Parameter Estimation of Heavy-tailed Random Walk Model from Incomplete Data

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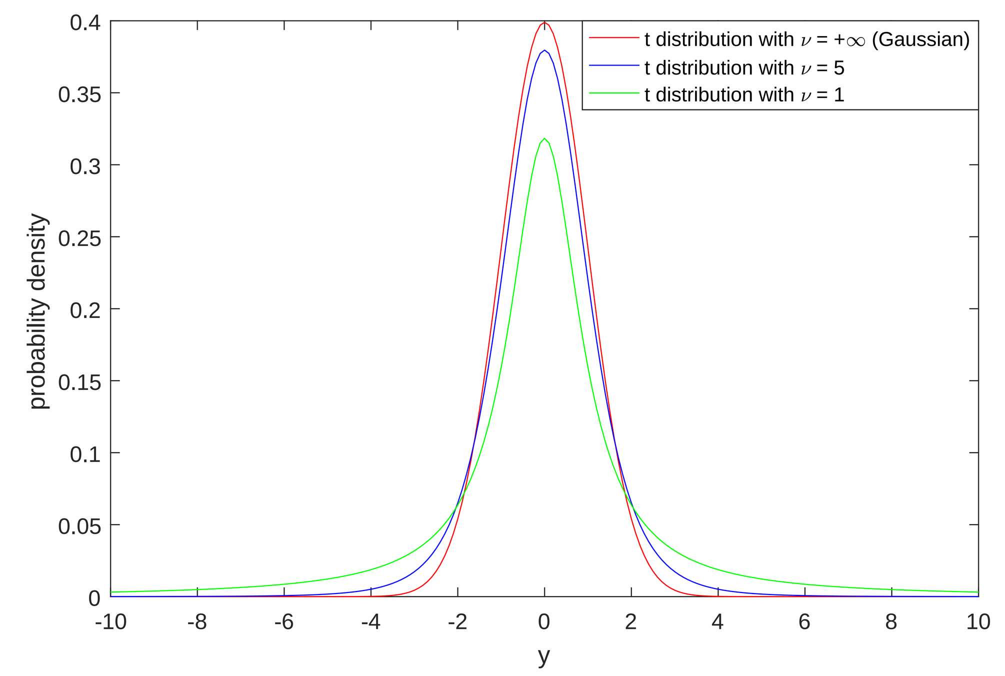
Background

- In the recent era of data deluge, many applications collect and process large amount of time series data for inference, learning, parameter estimation and decision making.
- Missing values frequently occur in the data recording process, e.g., some stocks may suffer a lack of liquidity resulting in no price recorded, observation devices like sensors breakdown, and weather or other conditions disturb sample taking schemes.
- Traditionally, the parameter estimation for time series from incomplete data has been considered under Gaussian noise. However, many real-world data follow heavy-tailed distributions, e.g., financial time series, brain fMRI, and animals movement.

Heavy-tailed Random Walk Model

- Student's t-distribution: $f_t\left(y;\mu,\sigma^2,\nu\right)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\sigma\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{(y-\mu)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}$.



- A univariate time series $Y_1, Y_2, \ldots, Y_T$ that follows a Student's t random walk model:

$$Y_t - Y_{t-1} \overset{i.i.d.}{\sim} t\left(\mu,\sigma^2,\nu\right).$$

Problem Formulation

- An observation of this time series with D missing blocks:

$$y_1,\ldots,y_{t_1},\text{NA},\ldots,\text{NA},y_{t_1+n_1+1},\ldots y_{t_d},\text{NA},\ldots,\text{NA},$$
$$y_{t_d+n_d+1},\ldots,y_{t_D},\text{NA},\ldots,\text{NA},y_{t_D+n_D+1},\ldots,y_T.$$

- Maximum likelihood estimation (MLE) problem for μ, σ^2, and ν:

$$\underset{\mu,\sigma^2,\nu>0}{\text{maximize}}\ \ l^{obs}\left(\{y_t\}_{t\in C_{obs}}|\mu,\sigma^2,\nu\right), \quad (1)$$

where

$$l^{obs}\left(\{y_t\}_{t\in C_{obs}}|\mu,\sigma^2,\nu\right)$$
$$=\log\left(\prod_{d=0}^{D}\prod_{t=t_d+n_d+2}^{t_{d+1}} f_t\left(y_t;\mu+y_{t-1},\sigma^2,\nu\right)\right)$$
$$+\log\left(\prod_{d=1}^{D}\int\!\!\cdots\!\!\int\prod_{t=t_d+1}^{t_d+n_d+1} f_t\left(y_t;\mu+y_{t-1},\sigma^2,\nu\right)\mathrm{d}y_{t_d+1}\cdots\mathrm{d}y_{t_d+n_d}\right).$$

- The objective function involves multiple integrals, and has no closed-form expression. It is difficult to optimize this problem directly.

Expectation-Maximization Algorithm

- A general MLE problems with missing data $\mathbf{Z}$, observed data $\mathbf{X}$, parameter $\boldsymbol{\theta}$:

$$\max_{\boldsymbol{\theta}}\ \log p(\mathbf{X}|\boldsymbol{\theta}).$$

- EM algorithm is a very popular iterative algorithm to solve this kind of problem.
  - E step: $Q\left(\boldsymbol{\theta}|\boldsymbol{\theta}^{(k)}\right)=\mathrm{E}_{\mathbf{Z}|\mathbf{X},\boldsymbol{\theta}^{(k)}}\log p\left(\mathbf{X},\mathbf{Z}|\boldsymbol{\theta}\right)$.
  - M step: $\boldsymbol{\theta}^{(k+1)}=\arg\max_{\boldsymbol{\theta}} Q\left(\boldsymbol{\theta}|\boldsymbol{\theta}^{(k)}\right)$.

Stochastic Expectation-Maximization Algorithm

- Stochastic EM (when E step is intractable)
  - Simulation step (S step): draw a realization $\mathbf{Z}^{(k)}\sim\mathbf{Z}|\mathbf{X},\boldsymbol{\theta}^{(k)}$.
  - M step: $\boldsymbol{\theta}_1^{(k+1)}=\arg\max_{\boldsymbol{\theta}}\log p\left(\mathbf{X},\mathbf{Z}^{(k)}|\boldsymbol{\theta}\right)$, and $\boldsymbol{\theta}^{(k+1)}=\left(1-\gamma^{(k)}\right)\boldsymbol{\theta}_1^{(k+1)}+\gamma^{(k)}\boldsymbol{\theta}^{(k)}$, where $\sum\gamma^{(k)}=\infty$ and $\sum\left(\gamma^{(k)}\right)^2<\infty$.

Gaussian Mixture Representation of Student's t-Distribution

- The Student's t-distribution can be regarded as a Gaussian mixture.

$$Y_t-Y_{t-1}\overset{i.i.d.}{\sim}t\left(\mu,\sigma^2,\nu\right)\Longleftarrow\begin{cases}Y_t-Y_{t-1}|\mu,\sigma^2,\tau_t\sim\mathcal{N}\left(\mu,\sigma^2/\tau_t\right)\\ \tau_t\sim Gamma\left(\nu/2,\ \nu/2\right)\end{cases}$$

- We regard $\{\tau_t\}$ and $\{y_t\}_{t\in C_{mis}}$ as missing variables, and apply the stochastic EM algorithm to solve problem (1).
- If we only regard $\{y_t\}_{t\in C_{mis}}$ as missing data, there would be no closed-form maximizer in the M step.

Posterior Distribution of Missing Data

- The posterior distribution of missing data (complicated):

$$f\left(y_{t_d+1},y_{t_d+2},\ldots,y_{t_d+n_d},\tau_{t_d+1},\tau_{t_d+2},\ldots,\tau_{t_d+n_d+1}|\left\{y_{t\in C_{obs}}\right\},\mu^k,\left(\sigma^k\right)^2,\nu^k\right)$$
$$\propto\prod_{t=t_d+1}^{t_d+n_d+1}\tau_t^{\frac{\nu^k-1}{2}}\exp\left(-\frac{\nu^k}{2}\tau_t-\frac{\left(\sigma^k\right)^{-2}\left(y_t-y_{t-1}-\mu^k\right)^2}{2}\tau_t\right).$$

- Gibbs sampling: instead of drawing the all components of the missing data jointly, draws realizations of each component sequentially based on its distribution conditional on all the other components.
- Conditional distributions (much simpler):

$$Y_t|\mu^{(k)},\sigma^{(k)},\nu^{(k)},\mathbf{Y}_{-t},\{\tau_t\}\sim\mathcal{N}\left(\frac{\tau_t\left(\mu^{(k)}+y_{t-1}\right)+\tau_{t+1}\left(y_{t+1}-\mu^{(k)}\right)}{\tau_t+\tau_{t+1}},\frac{\left(\sigma^{(k)}\right)^2}{\tau_t+\tau_{t+1}}\right),$$

$$\tau_t|\mu^{(k)},\sigma^{(k)},\nu^{(k)},\{y_t\},\mathcal{T}_{-t}\sim\text{Gamma}\left(\frac{\nu^{(k)}+1}{2},\frac{\left(\sigma^{(k)}\right)^{-2}\left(y_t-\mu^{(k)}-y_{t-1}\right)^2+\nu^{(k)}}{2}\right),$$

where $\mathcal{T}_{-t}$ is the set of all the mixture weights except τ_t,and $\mathbf{Y}_{-t}$ is the set of all the samples except y_t.

M step

- The resulting log-likelihood of the simulated complete data is

$$l\left(\left\{y_t^{(k)}\right\},\left\{\tau_t^{(k)}\right\}|\mu,\sigma^2,\nu\right)$$
$$=\sum_{t=2}^{T}\left\{-\frac{\tau_t^{(k)}}{2\sigma^2}\left(y_t^{(k)}-\mu-y_{t-1}^{(k)}\right)^2-\log\left(\sigma\right)\right.$$
$$\left.-\frac{\nu}{2}\tau_t^{(k)}+\frac{\nu}{2}\log\left(\frac{\nu}{2}\right)+\frac{\nu-1}{2}\log\left(\tau_t^{(k)}\right)-\log\left(\Gamma\left(\frac{\nu}{2}\right)\right)\right\}.$$

- Maximizer:

$$\mu_1^{(k+1)}=\frac{\sum_{t=2}^{T}\tau_t^{(k)}\left(y_t^{(k)}-y_{t-1}^{(k)}\right)}{\sum_{t=2}^{T}\tau_t^{(k)}}, \quad (2)$$

$$\left(\sigma_1^{(k+1)}\right)^2=\frac{\sum_{t=2}^{T}\tau_t^{(k)}\left(y_t^{(k)}-\mu^{(k+1)}-y_{t-1}^{(k)}\right)^2}{T-1}, \quad (3)$$

$$\nu_1^{(k+1)}=\arg\max_{\nu>0}l\left(\left\{y_t^{(k)}\right\},\left\{\tau_t^{(k)}\right\}|\mu^{(k)},\left(\sigma^k\right)^2,\nu\right). \quad (4)$$

Algorithm

1. Initialize $\mu^{(0)}$ and $\sigma^{(0)}$ as an arbitrary number, $\nu^{(0)}$ as an arbitrary positive number, and $k=0$.
2. Draw one realization $\left\{y_t^{(k)}\right\}_{t\in C_{mis}}$ and $\left\{\tau_t^{(k)}\right\}$ via Gibbs sampling method.
3. Compute $\mu_1^{(k+1)}$, $\left(\sigma_1^{(k+1)}\right)^2$, and $\nu_1^{(k+1)}$ according to (2)-(4), and then update

$$\mu^{(k+1)}=\mu^{(k)}+\gamma^{(k)}\left(\mu_1^{(k+1)}-\mu^{(k)}\right),$$
$$\left(\sigma^2\right)^{(k+1)}=\left(\sigma^2\right)^{(k)}+\gamma^{(k)}\left(\left(\sigma_1^2\right)^{(k+1)}-\left(\sigma^2\right)^{(k)}\right),$$
$$\nu^{(k+1)}=\nu^{(k)}+\gamma^{(k)}\left(\nu_1^{(k+1)}-\nu^{(k)}\right),$$

where $\gamma^{(k)}=\frac{1}{k}$.

4. Return to step 2 or stop if the stopping criterion is satisfied.

Numerical Results

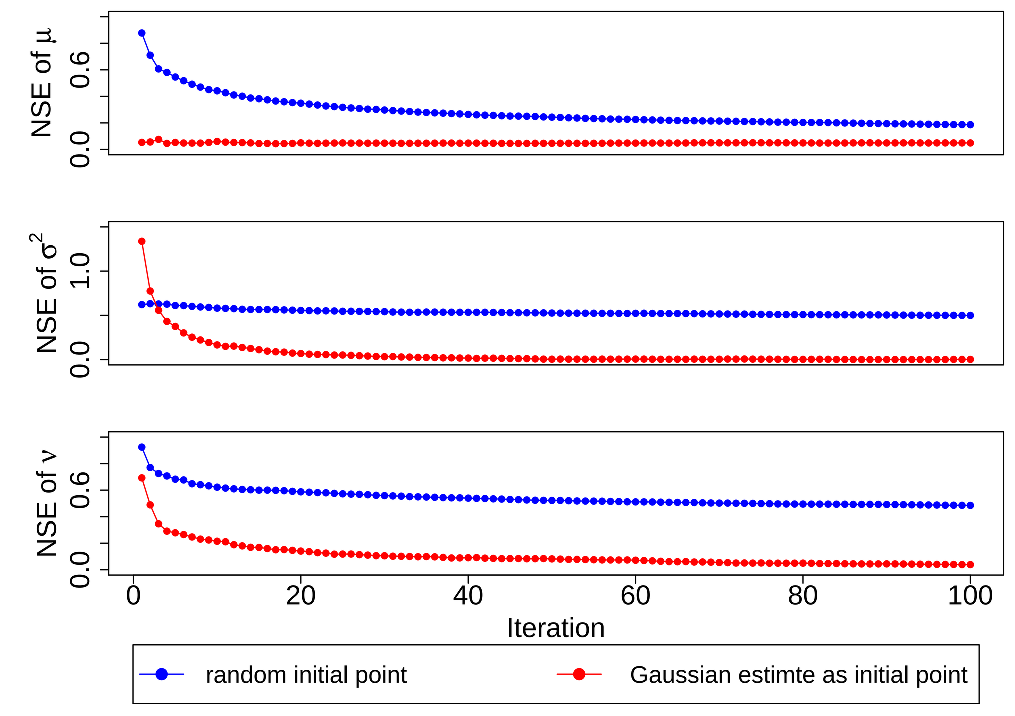


Figure 1: Estimation errors of parameters versus iterations.

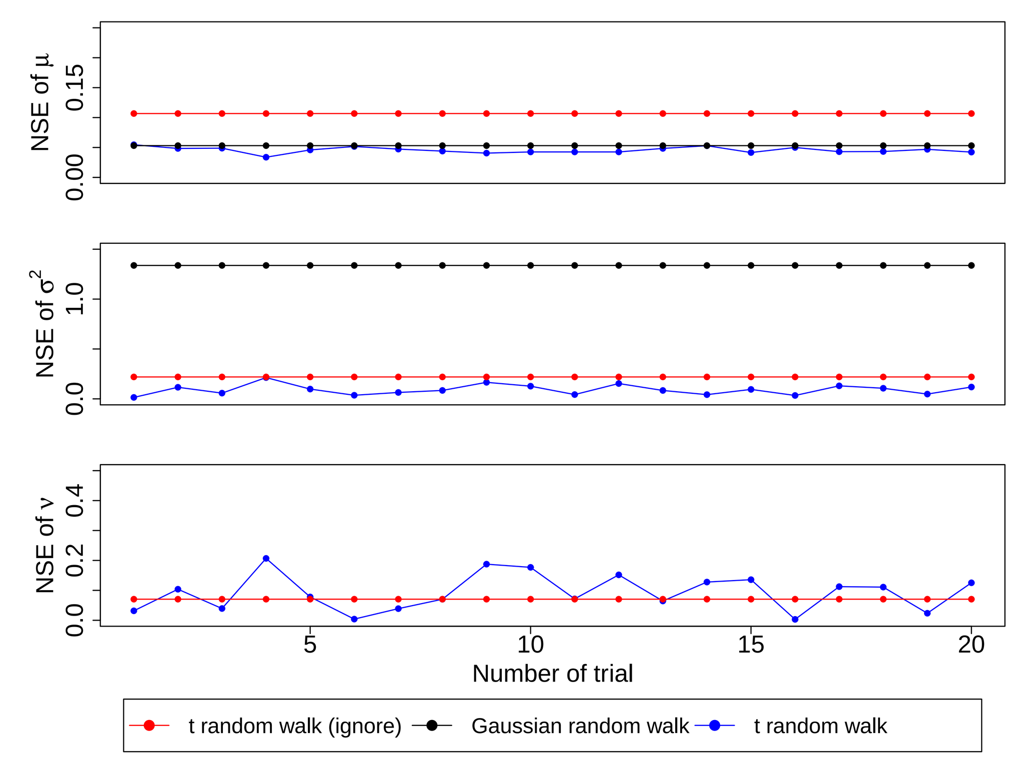


Figure 2: Final estimation errors of three different estimation approaches. t random walk (ignore) means MLE of Student's t random walk by ignoring the missing values (only use the available differences bwtween adjacent samples).

Conclusion

For heavy-tailed data, the traditional methods based on Gaussian distribution are too inefficient, and significant performance gain can be achieved by designing the algorithm under heavy-tailed model.

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