Importance Sampling Estimator of Outage Probability under Generalized Selection Combining Model

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Abstract

We consider the problem of evaluating outage probability (OP) values of generalized selection combining diversity receivers over fading channels. This is equivalent to computing the cumulative distribution function (CDF) of the sum of order statistics. Generally, closed-form expressions of the CDF of order statistics are unavailable for many practical distributions. Moreover, the naive Monte Carlo method requires a substantial computational effort when the probability of interest is sufficiently small. In the region of small OP values, we propose instead an efficient, yet universal, importance sampling (IS) estimator that yields a reliable estimate of the CDF with small computing cost. The main feature of the proposed IS estimator is that it has bounded relative error under a certain assumption that is shown to hold for most of the challenging distributions. Moreover, an improvement of this estimator is proposed for the Pareto and the Weibull cases. Finally, the efficiency of the proposed estimators are investigated through numerical experiments.

Motivation

- Order statistics play an important role in the performance analysis of wireless communication systems over fading channels.
  - The signal-to-noise-ratio (SNR) is expressed as the partial sum of ordered channel gains in a Generalized selection combining (GSC) model combined with maximum ratio combining (MRC) diversity technique
  - Sums of order statistics is encountered when GSC is combined with equal gain combining (EGC) diversity technique.

  Objective: Evaluate the CDF of the sum of ordered random variables (RVs) .
- Closed-form expressions of the CDF of partial sums of order RVs exist only for the exponential and Gamma RVs.
- Out of reach for many challenging distributions and still constitute open problems: Log-normal and Weibull variates.
- Naive Monte Carlo (MC) method is a good alternative to estimate the CDF of partial sums of ordered RVs.
- It requires a substantial amount of samples to yield an accurate estimate of the left-tail of the CDF.

Solution IS yields a very precise estimate of the CDF with small computing cost.

Problem Setting

- Consider a sequence of i.i.d RVs $X_1, X_2, \cdots, X_N$ with common probability density function (PDF) $f(\cdot)$.
- Propose efficient MC methods to evaluate the quantity

$$\ell = P\left(\sum_{k=1}^{L} X^{(k)} \leq \gamma_{th}\right), \tag{1}$$

$X^{(k)}$ is the k^{th} order statistic such that $X^{(1)} \geq X^{(2)} \geq \cdots \geq X^{(N)}$, and L is an integer satisfying $1 \leq L \leq N$.
- For small values of ℓ, IS techniques can deliver a reliable estimate with fewer number of runs compared to naive MC.
- Let $\hat{\ell}$ be an estimator of ℓ with $\mathbb{E}[\hat{\ell}] = \ell$, we say that $\hat{\ell}$ has bounded relative error when

$$\limsup_{\gamma_{th}\to 0} \frac{\mathrm{var}\left[\hat{\ell}\right]}{\ell^2} < \infty. \tag{2}$$

$\rightarrow$ the number of samples needed to achieve a given accuracy remains bounded regardless of how small ℓ is.

Importance Sampling Estimator

- Let $\mathbf{X} = (X_1, \cdots, X_N)'$ and $S = \{\boldsymbol{x} = (x_1, \cdots, x_N)' : \sum_{k=1}^{L} x^{(k)} \leq \gamma_{th}\}$.
- Consider another set S_1 that includes S with the assumption that $P(\mathbf{X} \in S_1)$ is known in closed form.
- The probability ℓ is re-written as

$$\ell = P(\mathbf{X} \in S) = P(\mathbf{X} \in S_1)\, P(\mathbf{X} \in S | \mathbf{X} \in S_1). \tag{3}$$

- ℓ is the product of a known approximate term $\ell_1 = P(\mathbf{X} \in S_1)$ and a non-rare event probability $P(\mathbf{X} \in S | \mathbf{X} \in S_1)$ that can be efficiently estimated through naive MC simulations.
- Alternatively, we write ℓ as

$$\ell = \mathbb{E}_g\left[\ell_1 \mathbf{1}_{(\mathbf{X}\in S)}\right] \triangleq \mathbb{E}_g\left[\hat{\ell}_{IS}\right], \tag{4}$$

where with $g(\cdot)$ is the PDF under which $\mathbf{X}$ is distributed according to its original PDF truncated over S_1,

- $\hat{\ell}_{IS}$ is an importance sampling estimator with biased PDF $g(\cdot)$.
- The variance of $\hat{\ell}_{IS}$ is given by

$$\mathrm{var}_g\left[\hat{\ell}_{IS}\right] = \ell_1 \ell - \ell^2. \tag{5}$$

- The closer ℓ_1 to ℓ, the smaller the variance of $\hat{\ell}_{IS}$ is, and hence the more efficient is the estimator $\hat{\ell}_{IS}$.

Universal IS Estimator

- The simplest choice of the set S_1 is

$$S_1 = \{\boldsymbol{x} = (x_1, \cdots, x_N)' : x^{(1)} \leq \gamma_{th}\}. \tag{6}$$

- The probability ℓ_1 is therefore given by

$$\ell_1 = \left(P\left(X_1 \leq \gamma_{th}\right)\right)^N \tag{7}$$

Proposition 1 *Assume* $P(X_1 < \gamma_{th}) / P(X_1 \leq \gamma_{th}/L) = \mathcal{O}(1)$ *as* $\gamma_{th} \to 0$, *we have*

$$\limsup_{\gamma_{th}\to 0} \frac{\ell_1}{\ell} < \infty \tag{8}$$

Hence, the bounded relative error property holds.

- The assumption holds for many challenging distributions: the Generalized Gamma (which includes the Gamma and the Weibull distributions), and the $\kappa - \mu$ distributions.
- Despite its general scope of applicability, the efficiency can be further improved if we settle for a particular distribution.

Pareto Case

- The PDF $f(\cdot)$ of X_i, $i = 1, \cdots, N$, is given as

$$f(x) = \alpha\left(1 + x\right)^{-(1+\alpha)},\ x \geq 0, \tag{9}$$

with $\alpha > 0$.
- $Y_i = \alpha \log\left(1 + X_i\right)$, $i = 1, \cdots, N$, are exponentially distribution with mean 1.

$$\ell = P\left(\sum_{k=1}^{L} \exp\left(Y^{(k)}/\alpha\right) \leq \gamma_{th} + L\right). \tag{10}$$

- Let $\lambda_i > 0$, $i = 1, 2, \cdots, L$, such that $\sum_{i=1}^{L} \lambda_i = 1$, S_1 is selected as

$$S_1 = \left\{\boldsymbol{y} : \sum_{k=1}^{L} \lambda_k y^{(k)} \leq \gamma_1 = \alpha(\log(\gamma_{th} + L) + \sum_{k=1}^{L} \lambda_k \log(\lambda_k))\right\}. \tag{11}$$

and

$$\ell_1 = P\left(\sum_{i=1}^{N} \beta_i Z_i \leq \gamma_1\right), \tag{12}$$

where $Z_1, \cdots, Z_N$ are i.i.d exponential RVs with mean 1 and

$$\beta_i = \begin{cases} \sum_{j=1}^{L} \lambda_j/(N-i+1) & i = 1 = 1, \cdots, N-L+1, \\ \sum_{j=1}^{N+1-i} \lambda_j/(N-i+1) & i = N-L+2, \cdots, N. \end{cases} \tag{13}$$

- A closed-form expression of ℓ_1 is

$$\ell_1 = 1 - (1, 0, \cdots, 0)\exp\left(\gamma_1 \mathbf{A}\right)(1, 1, \cdots, 1)', \tag{14}$$

with $\exp\left(\gamma_1 \mathbf{A}\right)$ being the matrix exponential of $\gamma_1 \mathbf{A}$ and

$$\mathbf{A} = \begin{pmatrix} -1/\beta_1 & 1/\beta_1 & 0 & \cdots & 0 \\ 0 & -1/\beta_2 & 1/\beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1/\beta_{N-1} & 1/\beta_{N-1} \\ 0 & \cdots & 0 & 0 & -1/\beta_N \end{pmatrix} \tag{15}$$

Proposition 2 *Let* $\lambda_k = 1/L$ *for all* $k \in \{1, \cdots, L\}$. *Then, we have*

$$\limsup_{\gamma_{th}\to 0} \frac{\ell_1}{\ell} < \infty. \tag{16}$$

Thus, the bounded relative error property holds.

Weibull Case

- The PDF of $X_1, \cdots, X_N$ is given as

$$f(x) = \frac{\alpha}{\eta}\left(\frac{x}{\eta}\right)^{\alpha-1} \exp\left(-\left(\frac{x}{\eta}\right)^{\alpha}\right),\ x > 0, \tag{17}$$

where η is the scale parameter, α is the shape parameter which is assumed $0 < \alpha < 1$.

- $Y_i = (X_i/\eta)^{\alpha}$, $i = 1, \cdots, N$, are i.i.d exponential with mean 1

$$\ell = P\left(\sum_{k=1}^{L}\left(Y^{(k)}\right)^{1/\alpha} \leq \gamma_{th}/\eta\right). \tag{18}$$

- Let $\lambda_i > 0$, $i = 1, \cdots, L$, such that $\sum_{i=1}^{L} \lambda_i = 1$, S_1 is selected as

$$S_1 = \left\{\mathbf{y} : \sum_{k=1}^{L} \lambda_k^{1-\alpha} Y^{(k)} \leq \left(\gamma_{th}/\eta\right)^{\alpha}\right\}. \tag{19}$$

and

$$\ell_1 = P\left(\sum_{i=1}^{N} \nu_i Z_i \leq \left(\gamma_{th}/\eta\right)^{\alpha}\right), \tag{20}$$

with

$$\nu_i = \begin{cases} \sum_{j=1}^{L} \lambda_j^{1-\alpha}/(N-i+1) & i = 1 = 1, \cdots, N-L+1, \\ \sum_{j=1}^{N+1-i} \lambda_j^{1-\alpha}/(N-i+1) & i = N-L+2, \cdots, N. \end{cases} \tag{21}$$

Proposition 3 *For* $0 < \alpha < 1$ *and arbitrary values of* λ_k, $k = 1, \cdots, L$, *we have*

$$\limsup_{\gamma_{th}\to 0} \frac{\ell_1}{\ell} < \infty. \tag{22}$$

Hence, the bounded relative error property holds.

Numerical Results

The relative error of an estimator $\hat{\ell}$ is defined as

$$RE(\hat{\ell}) = \frac{\sqrt{\mathrm{var}\left[\hat{\ell}\right]}}{\ell\sqrt{M}}. \tag{23}$$

Table 1: *CDF of the sum of order statistics for Pareto Case with* $N = 8$, $L = 4$, $\alpha = 1$ *and* $M = 5 \times 10^5$.

	IS estimator		Universal IS estimator	
γ_{th}	$\hat{\ell}_{IS}$	$RE(\hat{\ell}_{IS})\%$	$\hat{\ell}_{IS,u}$	$RE(\hat{\ell}_{IS,u})\%$
1.5	2.21×10^{-4}	6.06×10^{-2}	2.19×10^{-4}	1.23
1	2.06×10^{-5}	5.18×10^{-2}	2.11×10^{-5}	1.92
0.5	2.13×10^{-7}	3.85×10^{-2}	2.09×10^{-7}	3.82
0.1	1.29×10^{-12}	1.79×10^{-2}	1.29×10^{-12}	8.51

Table 2: *CDF of the sum of order statistics for Weibull Case with* $N = 8$, $L = 4$, $\alpha = 0.5$, $\eta = 1$ *and* $M = 5 \times 10^5$.

	IS estimator		Universal IS estimator	
γ_{th}	$\hat{\ell}_{IS}$	$RE(\hat{\ell}_{IS})\%$	$\hat{\ell}_{IS,u}$	$RE(\hat{\ell}_{IS,u})\%$
1	0.0029	9.96×10^{-2}	0.0029	0.4
0.5	3.37×10^{-4}	0.1	3.37×10^{-4}	0.49
0.1	1.27×10^{-6}	0.11	1.27×10^{-6}	0.66
0.05	9.79×10^{-8}	0.11	9.85×10^{-8}	0.71
0.01	2.06×10^{-10}	0.11	2.06×10^{-10}	0.8
0.005	1.38×10^{-11}	0.11	1.39×10^{-11}	0.81

Table 3: *DF of the sum of order statistics for Weibull Case with* $N = 8$, $L = 2$, $\alpha = 0.5$, $\eta = 1$ *and* $M = 5 \times 10^5$.

	IS estimator		Universal IS estimator	
γ_{th}	$\hat{\ell}_{IS}$	$RE(\hat{\ell}_{IS})\%$	$\hat{\ell}_{IS,u}$	$RE(\hat{\ell}_{IS,u})\%$
0.355	3.38×10^{-4}	4.37×10^{-2}	3.37×10^{-4}	0.28
0.07	1.28×10^{-6}	4.41×10^{-2}	1.28×10^{-6}	0.34
0.0069	2.03×10^{-10}	4.42×10^{-2}	2.04×10^{-10}	0.37
0.0035	1.44×10^{-11}	4.42×10^{-2}	1.45×10^{-11}	0.38

References

[1] N. Ben Rached, Z. Botev, A. Kammoun, M.-S. Alouini, and R. Tempone, "On the Sum of Order Statistics and Applications to Wireless Communication Systems Performances," Submitted to IEEE Transactions on Wireless Communications, Nov. 2017.