Frequency Estimation for a Mixture of Sinusoids: A Near-Optimal Sequential Approach

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Outline

- Introduction
- Proposed sequential algorithm
- Stopping criteria: CFAR
- Convergence
- Performance evaluation

You can download a MATLAB implementation of the algorithm here:
https://bitbucket.org/wcslspectralestimation/continuous-frequency-estimation/src/NOMP

Formulation

Unit norm sinusoid of frequency $\omega \in [0, 2\pi)$

$$\mathbf{x}(\omega) = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & e^{j\omega} & \dots & e^{j(N-1)\omega} \end{bmatrix}^T$$

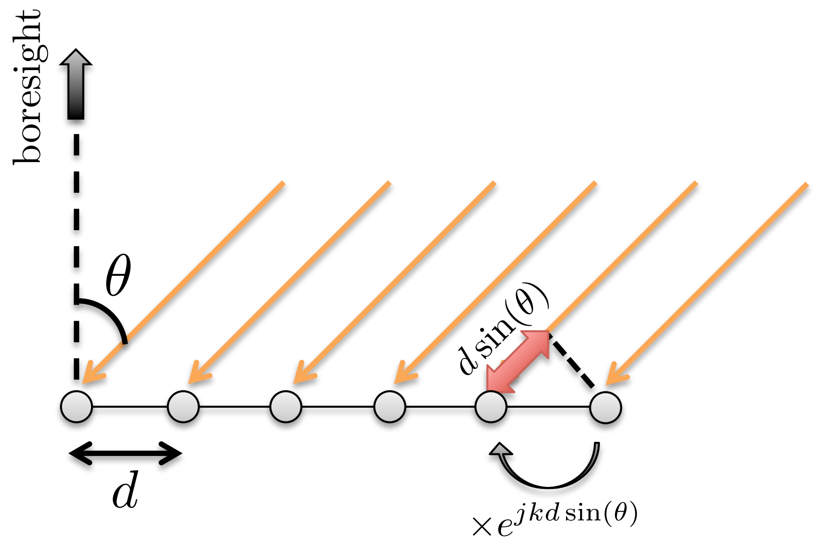
Mixture of sinusoids:

$$\mathbf{y} = \sum_{l=1}^{K} g_l \mathbf{x}(\omega_l) + \mathbf{z}$$

$\mathbf{y} \in \mathbb{C}^N$, $g_l \in \mathbb{C}$, $\mathbf{z} \sim CN(\mathbf{0}, \sigma^2 \mathbb{I})$

Goal: Estimate $\{(g_l, \omega_l), l = 1, 2, \dots, K\}$ and K

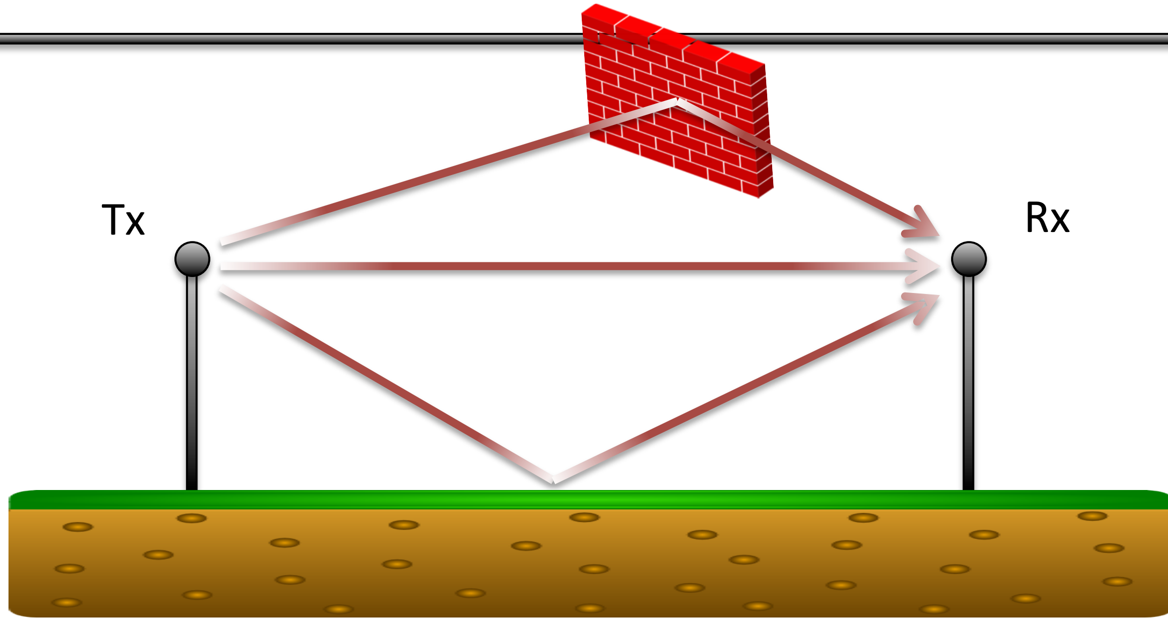
DoA Estimation



$$\omega = kd\sin(\theta) \rightarrow \mathbf{y} = g\mathbf{x}(\omega)$$

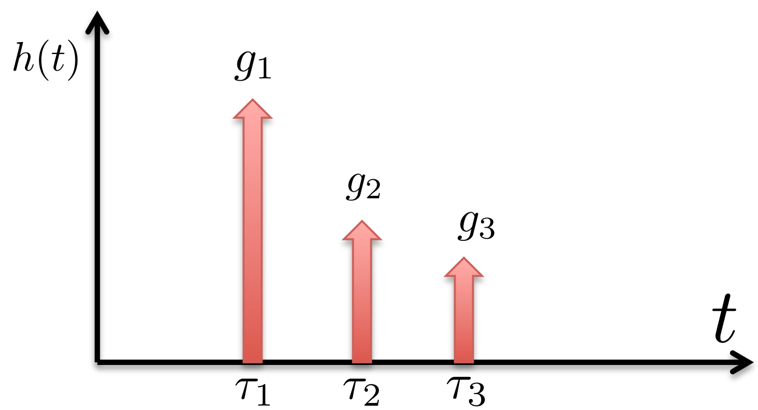
Direction of Arrival estimation $\rightarrow$ Frequency estimation problem

Multipath Channel Estimation



Channel impulse response: $h(t) = \sum_{l=1}^{K} g_l \delta(t - \tau_l)$

Channel transfer function: $H(f) = \sum_{l=1}^{K} g_l e^{-j2\pi f \tau_l}$



Sample Uniformly in Frequency Domain

$$H = \sum_{l=1}^{K} g_l \mathbf{x}(\omega_l) \qquad \omega_l = -2\pi \Delta f \tau_l$$

Single Frequency Estimation

Unknown parameters are frequency and complex gain.

Maximum Likelihood: $\max_{\{g,\omega\}} 2\Re\{\mathbf{y}^H g\mathbf{x}(\omega)\} - |g|^2||\mathbf{x}(\omega)||^2$

GLRT: first maximize over all possible complex gains, then maximize over frequencies.

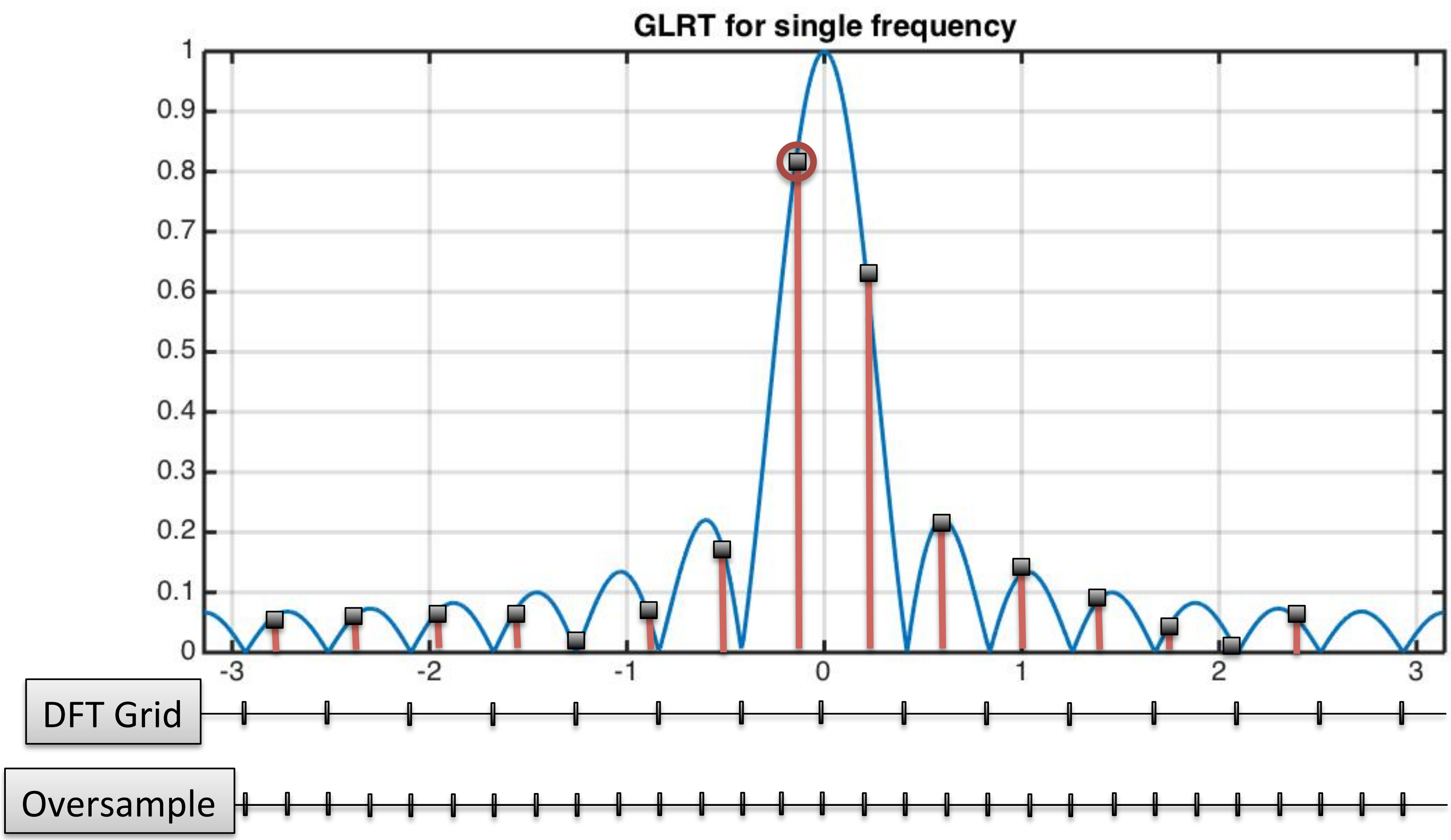
$$\max_{g} 2\Re\{\mathbf{y}^H g\mathbf{x}(\omega)\} - |g|^2||\mathbf{x}(\omega)||^2 \longrightarrow \hat{g} = \frac{\mathbf{x}(\omega)^H\mathbf{y}}{||\mathbf{x}(\omega)||^2}$$

Next, we maximize for the frequency: $\max_{\omega} G_{\mathbf{y}}(\omega)$

$$\text{Where,} \quad G_{\mathbf{y}}(\omega) = \frac{|\mathbf{x}(\omega)^H\mathbf{y}|^2}{||\mathbf{x}(\omega)||^2}$$

Grid & Refine

Grid & Refine



GLRT for single frequency
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-3
-2
-1
0
1
2
3
DFT Grid
Oversample

Newtonized OMP (NOMP)

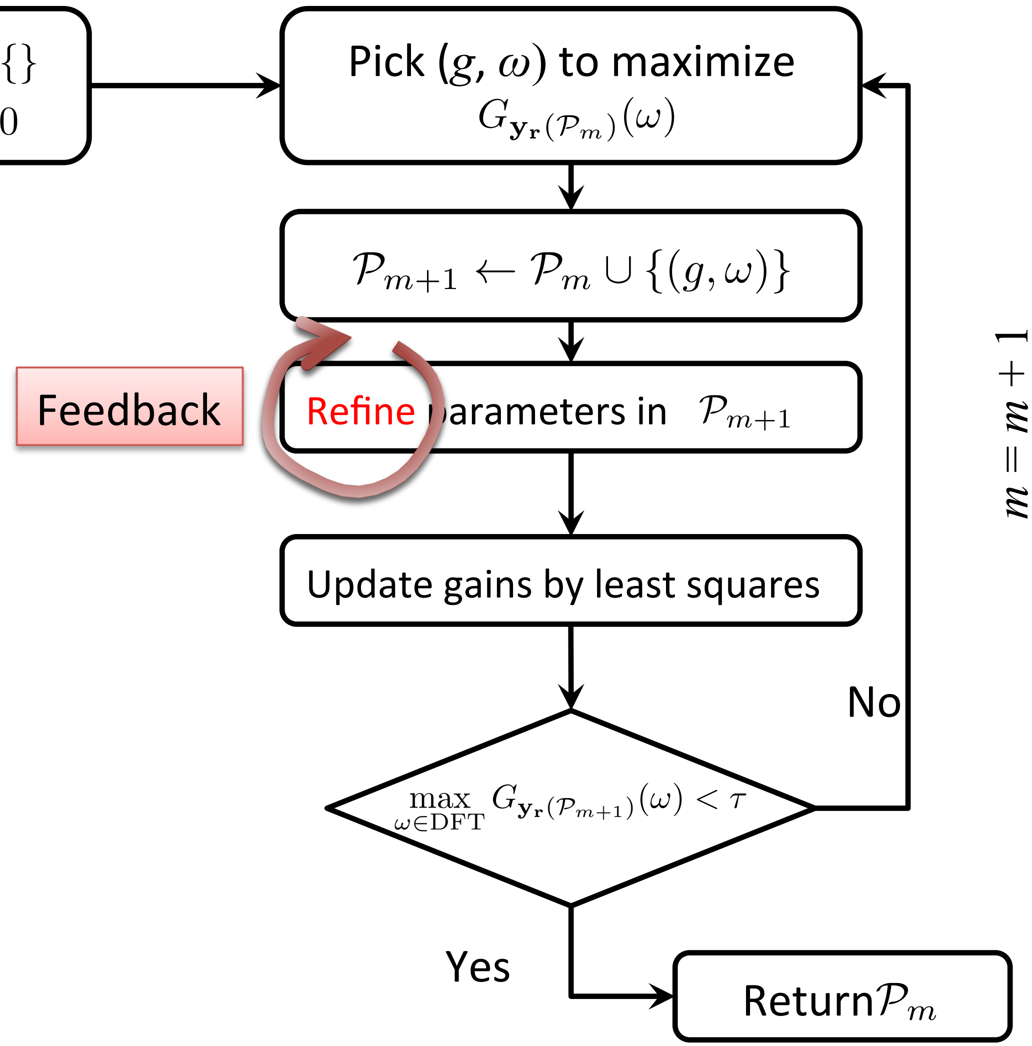
GLRT cost function

$$G_{\mathbf{y}}(\omega) = \frac{|\mathbf{x}(\omega)^H \mathbf{y}|^2}{||\mathbf{x}(\omega)||^2}$$

Residual response

$$\mathcal{P} = \{(g_l, \omega_l), l = 1, \ldots, k\}$$

$$\mathbf{y_r}(\mathcal{P}) = \mathbf{y} - \sum_{l=1}^{k} g_l \mathbf{x}(\omega_l)$$



[1] B. Mamandipoor, D. Ramasamy, U. Madhow, "Newtonized Orthogonal Matching Pursuit: Frequency Estimation over the Continuum,"arXiv preprint arXiv:1509.01942, 2015.

Stopping Criteria: CFAR

- Common strategy in detection problems → use noise model only
- Criteria: if noise can explain the observation → target does not exist!
- We develop a similar criteria for the frequency estimation algorithm:

$$\max_{\omega \in \mathrm{DFT}} G_{\mathbf{y_r}(\mathcal{P}_{m+1})}(\omega) < \tau \qquad \text{Assuming } \mathbf{y_r}(\mathcal{P}_{m+1}) \text{ is pure noise!}$$

$$||\mathcal{F}\mathbf{y_r}(\mathcal{P}_{m+1})||_\infty^2 < \tau$$

Probability of false alarm:

$$Pr\{||\mathcal{F}\mathbf{y_r}(\mathcal{P}_{m+1})||_\infty^2 > \tau\} = P_{\mathrm{fa}}$$

Stopping Criteria: CFAR

P_{fa} is the nominal false alarm rate.

$$\tau = \sigma^2 \log(N) - \sigma^2 \log\log\left(\frac{1}{1-P_{\text{fa}}}\right)$$

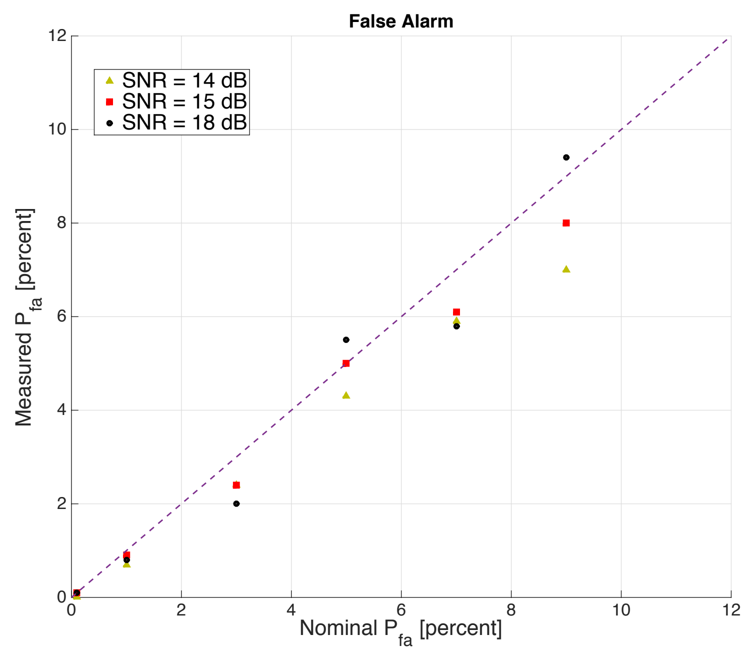
300 runs of NOMP

#sinusoids $K = 16$

#observations $N = 256$

fixed nominal SNR

$\Delta\omega_{\min} = 2.5\Delta_{\text{dft}}$



Simulation result: measured false alarm rate is in agreement with the the nominal value.

Probability of Miss and ROC

Taking into account the effect of noise

Ignoring the "interference" from other sinusoids

$$P_{\text{miss}} \approx 1 - Q_1\left(0.88\sqrt{2SNR}, \sqrt{2\tau/\sigma^2}\right)$$

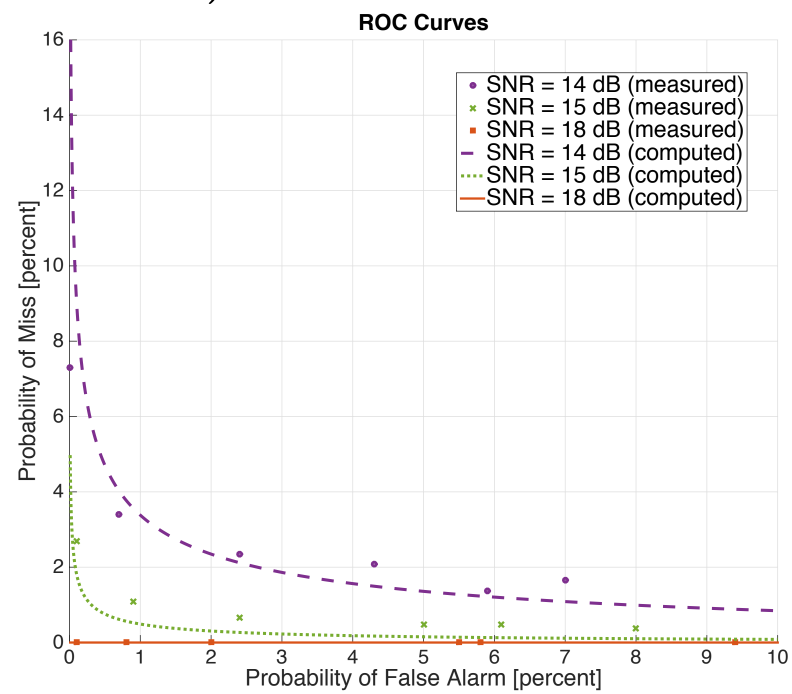
300 runs of NOMP

#sinusoids $K = 16$

#observations $N = 256$

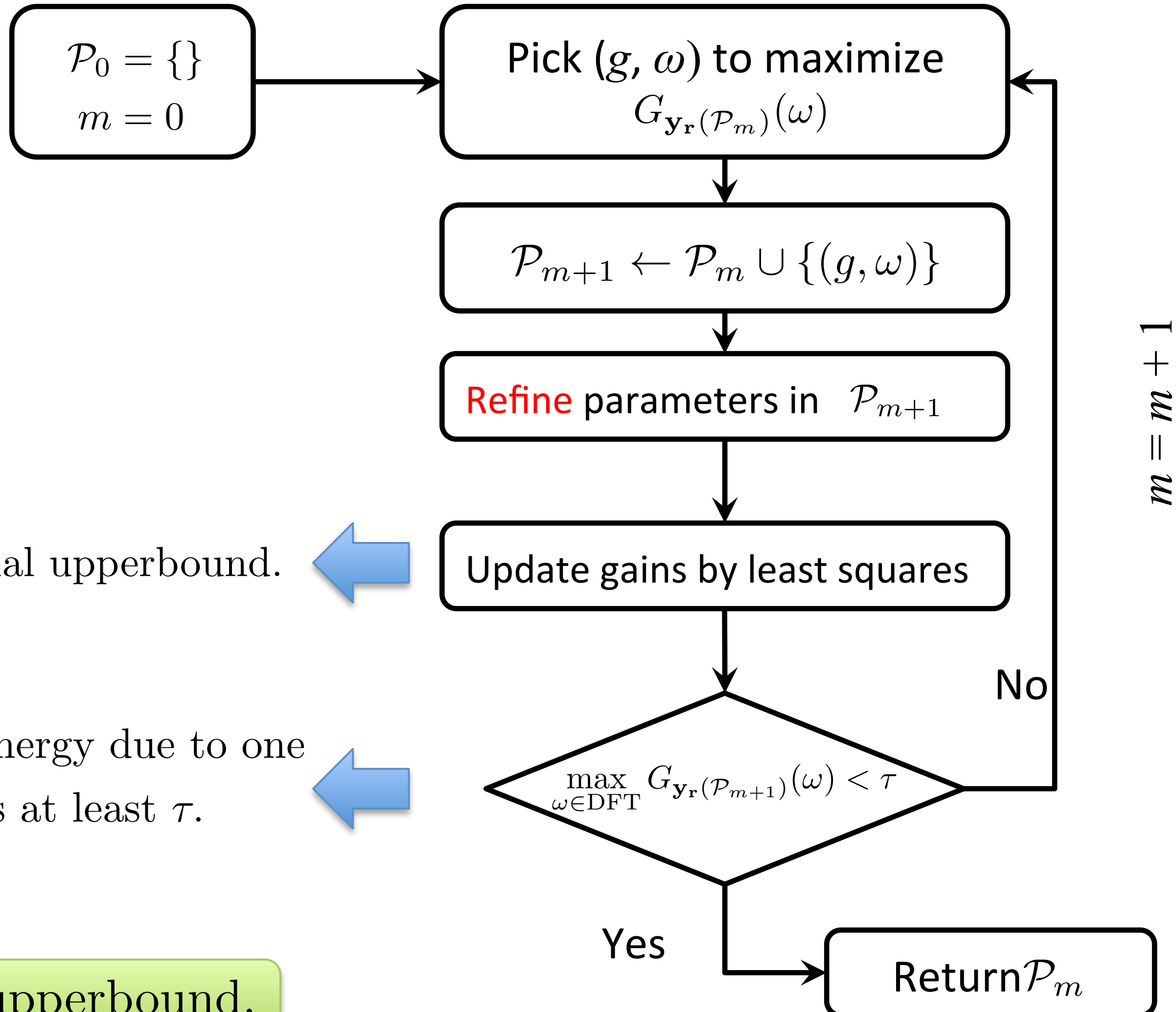
fixed nominal SNR

$\Delta\omega_{\text{min}} = 2.5\Delta_{\text{dft}}$



The resulting ROC turns out to be in remarkable agreement with simulations.

Convergence: bounding # iterations



#observations (N) is a trivial upperbound.

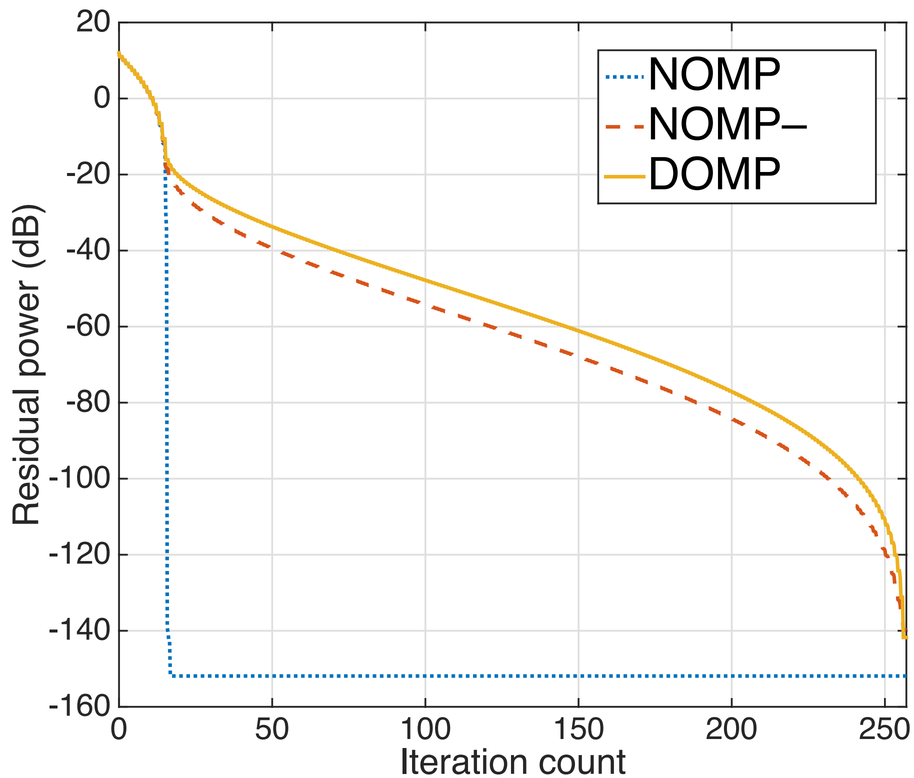
Reduction of the residual energy due to one iteration of the algorithm is at least τ.

$\min\{N, ||\mathbf{y}||^2/\tau\}$ is an upperbound.

Empirical Convergence Rate

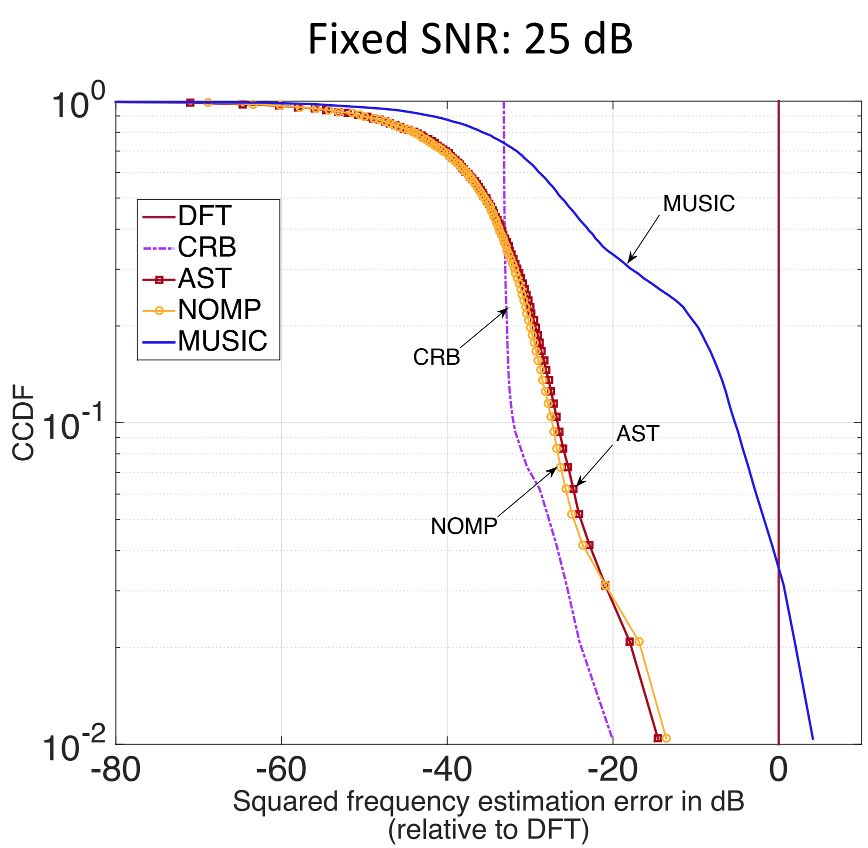
NOMP – : just a single refinement step for the newly detected sinusoid
DOMP : discretize the parameter space and apply OMP

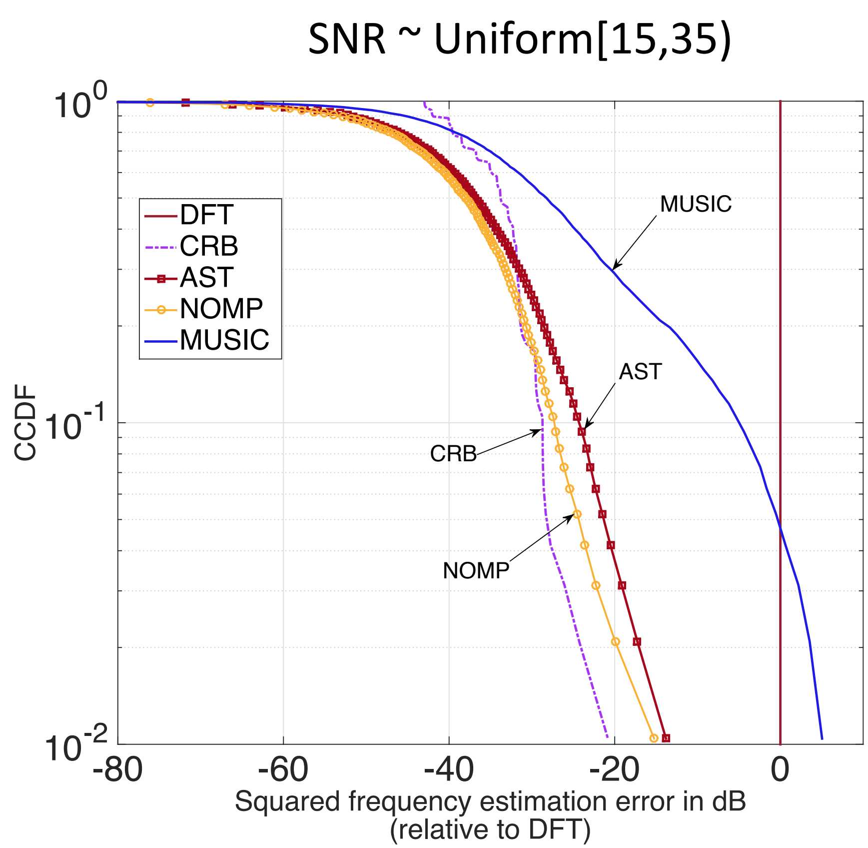
average over 1000 runs
#sinusoids $K = 16$
$\Delta\omega_{\min} = 2.5\Delta_{\text{dft}}$
No noise



Cyclic refinements has a significant impact on speeding up the convergence!

Performance – Accuracy



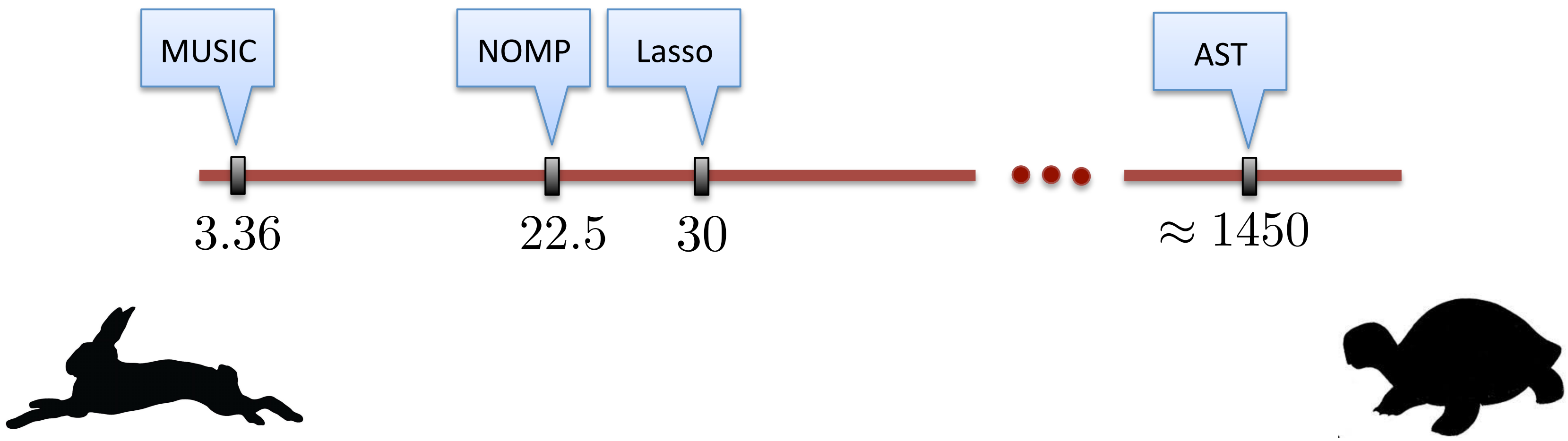


Comparison with state-of-the-art algorithm: Atomic norm Soft Thresholding (AST)

[2] B. N. Bhaskar, G. Tang, and B. Recht, "atomic norm denoising with applications to line spectral estimation," arXiv preprint arXiv:1204.0562, 2012

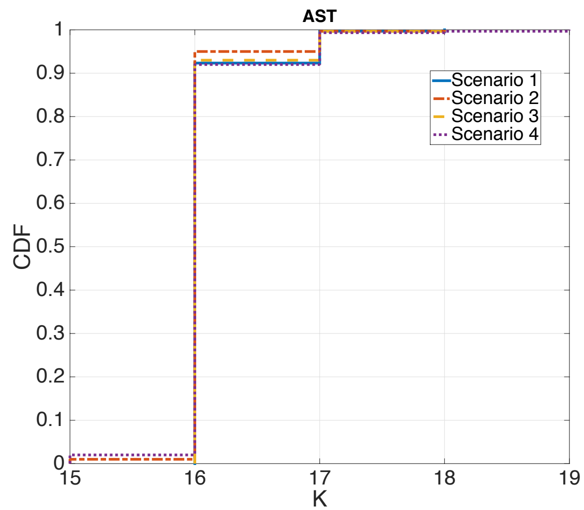
Performance – Speed

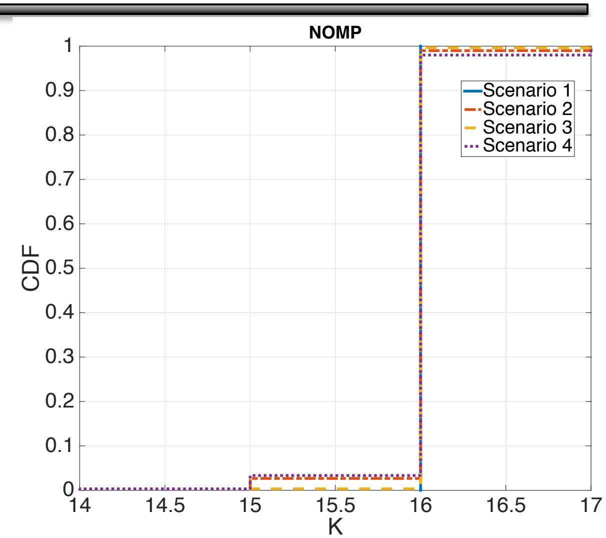
Run time of various algorithms over 300 simulation runs
(#sinusoids in the mixture =16)



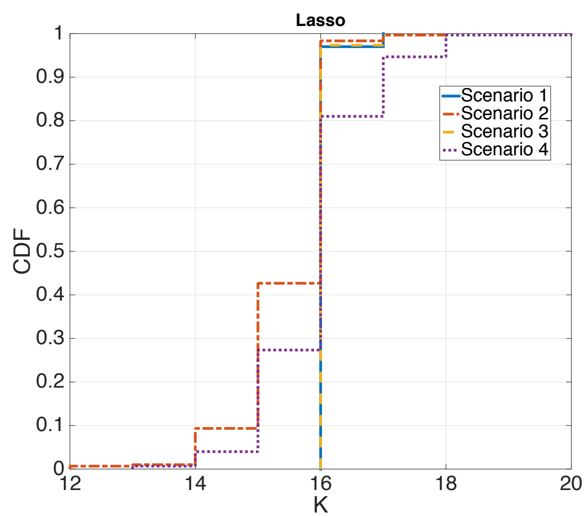
Model Order (K) Estimation

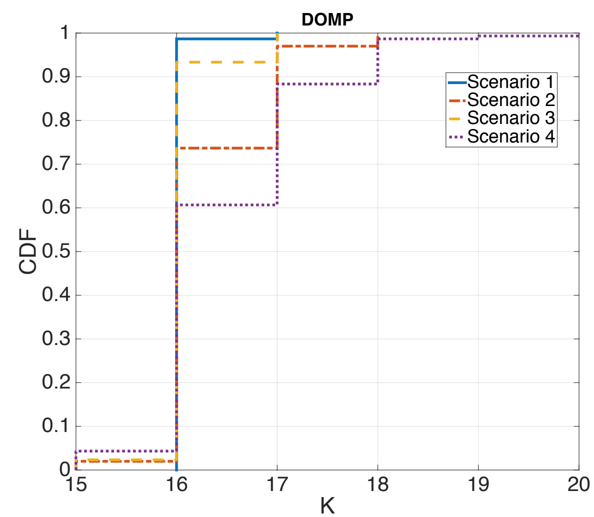
Scenarios	SNR (dB)	$\Delta\omega_{\min}/\Delta_{\text{dft}}$
1	25	2.5
2	25	0.5
3	Uniform[15, 35]	2.5
4	Uniform[15, 35]	0.5





When minimum separation between frequencies is small, Lasso and DOMP make errors!





Questions??