathcal{H}(\tilde{\mathbf{h}})>$ can be computed easily.
- However, when $G < M$, these matrices are block diagonal.
- To simplify the computation, we propose to approximate them with multiples of the Identity matrix,

$$\begin{aligned} \mathbf{C}_{\tilde{f}\tilde{f}} &\approx \frac{tr\{(<\mathcal{H}^H\mathcal{H}> + <\alpha>\mathbf{I})^{-\mathbf{1}}\}}{M}\mathbf{I}_M \\ \mathbf{C}_{\tilde{h}\tilde{h}} &\approx \frac{tr\{(<\mathcal{F}^H\mathcal{F}> + <\beta>\mathbf{I})^{-\mathbf{1}}\}}{N_h}\mathbf{I}_{N_h}. \end{aligned} \tag{17}$$

- We call this approach EC-VB (Expectation consistent[5] VB).

[5] Opper and Winther, "Expectation Consistent Approximate Inference".

VB and Alternating Maximum Likelihood (AML)

- By forcing the matrices $\mathbf{C}_{\tilde{f}\tilde{f}}$, $\mathbf{C}_{\tilde{h}\tilde{h}}$ to zero and α, β to zero, this algorithm reduces to the AML algorithm[6] [7].
- AML iteratively maximizes the likelihood by alternating between the desired parameters $\mathbf{f}$ and the nuisance parameters $\mathbf{h}$ in a deterministic (non-Bayesian) setting.

[6] Carvalho and Slock, "Semi–Blind Methods for FIR Multichannel Estimation".

[7] Carvalho, Omar, and Slock, "Performance and complexity analysis of blind FIR channel identification algorithms based on deterministic maximum likelihood in SIMO systems".

Simulation setup

- We assess numerically the performance of various calibration algorithms and also compare them against their CRBs.
- The Tx and Rx calibration parameters for the BS antennas are assumed to have random phases uniformly distributed over $[-\pi, \pi]$ and amplitudes uniformly distributed in the range $[1-\delta, 1+\delta]$.
- SNR is defined as the ratio of the average received signal power across channel realizations at an antenna and the noise power at that antenna.
- Transmission happens from one antenna at a time $(G = M)$.

Mean square error performance of the algorithms.

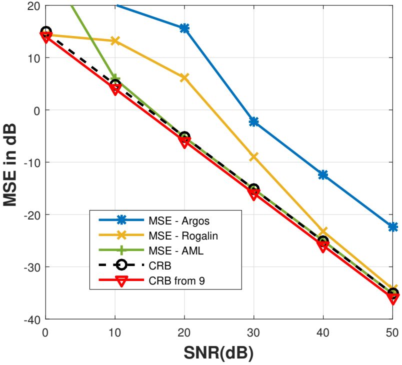

Figure: Comparison of single antenna transmit schemes with the CRB ($G = M = 16$).

- Curves generated over one realization of an i.i.d. Rayleigh channel. The first coefficient known constraint is used.
- Rogalin [8]method performs the LS as was presented earlier and improves over Argos by using all the bi-directional received data.
- AML outperforms the Rogalin performance at low SNR.

[8]Rogalin et al., "Scalable synchronization and reciprocity calibration for distributed multiuser MIMO".

CRB vs MSE.

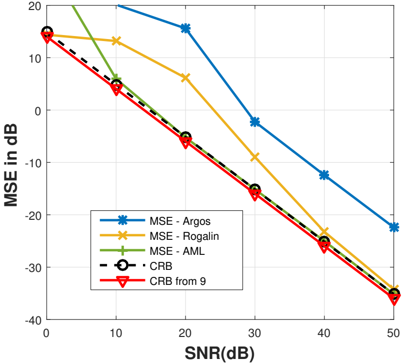

Figure: Comparison of single antenna transmit schemes with the CRB ($G = M = 16$).

- AML curve overlaps with the CRB at higher SNRs.
- Also plotted is the CRB as given in[9] assuming the internal propagation channel is fully known (the mean is known and the variance is negligible) and a (small) underestimation of the MSE can be observed as expected.

[9] Joao Vieira et al. "Reciprocity Calibration for Massive MIMO: Proposal, Modeling and Validation". In: *IEEE Trans. Wireless Commun.* (2017)

Convergence of the Optimal algorithms.

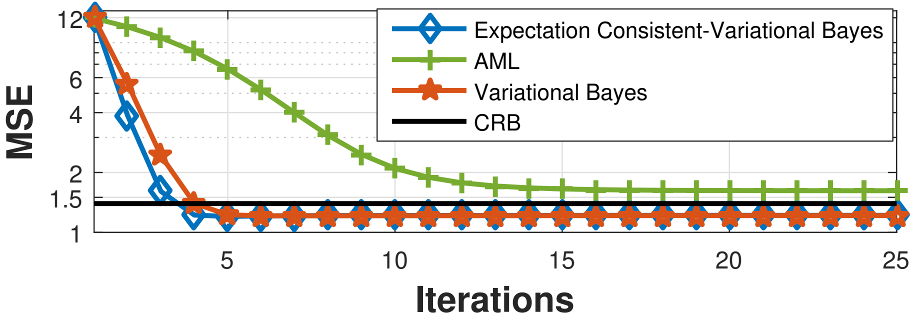

Figure: Convergence of the various iterative schemes for $M = 16$ and single antenna transmission.

Conclusion

- A simple and elegant derivation of the CRB has been presented for a general calibration framework.
- An optimal estimation algorithm based on VB is also introduced along with a variant.
- All these techniques have been compared via simulations in terms of both MSE performance and speed of convergence.

Thank You !!