Reconstruction of Graph Signals: Percolation from a Single Seeding Node

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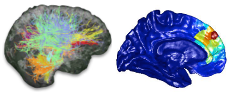
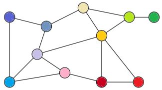
Graph signal processing

- Emergence of network science and big data
- Networks and graphs: structures that encode pairwise relationships
- Our interest, not in network itself, but in data associated with nodes
  - $\Rightarrow$ The object of study is a graph signal
- Graph SP: need to extend classical SP results to graph signals
  - $\Rightarrow$ Modify existing algorithms, gain intuition on concepts preserved/lost



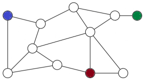


Graph $G = (\mathcal{V}, \mathcal{E}, W)$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_{|\mathcal{V}|} \end{bmatrix} = \begin{bmatrix} 0.6 \\ \vdots \\ 0.7 \end{bmatrix}$$

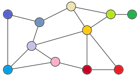
- Many relevant GSP problems: filter design, sampling, blind deconvolution

- Our focus in this paper: reconstruction of bandlimited graph signals
- Most related problems:
  - ⇒ Estimate the *unknown* signal $\mathbf{y}$ by observing a subset of nodes
- Our problem:
  - ⇒ Reconstruct the *known* signal $\mathbf{y}$ by acting on a subset of nodes
  - ⇒ Injection of a sparse signal followed by a low-pass graph filter



⇒ GRAPH FILTER ⇒

$$\mathbf{y} = \mathbf{H}\mathbf{x}$$

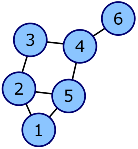


- Not only theoretical merits, also practical relevance
  - ⇒ Graph filters ⇒ percolation of local information
  - ⇒ Distributed nets, opinion formation, biological percolation processes

- Before being more specific: review of graph signals and filters

Graph signals and the need for graph topology

- (Node) graph signals are mappings $x : \mathcal{V} \to \mathbb{R}$
  - $\Rightarrow$ May be represented as a vector $\mathbf{x} \in \mathbb{R}^N$ (with $|\mathcal{V}| = N$)
  - $\Rightarrow$ DSP can be seen as a particular case of GSP $\Rightarrow$ directed cycle graph
- Graph $G = (\mathcal{V}, \mathcal{E}, W)$ is endowed with a graph-shift operator $\mathbf{S}$
  - $\Rightarrow$ Can be represented as a matrix $\mathbf{S} \in \mathbb{R}^{N \times N}$ satisfying:
  - $\Rightarrow$ $S_{ij} = 0$ for $i \neq j$ and $(i, j) \notin \mathcal{E}$ (captures local structure in G)
  - $\Rightarrow$ $\mathbf{S}$ can take nonzero values in the edges of G or in its diagonal



$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & 0 & 0 & S_{15} & 0 \\ S_{21} & S_{22} & S_{23} & 0 & S_{25} & 0 \\ 0 & S_{23} & S_{33} & S_{34} & 0 & 0 \\ 0 & 0 & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & 0 & S_{54} & S_{55} & 0 \\ 0 & 0 & 0 & S_{64} & 0 & S_{66} \end{pmatrix}$$

- Examples: Adjacency $\mathbf{A}$, Degree $\mathbf{D}$ and Laplacian $\mathbf{L}$

- **S** is a local linear operator, i.e., if $\mathbf{y} = \mathbf{S}\mathbf{x}$
  - $\Rightarrow y_i = \sum_j S_{ij} x_j = \sum_{j \in \mathcal{N}_i^+} S_{ij} x_j \Rightarrow$ only to 1-hop info
  - $\Rightarrow$ if $\mathbf{z} = \mathbf{S}^2 \mathbf{x} \Rightarrow \mathbf{z} = \mathbf{S}\mathbf{y} \Rightarrow 2$-hop info
- **S** (spectrum) useful to analyze **x**, here diagonalizable shifts $\mathbf{S} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$
  - $\Rightarrow \mathbf{V} = [\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_N]$ eigenvectors; $\mathbf{\Lambda}$ eigenvalues; if normal, $\mathbf{V}^{-1} = \mathbf{V}^H$
- Leverage **S** to define GFT and iGFT

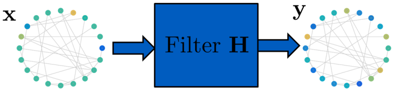
$$\tilde{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x} \qquad\qquad \mathbf{x} = \mathbf{V}\tilde{\mathbf{x}}$$

  - $\Rightarrow$ Bandlimited signals: $\tilde{\mathbf{x}}$ sparse; particular cases: DFT, PCA
- Key message: the two basic elements of GSP are **x** and **S**

Definition of a linear (shift-invariant) graph filter

- A graph filter $H : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a map between graph signals

Focus on linear filters
$\Rightarrow$ represent by a $N \times N$ matrix



- Filter $\mathbf{H}$ is a polynomial on $\mathbf{S}$ with coefficients h_l and degree L

$$\mathbf{H} := h_0\mathbf{S}^0 + h_1\mathbf{S}^1 + h_2\mathbf{S}^2 + \ldots = \sum_{l=0}^{L} h_l\mathbf{S}^l$$

- A graph filter represents a linear transformation that
  - $\Rightarrow$ Accounts for local structure of the graph
  - $\Rightarrow$ Can be implemented distributedly in L steps
  - $\Rightarrow$ Only requires information in the L-neighborhood

Frequency response of a graph filter

- Using $\mathbf{S} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$, we may write $\mathbf{H} = \sum_{l=0}^{L} h_l \mathbf{S}^l = \mathbf{V}\left(\sum_{l=0}^{L} h_l \boldsymbol{\Lambda}^l\right)\mathbf{V}^{-1}$

- Since $\boldsymbol{\Lambda}^l$ are diagonal, the GFT-iGT can be used to write $\mathbf{y} = \mathbf{H}\mathbf{x}$ as

$$\tilde{\mathbf{y}} = \text{diag}(\tilde{\mathbf{h}})\tilde{\mathbf{x}}$$

  $\Rightarrow$ Output at frequency k depends only on input at frequency k

  $\Rightarrow$ $\tilde{\mathbf{h}}$ is the frequency response of the filter $\mathbf{H}$

- Clearly $\tilde{h}_k = \sum_{l=0}^{L} h_l \lambda_k^l$, hence one can obtain $\tilde{\mathbf{h}}$ as $\tilde{\mathbf{h}} = \boldsymbol{\Psi}\mathbf{h}$, where

$$\boldsymbol{\Psi} := \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^L \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_N & \dots & \lambda_N^L \end{pmatrix}$$

  $\Rightarrow$ Since $\boldsymbol{\Psi}$ is Vandermonde, invertible if $\lambda_k \neq \lambda_{k'}$ $\Rightarrow$ $\mathbf{h} = \boldsymbol{\Psi}^{-1}\tilde{\mathbf{h}}$

  $\Rightarrow$ To be leveraged when designing (low-pass) graph filters

- Note that GFT for signals $\mathbf{V}^{-1}$ and filters $\boldsymbol{\Psi}$ is not the same

- We want to reconstruct a (K-bandlimited) graph signal $\mathbf{y}$
- Most existing problems
  - ⇒ Estimate the unknown signal $\mathbf{y}$ by observing a subset of nodes
- Our problem
  - ⇒ Reconstruct the known signal $\mathbf{y}$ by acting on a subset of nodes
- Approach: design a sparse input that is percolated by a graph filter
  - ⇒ We act on a node by injecting signal values
  - ⇒ Distributed implementation
- Examples include the reconstruction of:
  - ⇒ Global opinion in a social net by influencing a few people
  - ⇒ Brain state by exciting a few brain regions

Operation and problem statement

Operation: The reconstruction scheme proceeds in two phases

1. Seeding phase

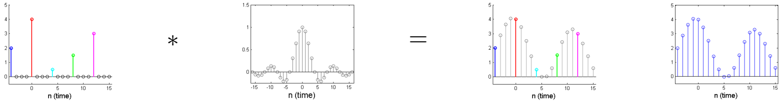
- Output is a sparse signal $\mathbf{x}$
  - $\Rightarrow$ In its simplest form we can act directly on $\{x_i\}_{i\in\mathcal{P}}$
  - $\Rightarrow$ Single seeding node, injects scalars $\{s^t\}_{t\in\mathcal{P}}$ diffused by $\mathbf{S}$ to form $\mathbf{x}$

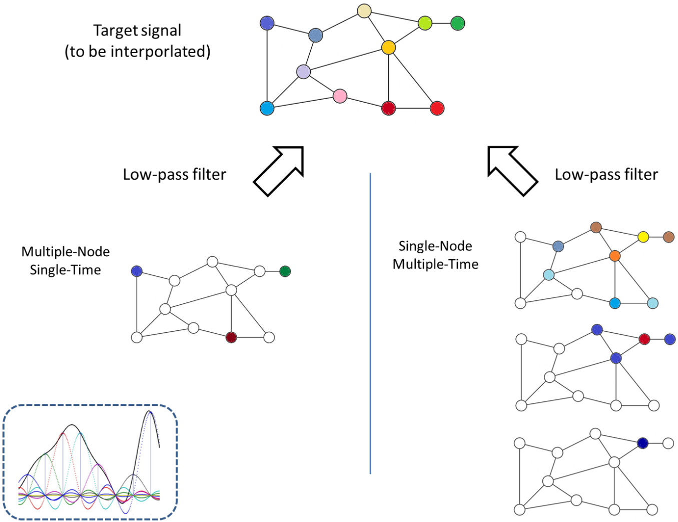
2. Filtering phase

- Use $\mathbf{x}$ as input
- Apply a low-pass graph filter $\mathbf{H}$ with freq. response $\tilde{\mathbf{h}} = [\tilde{\mathbf{h}}_K^T\,,\, \mathbf{0}]^T$
- Obtain the output signal $\mathbf{z} := \mathbf{H}\mathbf{x}$

Problem statement: How to design $\mathbf{x}$ and $\mathbf{H}$ such that $\mathbf{z} = \mathbf{y}$?

- Resembles (uniform) time interpolation





- We will focus on Single Node - Multiple Time seeding

- A single node (say 1) injects P scalar seeding signals s^t, one per time t
- Collect those P signals in $\mathbf{s}_P := [s^{P-1}, \ldots, s^0]^T$ and define $\mathbf{s}^t = [\mathbf{s}_P]_t \mathbf{e}_1$
- Goal of $\mathbf{y} = \mathbf{H}\mathbf{x}$, rewritten in the frequency domain

$$\tilde{\mathbf{y}} = \mathbf{V}^{-1}\mathbf{H}\mathbf{x} = \mathbf{V}^{-1}\mathbf{V}\text{diag}(\mathbf{\Psi}\mathbf{h})\mathbf{V}^{-1}\mathbf{x} = \text{diag}\big(\mathbf{\Psi}\mathbf{h}\big)\tilde{\mathbf{x}}$$

$\Rightarrow$ Bilinear problem in $\tilde{\mathbf{x}}$ and $\mathbf{h}$

- Split the system of equations in two $\Rightarrow$ $\mathbf{E}_K$ first K canonical vectors

$$\tilde{\mathbf{y}}_K = \mathbf{E}_K^T \text{diag}(\mathbf{\Psi}\mathbf{h})\,\tilde{\mathbf{x}}, \qquad (1)$$

$$\mathbf{0}_{N-K} = \bar{\mathbf{E}}_K^T \text{diag}(\mathbf{\Psi}\mathbf{h})\,\tilde{\mathbf{x}}. \qquad (2)$$

- Equation (2) holds for every K-bandlimited signal $\mathbf{y}$

  $\Rightarrow$ Design $\mathbf{h}$ to solve (2) and $\tilde{\mathbf{x}}$ to solve (1)

- If degree of $\mathbf{H}$ no smaller than distinct eigenvalues in $\{\lambda_i\}_{i=K+1}^N$

  $\Rightarrow$ $\mathbf{h}^*$ solving (2) can always be found

- What is the relation between $\tilde{\mathbf{x}}$ and the injected values $\mathbf{s}_P$?

- Each seed percolated using $\mathbf{S}$ $\Rightarrow$ output of the seeding phase is

$$\mathbf{x} = \sum_{t=1}^{P} \mathbf{S}^{t-1}\mathbf{s}^{t} = \sum_{t=1}^{P} [\mathbf{s}_P]_t \mathbf{S}^{t-1}\mathbf{e}_1$$

- Same form of a filter with input $\mathbf{e}_1$ and coefficients $\mathbf{s}_P$

$$\tilde{\mathbf{x}} = \text{diag}(\mathbf{\Psi}\mathbf{s}_P)\tilde{\mathbf{e}}_1 = \text{diag}(\tilde{\mathbf{e}}_1)\mathbf{\Psi}\mathbf{s}_P$$

- U_1 # zeros in $\{[\tilde{\mathbf{e}}_1]_k\}_{k=1}^{K}$ and D_1 # of repeated values in $\{\lambda_k\}_{k=1}^{K}$

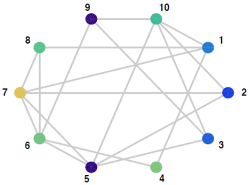
Perfect reconstruction in SN-MT seeding

Perfect reconstruction of $\mathbf{y}$ is guaranteed via SN-MT seeding if:
i) $\lambda_{k_1} \neq \lambda_{k_2}$ for all $(\lambda_{k_1}, \lambda_{k_2})$ such that $k_1 \leq K$ and $k_2 > K$,
ii) $U_1 = 0$ and $D_1 = 0$.

- Seeding $\mathbf{s}_P^* = \mathbf{\Psi}^{-1}\text{diag}^{-1}(\tilde{\mathbf{e}}_1)\text{diag}^{-1}(\tilde{\mathbf{h}}_K^*)\tilde{\mathbf{y}}_K$
  - $\Rightarrow$ Cond. *i)* ensures that $\mathbf{h}^*$ does not eliminate any active frequency
  - $\Rightarrow$ $U_1 = 0$: seeding node acts on every active frequency
  - $\Rightarrow$ $D_1 = 0$: every active frequency is distinguishable from each other

- SN-MT recovery depends on $\mathbf{\Lambda}$ and rows of $\mathbf{V}^{-1}$ being non-zero
  - ⇒ Easy to look for a good seeding node: as in sampling [Marques15]
- MN-ST recovery depends on rank of submatrix of $\mathbf{V}^{-1}$ (Not shown here)
  - ⇒ No clear way to check a priori: as in sampling [Chen15]
- Extensions to Multiple Node -Multiple Time seeding developed too
- Approximate (imperfect) reconstruction settings
  - ⇒ Insufficient amount of seeding values or filter degree
  - ⇒ Noisy seeding value injections
- The set of seeding nodes has a significant impact on robustness
  - ⇒ Optimal design to minimize mean or worst-case error

- Erdős-Rényi graph with $p = 0.2$ and $N = 10$, single seeding node



Node Index	$\mathbf{y}$	Frequency Index	$\tilde{\mathbf{y}}$
1	-0.41	1	-1.35
2	-0.42	2	0.36
3	-0.50	3	-0.45
4	-0.10	4	0
5	-0.27	5	0
6	-0.81	6	0
7	-0.59	7	0
8	-0.21	8	0
9	-0.07	9	0
10	-0.66	10	0

Bandwidth: $K = 3$
$\Rightarrow$ Seeding phase of length 3

- Evolution of the signal (space and frequency) for every shift

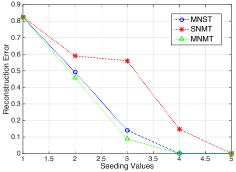
Node Index \ Time	0	1	2	3	4	5	6	7	8	9
1	-0.04	0.05	0.23	0.45	1.01	1.68	2.21	2.57	3.31	-0.41
2	0	0	-0.07	-0.09	-0.24	-0.51	-0.79	-0.61	0.42	-0.42
3	0	0	-0.04	-0.10	-0.41	-1.09	-2.12	-2.52	-1.86	-0.50
4	0	-0.04	0.05	0.19	0.53	1.08	1.77	2.13	2.70	-0.10
5	0	0	-0.07	-0.24	-0.53	-0.97	-1.14	-0.96	1.09	-0.27
6	0	0	-0.11	-0.16	-0.19	0.22	1.17	2.47	2.82	-0.81
7	0	-0.04	0.02	0.02	0.15	0.63	1.57	2.72	3.76	-0.59
8	0	-0.04	0.02	0.16	0.51	1.23	2.38	3.27	4.17	-0.21
9	0	0	-0.04	-0.14	-0.34	-0.78	-1.28	-1.65	-0.27	-0.07
10	0	-0.04	0.05	0.09	-0.01	-0.52	-1.59	-2.01	-0.99	-0.66

Frequency Index \ Time	0	1	2	3	4	5	6	7	8	9
1	-0.01	-0.03	0.00	0.01	0.03	0.12	0.48	1.60	4.74	-1.35
2	0.02	-0.06	0.00	-0.01	0.01	-0.02	0.05	-0.15	0.57	0.36
3	0.01	-0.04	-0.01	0.01	-0.02	0.03	-0.07	0.21	-0.74	-0.45
4	-0.01	-0.01	0.10	0.34	1.10	2.76	5.22	6.86	6.35	0
5	-0.01	0.03	0.06	0	0	0	0	0	0	0
6	0.01	-0.02	-0.04	-0.01	0	0	0	0	0	0
7	-0.00	0.00	0.04	0.10	0.24	0.38	0.37	0.14	0	0
8	-0.02	0.02	0.25	0.51	0.99	1.19	0.68	0	0	0
9	0.00	-0.01	-0.03	-0.02	-0.02	0	0	0	0	0
10	-0.00	0.00	0.02	0.03	0.05	0.03	0	0	0	0

- Perfect recovery is achieved after seeding and filtering

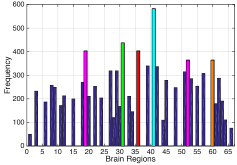
Opinion formation

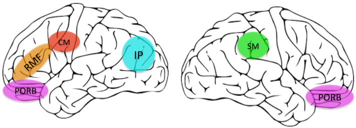
- Induce a desired opinion profile
  - $\Rightarrow$ Zachary's Karate club graph
- Study robustness of different seeding strategies
  - $\Rightarrow$ Insufficient seeding nodes
- Better to convince several people once (MN-ST)
  - $\Rightarrow$ than same person multiple times (SN-MT)



Brain state induction

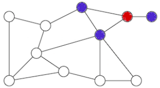
- Take brain to desired state by exciting a few (or just one) regions
- Study robustness of different seeding sets
  - $\Rightarrow$ Noise in the signal injection
- Corroborate neurophysiological meaning of the findings



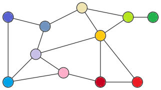


Conclusions and take-home messages

- Specific contribution in this paper
  - ⇒ Reconstruction of bandlimited graph signals from single node inputs
  - ⇒ Successive inputs followed by a graph filter
  - ⇒ Conditions for recovery
  - ⇒ Extensions to multiple nodes injecting multiple values



⇒ GRAPH FILTER ⇒



- From a more general point of view
  - ⇒ Decoupling betw. estimating unobserved values and low-pass filtering
  - ⇒ Graph filters can be viewed as linear network operators
  - ⇒ Strong relation between GSP and diffusion/percolation processes