# On the SNR Variability in Noisy Compressed Sensing


Anastasia Lavrenko*, Florian Römer‡, Giovanni Del Galdo* † and Reiner Thomä* †

*Institute for Information Technology, TU Ilmenau, Ilmenau

†Fraunhofer Institute for Integrated Circuits IIS, Ilmenau, Germany

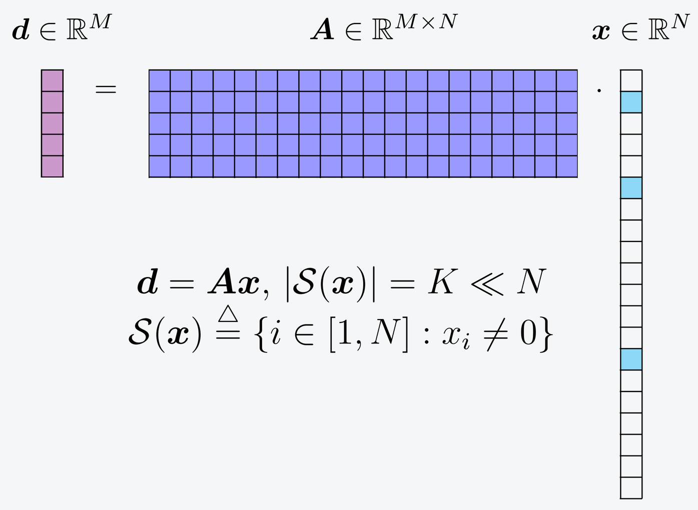
‡Fraunhofer Institute for Nondestructive Testing IZFP, Ilmenau, Germany


## Introduction

- Compressed Sensing (CS) aims at solving

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{n} = \boldsymbol{d} + \boldsymbol{n}, \text{ where} \quad (1)$$

  - $\boldsymbol{y}$ is a (known) $M \times 1$ vector of **measurements**
  - $\boldsymbol{x}$ is an (unknown) $N \times 1$ $K$-sparse **input** signal
  - $\boldsymbol{A}$ is an $M \times N$ the **sensing** matrix with $M < N$
  - $\boldsymbol{n}$ represents additive noise




**Problem**
The application of the sensing matrix affects the signal power and, in the presence of noise, the effective SNR

- Output signal power

$$\|\boldsymbol{d}\|_2^2 = \|\boldsymbol{A}\boldsymbol{x}\|_2^2 = \sum_{m=1}^{M}\left(\sum_{i\in\mathcal{S}(\boldsymbol{x})} a_{m,i}x_i\right)^2 \quad (2)$$

**depends on** the entries of $\boldsymbol{A}$ corresponding to the **support** of the input $\boldsymbol{x}$

- Noise power

In the CS setting, two types of additive noise can occur

  - signal or input noise $\boldsymbol{n}_{\mathrm{s}}$ $\rightarrow$ acts on the input signal $\boldsymbol{x}$
  - measurement noise $\boldsymbol{n}_{\mathrm{m}}$ $\rightarrow$ acts on the measured signal $\boldsymbol{d}$

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{n} = \boldsymbol{A}(\boldsymbol{x} + \boldsymbol{n}_{\mathrm{s}}) + \boldsymbol{n}_{\mathrm{m}} \rightarrow \boldsymbol{n} = \boldsymbol{A}\boldsymbol{n}_{\mathrm{s}} + \boldsymbol{n}_{\mathrm{m}} \quad (3)$$

Assuming the elements of $\boldsymbol{n}_{\mathrm{s}}$ and $\boldsymbol{n}_{\mathrm{m}}$ to be i.i.d. zero-mean normal variables with variance $\sigma_{\mathrm{s}}^2$ and $\sigma_{\mathrm{m}}^2$, respectively,

$$\mathbb{E}\{\|\boldsymbol{n}\|_2^2\} = \mathrm{trace}\{\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}\}\sigma_{\mathrm{s}}^2 + M\sigma_{\mathrm{m}}^2 = M\sigma_0^2 \quad (4)$$

  - $\sigma_0^2 = \frac{1}{M}\mathrm{trace}\{\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}\}\sigma_{\mathrm{s}}^2 + \sigma_{\mathrm{m}}^2$

## Output vs recovered SNR

- Output SNR (OSNR)

$$\eta_{\mathrm{O}} \triangleq \frac{\|\boldsymbol{A}\boldsymbol{x}\|_2^2}{\mathbb{E}\{\|\boldsymbol{n}\|_2^2\}} = \frac{\|\boldsymbol{A}\boldsymbol{x}\|_2^2}{\mathbb{E}\{\|\boldsymbol{A}\boldsymbol{n}_{\mathrm{s}} + \boldsymbol{n}_{\mathrm{m}}\|_2^2\}} = \frac{1}{M\sigma_0^2}\sum_{m=1}^{M}\left(\sum_{i\in\mathcal{S}(\boldsymbol{x})} a_{m,i}x_i\right)^2 \quad (5)$$

  - expresses the ratio between the total output signal power to the total noise power $\rightarrow$ **defines** the reconstruction **performance**

OSNR is a **function of** the non-zeros $x_i$ and **the support** $\mathcal{S}(\boldsymbol{x})$ via the corresponding values $a_{m,i}$ of the sensing matrix $\boldsymbol{A}$

**Difference with Nyquist-rate setting**
The effective output SNR might vary depending on the support of $\boldsymbol{x}$ $\rightarrow$ potentially non-uniform system performance!

- Recovered SNR (RSNR)

$$\eta_{\mathrm{R}} \triangleq \frac{\|\boldsymbol{x}\|_2^2}{\mathbb{E}\{\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_2^2\}}, \quad (6)$$

  - accounts for the ratio of the signal power to the power of the residual noise present after reconstruction $\rightarrow$ provides a measure of the reconstruction performance

RSNR for an **oracle-assisted** recovery evaluates best-case performance $\rightarrow$ a **benchmark** for any practical recovery method

$$\eta_{\mathrm{R}} = \frac{\|\boldsymbol{x}\|_2^2}{\mathbb{E}\left\{\|\boldsymbol{A}_{\mathcal{S}(x)}^{\dagger}\boldsymbol{y} - \boldsymbol{x}\|_2^2\right\}} = \frac{\|\boldsymbol{x}\|_2^2}{\mathbb{E}\left\{\|\boldsymbol{A}_{\mathcal{S}(x)}^{\dagger}\boldsymbol{n}\|_2^2\right\}} \quad (7)$$

  - $(\cdot)^{\dagger}$ denotes the matrix pseudo-inverse
  - $\boldsymbol{A}_{\mathcal{S}(x)}$ contains the columns of $\boldsymbol{A}$ indexed by $\mathcal{S}(\boldsymbol{x})$

- The ratio of RSNR to OSNR

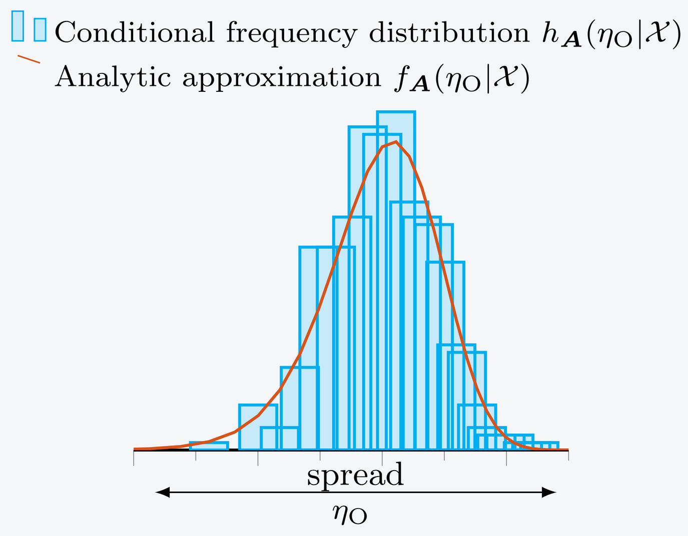
$$\left(\frac{1-\delta}{1+\delta}\right)\frac{M}{K} \leq \frac{\eta_{\mathrm{R}}}{\eta_{\mathrm{O}}} \leq \left(\frac{1+\delta}{1-\delta}\right)\frac{M}{K}, \ \delta \in (0,1) \text{ is the RIP constant} \quad (8)$$

**The variation of the OSNR over the signal support results in a corresponding variation of the bounds on RSNR**

## OSNR Spread Analysis

- For any given sensing matrix $\boldsymbol{A}$ and input $\mathcal{X} = \{x_{i_1}, \ldots, x_{i_K}\}$ one could evaluate the spread by
  - computing a conditional frequency distribution $h_{\boldsymbol{A}}(\eta_{\mathrm{O}}|\mathcal{X})$
  - repeating this procedure for different $\mathcal{X}$, i.e., combinations of signal magnitudes
  - averaging the results to obtain $h_{\boldsymbol{A}}(\eta_{\mathrm{O}})$

**Time consuming - requires combinatorial search!**




*CoV: coefficient of variation

Instead: when the **elements** of $\boldsymbol{A}$ are **drawn** from some **distribution** $f(\alpha)$ and $N$ is large

- model $a_{m,n}$ as i.i.d. random variables
- approximate $h_{\boldsymbol{A}}(\eta_{\mathrm{O}}|\mathcal{X})$ by the (analytic) distribution $f_{\boldsymbol{A}}(\eta_{\mathrm{O}}|\mathcal{X})$ and/or
- compute mean and variance of $(\eta_{\mathrm{O}})$
- evaluate the OSNR spread by computing

$$\mathrm{CoV}^*: \ c_{\mathrm{v}}(\eta_{\mathrm{O}}) = \sqrt{\frac{\mathrm{var}\{\eta_{\mathrm{O}}\}}{(\mathbb{E}\{\eta_{\mathrm{O}}\})^2}}$$

### Generic approach

- When $a_{m,n}$ are independently drawn from some probability distribution $f(\alpha)$

$$\eta_{\mathrm{O}} = \frac{1}{M\sigma_0^2}\sum_{m=1}^{M}\Big(\underbrace{\sum_{i\in\mathcal{S}(x)} a_{m,i}x_i}_{d_m}\Big)^2 = \frac{1}{M\sigma_0^2}\sum_{m=1}^{M} d_m^2$$

is a **random variable** whose distribution is defined by that of $\sum_{m=1}^{M} d_m^2$

- Given large enough $N$ (each row of $\boldsymbol{A}$ is a large enough sample of $f(\alpha)$) we
  - approximate $h_{\boldsymbol{A}}(\eta_{\mathrm{O}}|\mathcal{X})$ assuming that $a_{m,n} \sim f(\alpha)$
  - compute mean $\mathbb{E}\{\eta_{\mathrm{O}}\} = \mathbb{E}_{\mathcal{X}}\{\mathbb{E}_{\boldsymbol{A}}\{\eta_{\mathrm{O}}|\mathcal{X}\}\}$ and variance $\mathrm{var}\{\eta_{\mathrm{O}}\} = \mathbb{E}\{\eta_{\mathrm{O}}^2\} - \mathbb{E}^2\{\eta_{\mathrm{O}}\}$

### Examples: Gaussian, Bernoulli and Rademacher $A$

Gaussian sensing matrix

- Suppose $a_{m,n} \sim \mathcal{N}(0, 1/M)$, then
  - $d_m \sim \mathcal{N}(0, \frac{\|\boldsymbol{x}\|_2^2}{M})$ and $\frac{M}{\|\boldsymbol{x}\|_2^2}\sum_{m=1}^{M} d_m^2 \sim \chi_M^2$

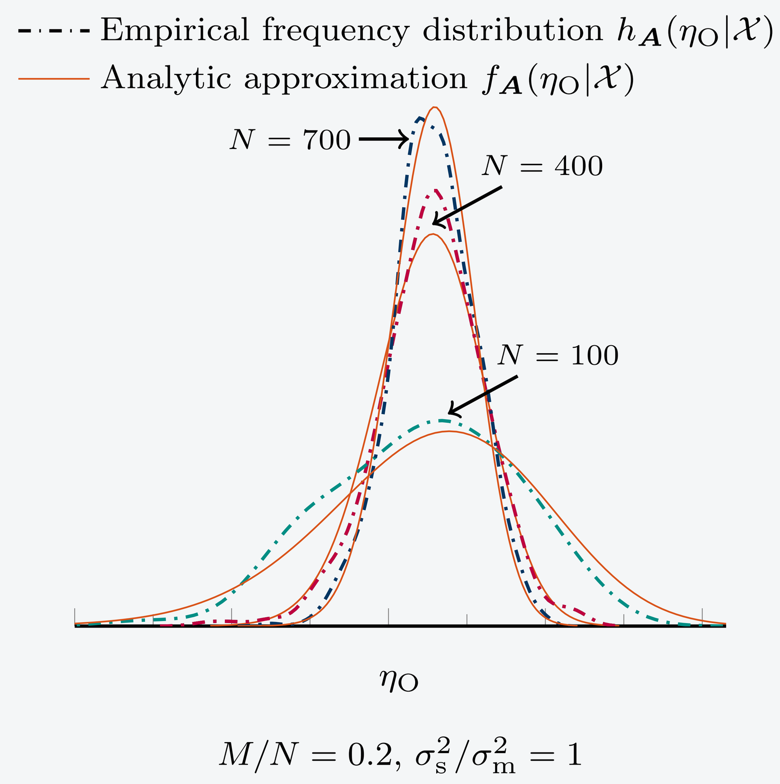
$$\rightarrow \quad f_{\boldsymbol{A}}(\eta_{\mathrm{O}}|\mathcal{X}) = \Gamma\left(\frac{M}{2}, \frac{2\|\boldsymbol{x}\|_2^2}{M^2\sigma_0^2}\right)$$

- We can also compute
  - Mean of $\eta_{\mathrm{O}}$

$$\mathbb{E}\{\eta_{\mathrm{O}}\} = \mathbb{E}_{\mathcal{X}}\left\{\frac{\|\boldsymbol{x}\|_2^2}{M\sigma_0^2}\right\} = \frac{\mathbb{E}\{\|\boldsymbol{x}\|^2\}}{M\sigma_0^2} = \vartheta$$

  - Variance of $\eta_{\mathrm{O}}$

$$\mathrm{var}\{\eta_{\mathrm{O}}\} = \left(1 + \frac{2}{M}\right)\vartheta^2 - \vartheta^2 = \frac{2}{M}\vartheta^2.$$




- CoV is given by $c_{\mathrm{v}}(\eta_{\mathrm{O}}) = \sqrt{2\vartheta^2/M\vartheta^2} = \sqrt{2/M}$ - depends only on $M$

Exact derivations for equal signal magnitudes, i.e., $\forall i \in \mathcal{S}(\boldsymbol{x})\ x_i = c$

Bernoulli sensing matrix

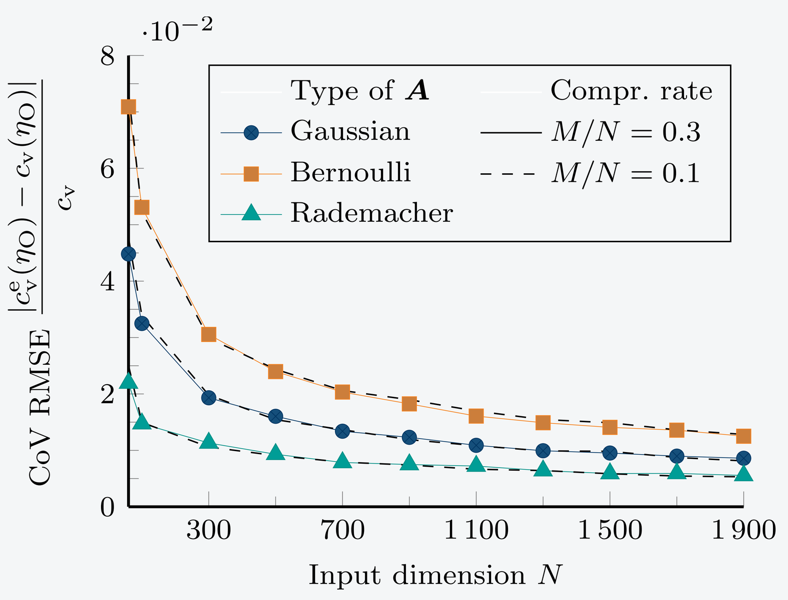
$$c_{\mathrm{v}}(\eta_{\mathrm{O}}) = \sqrt{\frac{1}{M}\frac{1 + 2p(K-1)\left((2K-3)p + 3\right)}{K\left((K-1)p+1\right)^2 p}(1-p)}$$

Rademacher sensing matrix

$$c_{\mathrm{v}}(\eta_{\mathrm{O}}) = \sqrt{\frac{1}{M}\frac{2(K-1)}{K}}$$

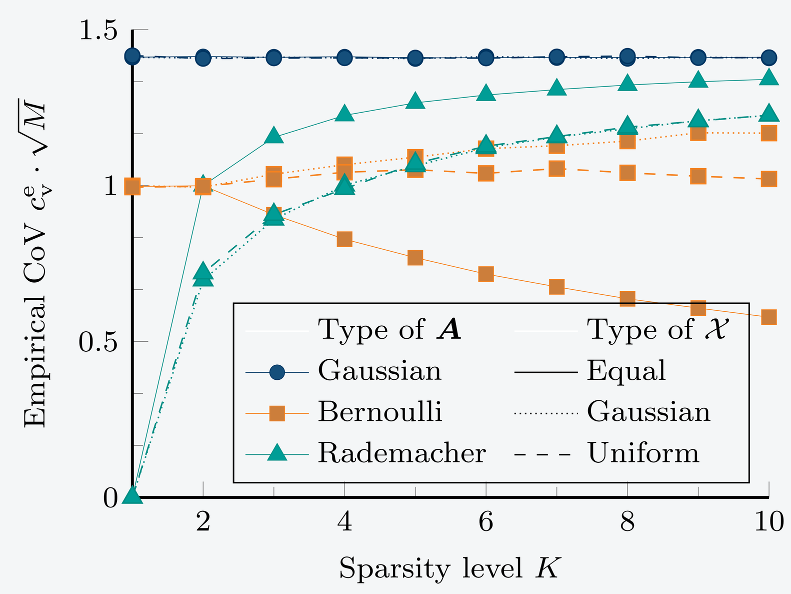
## Numerical validation

**RMSE: numerical vs analytic CoV**




- Gaussian, Bernoulli and Rademacher $\boldsymbol{A}$
- Equal input entries: $\forall i \in \mathcal{S}(\boldsymbol{x})\ x_i = c$

**CoV vs choice of $\mathcal{X}$**




- Gaussian: CoV independent of $K$, $\mathcal{X}$
- Bernoulli and Rad.: behave differently

**The SNR spread in CS can be significant!**




Technische Universität Ilmenau
Institute for Information Technology
Contact: anastasia.lavrenko@tu-ilmenau.de