GENERALIZED UNCERTAINTY PRINCIPLES FOR THE TWO-SIDED QUATERNION LINEAR CANONICAL TRANSFORM

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I. Introduction

An uncertainty principle (UP), which provides a lower bound on the spreads of two specific transform domains, is of importance in various scientific fields such as mathematics, signal processing and information theory. On the mathematical side, the classical Heisenberg UP in the time-frequency plane is given by

$$\frac{\int_{\mathbb{R}}(t-t_0)^2\left|f(t)\right|^2 dt}{\int_{\mathbb{R}}\left|f(t)\right|^2 dt}\cdot\frac{\int_{\mathbb{R}}(w-w_0)^2\left|F(w)\right|^2 dw}{\int_{\mathbb{R}}\left|F(w)\right|^2 dw}\ge\frac{1}{4}$$

- t_0, w_0: mean time and mean frequency; $E=\int_{\mathbb{R}}|f(t)|^2dt$: the energy of signal;
- UP means a function and its FT cannot both be higher concentrated.

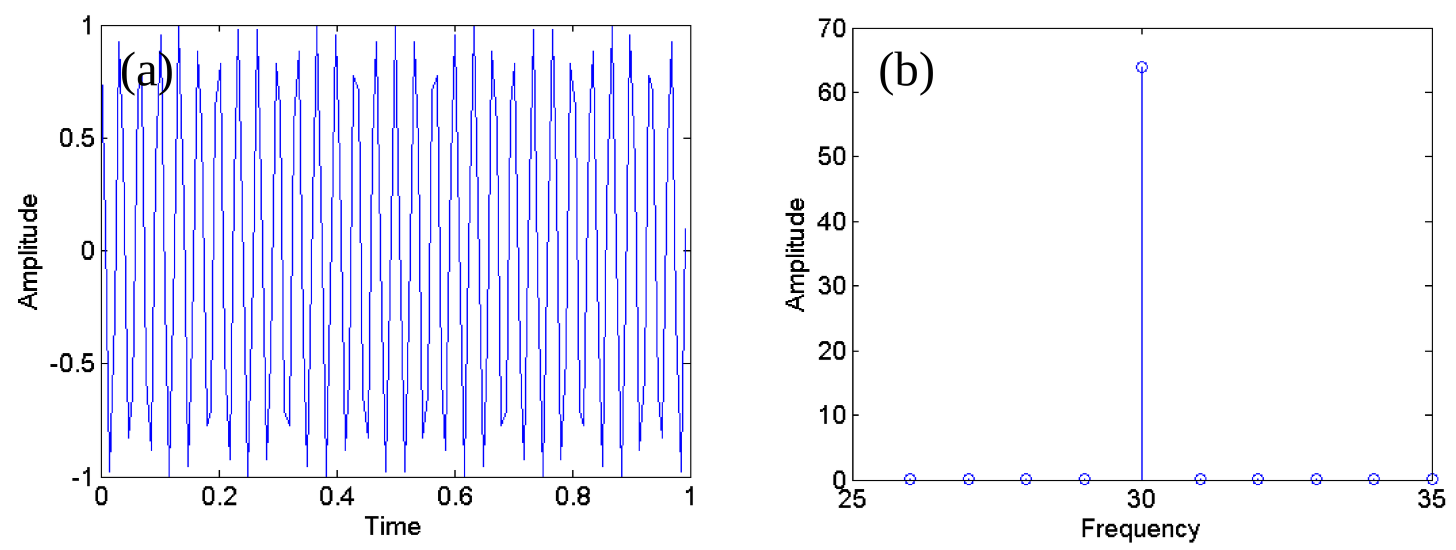

Fig. 1 (a) cosine signal with 30Hz; (b) signal after Fourier transform (FT).

We mainly investigate the mathematical aspects of UPs for the two-sided quaternion linear canonical transform (QLCT) in this paper.

II. Linear Canonical Transform (LCT)

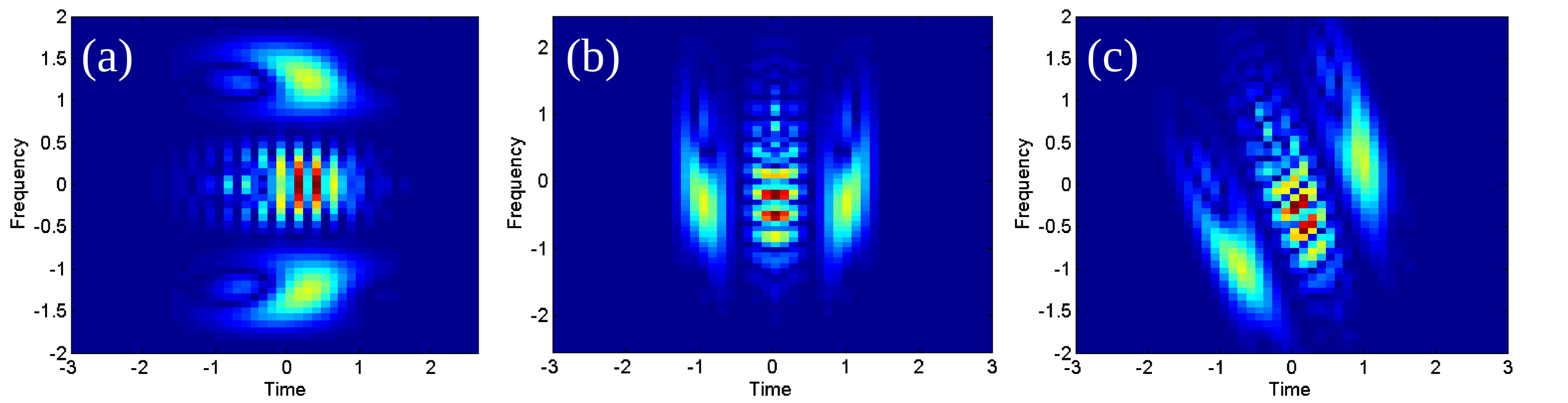

Fig. 2 Time-frequency distributions of (a) Gaussian-weighted sinusoidal signal; (b) signal after the FT; (c) signal after the LCT. In the time-frequency plane, it is noted that FT means a 90 rotation and the LCT is an affine transformation.

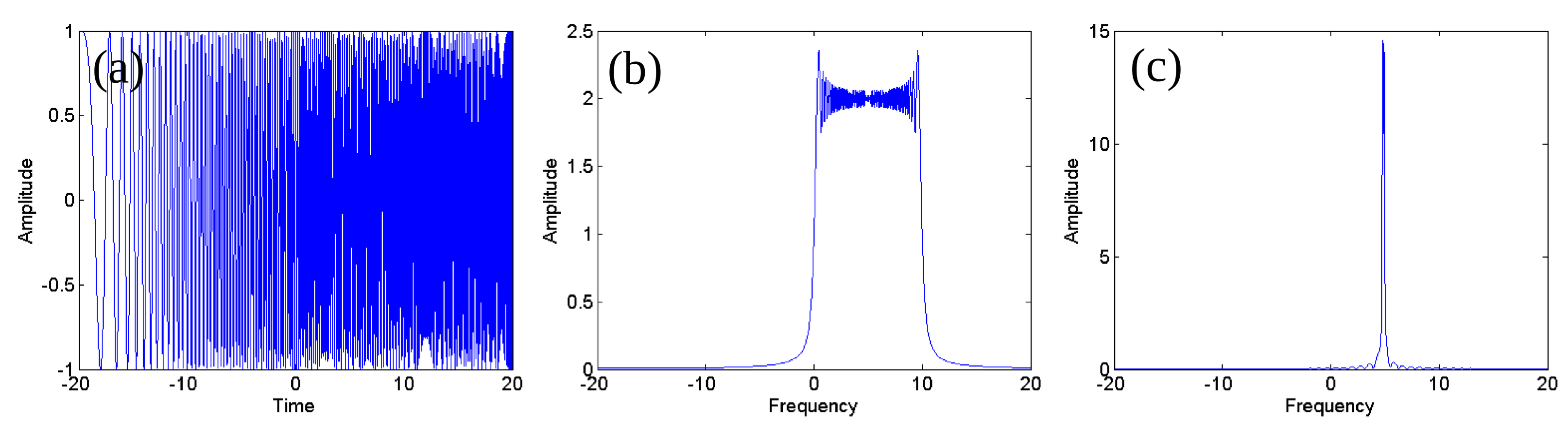

Fig. 3 (a) waveform of a chirp signal; (b) signal after the FT; (c) signal after the LCT.

- UP related to the LCT:

$$\int_{\mathbb{R}}u^2\left|F_{A_1}(u)\right|^2du\cdot\int_{\mathbb{R}}v^2\left|F_{A_2}(v)\right|^2dv\ge\frac{(a_1b_2-a_2b_1)^2}{4}$$

- Definition of the LCT:

$$F_A(u)=\left\langle f(t),K_A^*(u,t)\right\rangle=\frac{1}{\sqrt{j2\pi b}}\int_{\mathbb{R}}f(t)e^{j\frac{1}{2b}\left(du^2-2ut+at^2\right)}dt, b\neq 0, \det(A)=1$$

III. Quaternion Linear Canonical Transform (QLCT)

- Quaternion algebra:

$$\mathbb{H}=\left\{q\,|\,q=q_0+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}, q_0,q_1,q_2,q_3\in\mathbb{R}\right\}$$

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-1, \mathbf{ij}=-\mathbf{ji}=\mathbf{k}, \mathbf{jk}=-\mathbf{kj}=\mathbf{i}, \mathbf{ki}=-\mathbf{ik}=\mathbf{j}$$

Remark:

The main difference between common complex and quaternion number is that the latter has the non-commutative property of quaternion multiplication.

- Quaternion linear canonical transform:

$$L_{A_1,A_2}^{\mathbb{H}}\{f\}(\mathbf{u})=\int_{\mathbb{R}^2}K_{A_1}^{\mathbf{i}}(u_1,x_1)f(\mathbf{x})K_{A_2}^{\mathbf{j}}(u_2,x_2)d\mathbf{x} \quad \text{(Two-sided QLCT)}$$

$$L_{A_1,A_2}^{\mathbb{H}}\{f\}(\mathbf{u})=\int_{\mathbb{R}^2}f(\mathbf{x})K_{A_1}^{\mathbf{i}}(u_1,x_1)K_{A_2}^{\mathbf{j}}(u_2,x_2)d\mathbf{x} \quad \text{(Right-sided QLCT)}$$

$$L_{A_1,A_2}^{\mathbb{H}}\{f\}(\mathbf{u})=\int_{\mathbb{R}^2}K_{A_1}^{\mathbf{i}}(u_1,x_1)K_{A_2}^{\mathbf{j}}(u_2,x_2)f(\mathbf{x})d\mathbf{x} \quad \text{(Left-sided QLCT)}$$

- **Lemma 1** (*component-wise UP of the QFT*)

$$\int_{\mathbb{R}^2}x_k^2\left|f(\mathbf{x})\right|^2d\mathbf{x}\cdot\int_{\mathbb{R}^2}u_k^2\left|F\{f\}(\mathbf{u})\right|^2d\mathbf{u}\ge\frac{1}{4}\left(\int_{\mathbb{R}^2}\left|f(\mathbf{x})\right|^2d\mathbf{x}\right)^2, k=1,2$$

IV. Main Results

- **Theorem 1** (*component-wise UP of the QLCT*)

$$\int_{\mathbb{R}^2}u_1^2\left|L_{A_1,A_2}^{\mathbb{H}}\{f\}(\mathbf{u})\right|^2d\mathbf{u}\cdot\int_{\mathbb{R}^2}v_1^2\left|L_{A_3,A_4}^{\mathbb{H}}\{f\}(\mathbf{v})\right|^2d\mathbf{v}\ge\frac{(a_3b_1-a_1b_3)^2}{4}$$

$$\int_{\mathbb{R}^2}u_2^2\left|L_{A_1,A_2}^{\mathbb{H}}\{f\}(\mathbf{u})\right|^2d\mathbf{u}\cdot\int_{\mathbb{R}^2}v_2^2\left|L_{A_3,A_4}^{\mathbb{H}}\{f\}(\mathbf{v})\right|^2d\mathbf{v}\ge\frac{(a_4b_2-a_4b_2)^2}{4}$$

- **Theorem 2** (*directional UP of the QLCT*)

$$\int_{\mathbb{R}^2}\mathbf{u}^2\left|L_{A_1,A_2}^{\mathbb{H}}\{f\}(\mathbf{u})\right|^2d\mathbf{u}\cdot\int_{\mathbb{R}^2}\mathbf{v}^2\left|L_{A_3,A_4}^{\mathbb{H}}\{f\}(\mathbf{v})\right|^2d\mathbf{v}\ge\frac{\left(|a_3b_1-a_1b_3|+|a_4b_2-a_4b_2|\right)^2}{4}$$

- **Corollary 1** (*directional UP of the QLCT*)

$$\int_{\mathbb{R}^2}\mathbf{x}^2\left|f(\mathbf{x})\right|^2d\mathbf{x}\cdot\int_{\mathbb{R}^2}\mathbf{u}^2\left|L_{A_3,A_4}^{\mathbb{H}}\{f\}(\mathbf{u})\right|^2d\mathbf{u}\ge\frac{\left(|b_1|+|b_2|\right)^2}{4}$$

Remark:

- The main method to derive the **Theorem 1** and **Theorem 2** is based on the relationship between the QFT and the QLCT.
- When parameter matrices of the QLCT reduce to some special cases, we can derive corresponding forms of UPs in the QFT and the QFRFT domain.

V. Potential Applications

In this section, we mainly set an example to explain the potential applications of the UP in the QLCT domain. It is noted that the QLCT of the quaternionic Gaussian function is another Gaussian quaternionic function.

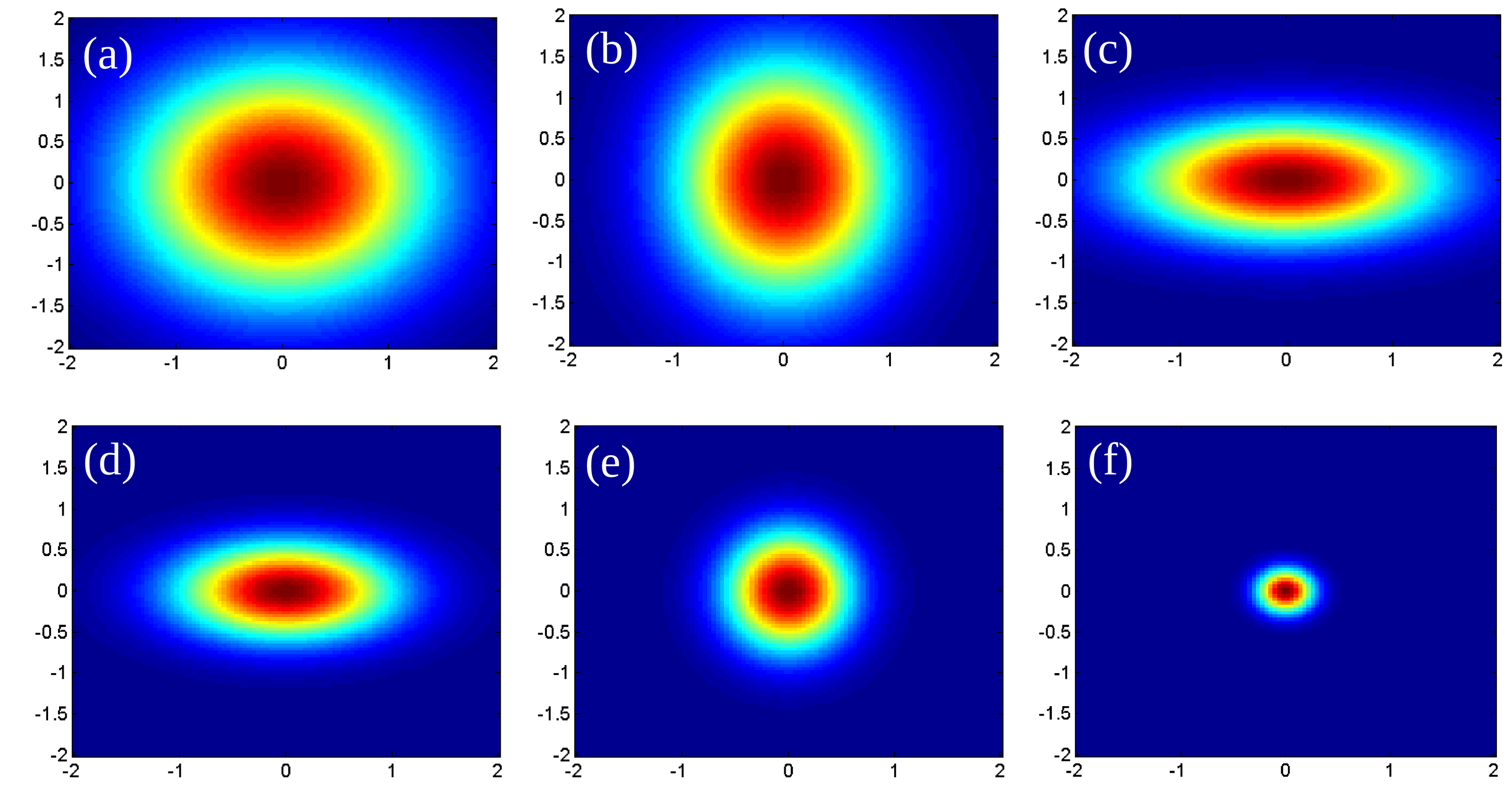

Fig. 4 2-D Gaussian signal in the QLCT domain under different parameter matrices. According to the flexiblity of the QLCT, it can provide much more information of quaternionic signal, and the UP can achieve tighter results than that of the QFT.

Contributions

- Two novel UPs of the two-sided QLCT are presented. One is a general form of the component-wise UP and the other is a generalization of the directional UP.
- Both of new derived results describe lower bounds of spreads of a quaternion-valued signal in arbitrary two different QLCT domains, which include those of UPs in the spatial and frequency domain as its special cases.
- The potential applications of the derived results mainly exist in signal energy concentration, time-frequency analysis and signal recovery in the quaternion setting.

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