UNCERTAINTY PRINCIPLE FOR RATIONAL FUNCTIONS IN HARDY SPACES

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INTRODUCTION

- Uncertainty principles in finite dimensional vector space cannot be applied to sparse representation of rational functions.
- This paper considers the sparse representation for a rational function under a pair of orthonormal rational function **(ORF)** bases.
- Two main results: uncertainty principle concerning pairs of ORF bases & the uniqueness of compressible representation using such pairs.

MOTIVATION

The uncertainty principle has encountered many parallel evolutions and generalizations in different domains.

- Robertson-Schrödinger inequality.
- entropic uncertainty principle.
- Heisenberg's uncertainty principle.
- $\cdots$
- uncertainty principle in finite dimensional settings (Donoho [1], Elad [2]), connecting with sparse representation and compressed sensing, cannot work for the sparse representation and reconstruction of rational functions which have infinite impulse response.

CONTRIBUTION

1. The proposal of the definition of ε-sparsity for infinite sequence.
2. The establishment of uncertainty principle for rational functions in Hardy Space.
3. Uniqueness of rational functions with ε-sparse representation in pairs of orthonormal rational function bases.

PROBLEM DESCRIPTION

The **Hardy** space H_2 is a Hilbert space with the inner product between two rational functions $F(z)$ and $G(z)$ defined as

$$\langle F(z), G(z)\rangle = \frac{1}{2\pi i}\oint_{\mathbb{T}} F(z)\overline{G(z)}\frac{dz}{z},$$

where $\mathbb{T} = \{z| |z| = 1\}$.
Given a rational function $H(z) \in H_2$, can we get a much sparser representation of $H(z)$ in a joint, overcomplete set of ORF bases $\{\Phi(z), \Psi(z)\} = \{\phi_1(z), \phi_2(z), \cdots, \psi_1(z), \psi_2(z), \cdots\}$?

BASIC DEFINITION

For a fixed threshold $\varepsilon > 0$ and an infinite sequence $\alpha = [\alpha_1, \alpha_2, \cdots]^T$ in l_1, i.e. $\|\alpha\|_1 = \sum_{k=1}^{\infty}|\alpha_k| < \infty$. Let $N_\varepsilon(\alpha) = \min\{K : \sum_{k=K}^{\infty}|\alpha_k| \leq \varepsilon\}$. The ε-**support** of α is defined as

$$\Gamma_\varepsilon(\alpha) = \{k : |\alpha_k| \neq 0, 1 \leq k < N_\varepsilon(\alpha)\},$$

and the cardinality of $\Gamma_\varepsilon(\alpha)$ as the ε-**0 norm** of α, denoted by $\|\alpha\|_{0(\varepsilon)}$.

- The coefficient α is (ε, s)-sparse if $\|\alpha\|_{0(\varepsilon)} \leq s$ for a given s.
- A rational function is (ε, s)-sparse if the representation coefficient under an ORF basis is (ε, s)-sparse.

RESULTS 1: UNCERTAINTY PRINCIPLE

Let $H(z) \in H_2$ be a rational function that can be represented as

$$H(z) = \sum_{k=1}^{\infty}\alpha_k\phi_k(z) = \sum_{l=1}^{\infty}\beta_l\psi_l(z).$$

Denote $\alpha = [\alpha_1, \alpha_2, \cdots]^T$ and $\beta = [\beta_1, \beta_2, \cdots]^T$. For a fixed threshold ε, $\|\alpha\|_{0(\varepsilon)}$ and $\|\beta\|_{0(\varepsilon)}$ are the ε-0 norm of α and β, respectively, then for all such pairs of representation we have

$$\left(\sqrt{\|\alpha\|_{0(\varepsilon)}} + \varepsilon\right)^2 + \left(\sqrt{\|\beta\|_{0(\varepsilon)}} + \varepsilon\right)^2 \geq \frac{2}{\mu},$$

where

$$\mu = \sup_{k,l}|\langle\phi_k(z), \psi_l(z)\rangle|.$$

RESULT 2: UNIQUENESS

For a rational function $H(z)$ represented under a pair of orthonormal rational function bases $\{\phi_k(z)\}$ and $\{\psi_l(z)\}$ as

$$H(z) = \sum_{k=1}^{\infty}\theta_k^{\phi}\phi_k(z) + \sum_{l=1}^{\infty}\theta_l^{\psi}\psi_l(z), \tag{1}$$

denote $\theta_1 = [\theta_1^{\phi}, \theta_2^{\phi}, \cdots]^T$ and $\theta_2 = [\theta_1^{\psi}, \theta_2^{\psi}, \cdots]^T$, respectively. $H(z)$ is called (ε, s)-**sparse** if $\|\theta_1\|_{0(\varepsilon)} + \|\theta_2\|_{0(\varepsilon)} \leq s$.

For $\varepsilon > 0$, the representation (1) is unique if

$$\left(\sqrt{\|\theta_1\|_{0(\varepsilon)}} + \varepsilon\right)^2 + \left(\sqrt{\|\theta_2\|_{0(\varepsilon)}} + \varepsilon\right)^2 < \frac{1}{\mu}.$$

CONCLUSION

- A novel uncertainty principle for sparse representation of rational functions in infinite dimensional function space is presented.
- The bound which guarantees the uniqueness of the sparse representation is presented using mutual coherence as a measure.
- Since rational functions are widely used to model both signals and dynamic systems, the consequence in this paper can be applicable to sparse representation of signal and systems using pairs of ORF bases.

REFERENCES

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