On the Computability of System Approximations under Causality Constraints

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Causal LTI Systems

Discrete Linear Time-Invariant (LTI) Systems

- A *causal* LTI systems is an operator $\mathrm{S} : \ell^2 \to \ell^2$ mapping input sequences $\boldsymbol{x} = \{x[n]\}_{n\in\mathbb{Z}}$ onto output sequences $\boldsymbol{y} = \{y[n]\}_{n\in\mathbb{Z}}$ according to

$$y[n] = (\mathrm{S}\boldsymbol{x})[n] = \sum_{k=0}^{\infty} f[k]\,x[n-k]\,, \qquad n \in \mathbb{Z}\,.$$

- The sequence $\{f[k]\}_{k=0}^{\infty}$ is said to be the impulse response of S.
- The $\mathcal{Z}$-transform of $\boldsymbol{f}$ is given by

$$f(z) = (\mathcal{Z}\boldsymbol{f})(z) = \sum_{k=0}^{\infty} f[k]\,z^k\,, \qquad z \in \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}\,.$$

- The transfer function of S is then given by $f(e^{i\omega})$, $\omega \in [-\pi, \pi)$.

▷ S is causal $\Leftrightarrow$ $f(z)$ is analytic in $z \in \mathbb{D}$ $\Leftrightarrow$ $f \in H(\mathbb{D})$

▷ S is stable $\Leftrightarrow$ $\|f\|_\infty = \max_{\omega\in[-\pi,\pi)} |f(e^{i\omega})| < \infty$ $\Leftrightarrow$ $f \in L^\infty(\mathbb{T})$

We consider spaces $\mathcal{B} \subset H^\infty(\mathbb{D})$ of stable causal LTI systems.

Design of LTI Systems – Motivation

- The design of LTI systems is often based on optimization techniques.
- They derive the transfer function f of the system based on certain optimality criteria (optimal filtering, pre-whitening, etc.).

▷ The optimal f has often a very complicated structure without any closed form analytic representation.

▷ The optimal f is often only given by its values on a certain discrete sampling set $\mathcal{Z} = \{e^{i\omega_m} : m = 1, \dots, M\}$.

Goal/Approach: Approximate the optimal transfer functions f by simpler stable systems $\widetilde{f}$ which are known analytically.

Approximation by Basis Expansion

A natural and common approach is to represent $f \in \mathcal{B}$ in a basis $\{\varphi_n\}_{n=0}^{\infty}$ of $\mathcal{B}$

$$f = \textstyle\sum_{n=0}^{\infty} c_n(f)\,\varphi_n \qquad \text{with coefficient sequence } \{c_n(f)\}_{n=0}^{\infty} \subset \mathbb{C}\,.$$

Approximations $\widetilde{f}_N$ are obtained by restricting the sum to its first $N \in \mathbb{N}$ terms:

$$\widetilde{f}_N = \mathrm{P}_N f = \textstyle\sum_{n=0}^{N-1} c_n(f)\,\varphi_n\,, \qquad N = 1, 2, 3, \dots$$

Since $\boldsymbol{\varphi}$ is a basis for $\mathcal{B}$, one has

$$\lim_{N\to\infty} \|f - \widetilde{f}_N\|_{\mathcal{B}} = 0 \qquad \text{for all } f \in \mathcal{B}\,.$$

Problem

Given finitely many samples $\{f(e^{i\omega_m})\}_{m=1}^{M}$ of a transfer function $f \in \mathcal{B}$. Find a procedure to determine approximate coefficients, i.e. find a mapping

$$\mathrm{A}_M \;:\; \{f(e^{i\omega_m}) : m = 1, \dots, M\} \;\mapsto\; \{c_{N,n}(f) : n = 0, \dots, N(M) - 1\} \quad (1)$$

such that

$$\lim_{N\to\infty} \big\|f - \textstyle\sum_{n=0}^{N-1} c_{N,n}(f)\,\varphi_n\big\|_{\mathcal{B}} = 0 \qquad \text{for all } f \in \mathcal{B}\,.$$

Banach Spaces with Energy Constraint

- Let $\mathcal{A}(\mathbb{D})$ be the disk algebra of all $f \in H^\infty(\mathbb{D})$ which are continuous in $\overline{\mathbb{D}}$.
- On $\mathcal{A}(\mathbb{D})$, we define for every $\beta \geq 0$ the seminorm

$$\|f\|_\beta = \Big(\textstyle\sum_{n=1}^{\infty} n\,(1 + \log n)^\beta |f[n]|^2\Big)^{1/2}$$

with the Fourier coefficients $f[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\omega})\,e^{-in\omega}\,d\omega$.

- Therewith, we define a scale of Banach spaces

$$\mathcal{B}_{\alpha,\beta} = \{f \in \mathcal{A}(\mathbb{D}) \;:\; \|f\|_\beta < +\infty\}\,, \qquad \beta \geq 0\,,$$

equipped with the norm $\|f\|_{\mathcal{B}_\beta} = \max\big(\|f\|_\infty, \|f\|_\beta\big)$.

Properties

▷ $\mathcal{B}_{\beta_2} \subset \mathcal{B}_{\beta_1} \subset \mathcal{B}_0 \subset \mathcal{A}(\mathbb{D})$ for all $\beta_2 > \beta_1 > 0$.

▷ $\|f\|_0$ corresponds to the (Dirichlet) energy of f $\Rightarrow$ Spaces of finite energy.

▷ Parameter β characterize the smoothness (or energy concentration) of f.

Lemma: For every $\beta \geq 0$ the set $\boldsymbol{\psi} = \{\psi_n(z) = z^n\}_{n=0}^{\infty}$ is a basis (the Fourier basis) of $\mathcal{B}_\beta$ with coefficient functionals

$$c_n(f) = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\omega})\,\overline{\psi_n(e^{i\omega})}\,d\omega = f[n]\,. \qquad (2)$$

I. Approximation by Numerical Integration

- Apply numerical integration methods to evaluate approximations $c_{N,n}(f)$ of the coefficients (2) based on given samples $\{f(z_{N,k})\}$ of f.
- Use the approximated coefficients $c_{N,n}(f)$ instead of the unknown true values $c_n(f)$ and determine the approximations

$$\widetilde{f}_N = \mathrm{Q}_N f = \textstyle\sum_{n=0}^{N-1} c_{N,n}(f)\,\psi_n\,, \qquad N \in \mathbb{N}\,. \qquad (3)$$

Question: Can we find an approximation procedure (1) such that

$$\lim_{N\to\infty} \|f - \mathrm{Q}_N f\|_{\mathcal{B}_\beta} = 0 \qquad \text{for all } f \in \mathcal{B}_\beta\,.$$

Example – Integration by Riemann Sums

Approximate the integral in (2) using the rectangular formula of the Riemann sum bases on $M = 2N$ equidistant samples of f on $\mathbb{T}$, i.e.

$$c_n(f) \approx c_{N,n}(f) = \tfrac{1}{M}\textstyle\sum_{m=1}^{M} f(z_{N,m})\,\psi_n(z_{N,m}) \qquad (4)$$

based on the sampling set $\mathcal{Z}_N = \big\{z_{N,m} = e^{i\frac{2\pi}{M}(m-1)}\big\}_{m=1}^{M} \subset \mathbb{T}$.

- Other methods: different quadrature formulas and/or other sampling sets.

Requirements on Integration Method

Two natural assumptions on the integration methods:

(i) To every approximation degree $N \in \mathbb{N}$ and there exists an $M(N) \in \mathbb{N}$ and a sampling set $\mathcal{Z}_N = \{z_{N,1}, \dots, z_{N,M(N)}\} \subset \mathbb{T}$ such that the functionals $c_{N,n}(f)$ are uniquely determined by the values of f on $\mathcal{Z}_N$.

(ii) The approximation operators Q_N defined by (3) satisfies

$$\lim_{N\to\infty} \|\mathrm{Q}_N\psi_n - \psi_n\|_\infty = 0 \qquad \text{for all } n = 0, 1, 2, \dots\,.$$

No Convergent Method on $\mathcal{B}_\beta$ with $0 \leq \beta \leq 1$

Theorem: Let $0 \leq \beta \leq 1$ be arbitrary and let $\boldsymbol{\psi} = \{\psi_n\}_{n=0}^{\infty}$ be the Fourier basis of $\mathcal{B}_\beta$. Let $\{\mathrm{Q}_N\}_{N=1}^{\infty}$ be the sequence (3) of operators associated with $\boldsymbol{\psi}$ and which satisfies Conditions (i) and (ii), then

$$\lim_{N\to\infty} \|\mathrm{Q}_N\|_{\mathcal{B}_\beta\to\mathcal{B}_\beta} = +\infty\,.$$

Corollary: Let $0 \leq \beta \leq 1$ be arbitrary and let $\{\mathrm{Q}_N\}_{N\in\mathbb{N}}$ be as in the previous theorem. Then there exists a residual subset $\mathcal{R} \subset \mathcal{B}_\beta$ such that

$$\limsup_{N\to\infty} \|\mathrm{Q}_N f\|_\infty = +\infty \qquad \text{for all } f \in \mathcal{R}\,.$$

Convergent Methods on $\mathcal{B}_\beta$ with $\beta > 1$

Theorem: Let $\beta > 1$ be arbitrary and let $\{\mathrm{Q}_N\}_{N=1}^{\infty}$ be the sequence (3) of approximation operators with coefficients $c_{N,n}(f)$ calculated by (4) then

$$\lim_{N\to\infty} \|\mathrm{Q}_N f - f\|_{\mathcal{B}_\beta} = 0 \qquad \text{for all } f \in \mathcal{B}_\beta\,.$$

II. Computational Bases

Definition: Let $\mathcal{B} \subset \mathcal{C}(\mathbb{T})$ be a Banach space of continuous functions on $\mathbb{T}$, and let $\boldsymbol{\varphi} = \{\varphi_n\}_{n=0}^{\infty}$ be a basis for $\mathcal{B}$. We call $\boldsymbol{\varphi}$ a computational basis if the corresponding coefficient functionals $\{c_n(f)\}$ of $\boldsymbol{\varphi}$ have the following property:

▷ To every $n = 0, 1, 2, \dots$ there exists an $\mu(n) \in \mathbb{N}$ and distinct numbers $z_{n,1}, \dots, z_{n,\mu} \in \mathbb{T}$ such that $c_n(f)$ does only depend on the values $f(z_{k,n})$, $1 \leq k \leq \mu(n)$ for every $f \in \mathcal{B}$.

- So the nth coefficient $c_n(f)$ can be determined exactly from only finitely many values $\{f(z_{n,1}), \dots, f(z_{n,\mu(n)})\}$ of f, for all $f \in \mathcal{B}$.
- If a computable basis is known then $\lim_{N\to\infty} \|\mathrm{Q}_N f - f\|_{\mathcal{B}} = 0$ for all $f \in \mathcal{B}$.
- Example: Spline basis for $\mathcal{C}(\mathbb{T})$.

No Computable Basis on $\mathcal{B}_\beta$ with $0 \leq \beta \leq 1$

Theorem: The spaces $\mathcal{B}_\beta$ with $0 \leq \beta \leq 1$ possess no computational basis.

Discussion and Remarks

- The same results hold for the space of all $f \in \mathcal{A}(\mathbb{D})$ which posses a uniformly converging power series.
- Similar results for Hilbert transform approximations $\Rightarrow$ Causality!
- We gave a precise characterization of subspaces of $\mathcal{A}(\mathbb{D})$ with finite energy on which basis expansions are practically feasible.
- We derived a general axiomatic theory showing that all sampling based methods fail on these spaces.

- H. Boche, V. Pohl, "On the computability of system approximations under causality constraints," *IEEE ICASSP*, Calgary, Canada, April 2018.
- H. Boche, V. Pohl, "Investigations on the approximability and computability of the Hilbert transform with applications ," Jan. 2018, sub. for publ.