Generalized Graph Signal Sampling and Reconstruction

Xiaohan Wang, Jiaxuan Chen, Yuantao Gu

Dept. EE, Tsinghua University

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Signals on Graphs

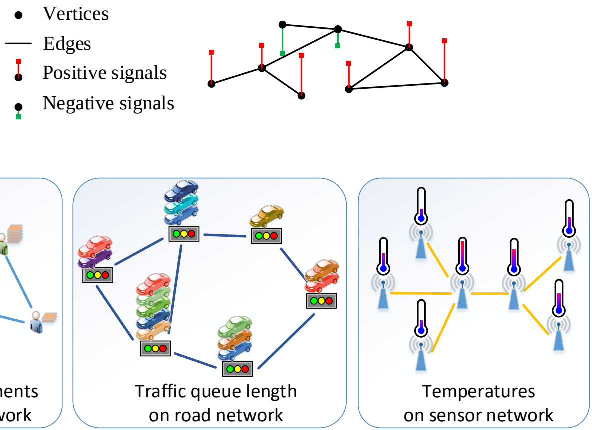

Figure: Examples of graph signals.

Graphs and graph signals

- Graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$
- Graph signal $\mathbf{f} \in \mathbb{R}^N$, or a mapping $f : \mathcal{V} \to \mathbb{R}$

Graph signal sampling and reconstruction

- Sampling set $\mathcal{S} \subseteq \mathcal{V}$
- Reconstruct $\mathbf{f}$ from the known samples $\{f(u)\}_{u \in \mathcal{S}}$
- Conditions: Smooth or bandlimited signals on graph

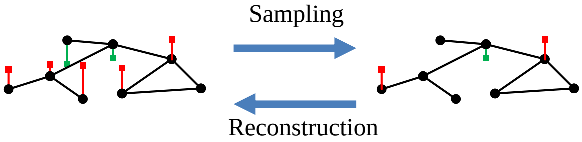

Figure: Sampling and reconstruction of graph signals.

The frequency domain of graph signals

- Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A}$
- Frequencies $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$
- Fourier basis $\{\mathbf{u}_k\}_{1 \le k \le N}$

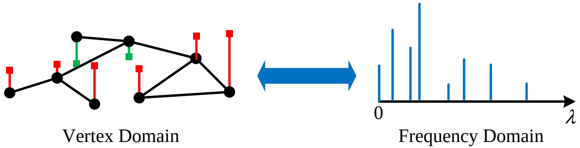

Figure: Vertex and frequency domains of a graph signal.

Bandlimited graph signals

- The subspace of ω-bandlimited signals is called Paley-Wiener space $PW_\omega(\mathcal{G}) \triangleq \text{span}\{\mathbf{u}_i | \lambda_i \le \omega\}$
- Bandlimited graph signal $\mathbf{f} \in PW_\omega(\mathcal{G})$

Existing Works on Graph Signal Sampling and Reconstruction

- Uniqueness set and Paley-Wiener space (I. Pesenson, 2008)
- Least-square reconstruction (S. Narang, A. Gadde and A. Ortega, 2013)
- Iterative least square reconstruction (S. Narang, A. Gadde, E. Sanou and A. Ortega, 2013)
- TV-minimization reconstruction (S. Chen, A. Sandryhaila, J. Moura and J. Kovacevic, 2014)
- Local-set-based reconstruction (X. Wang, P. Liu and Y. Gu, 2014)
- Sampling theorem (A. Anis, A. Gadde and A. Ortega, 2014; S. Chen, A. Sandryhaila, J. Moura and J. Kovacevic, 2015)
- Distributed algorithms (X. Wang, M. Wang and Y. Gu, 2015; S. Chen, A. Sandryhaila, and J. Kovacevic, 2015)
- Reconstruction through percolation (S. Segarra, A. Marques, G. Leus, A. Ribeiro, 2015)
- ......

Definitions

Centerless local sets

For a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, assume that $\mathcal{V}$ is divided into disjoint local sets $\{\mathcal{N}_i\}_{i\in\mathcal{I}}$ satisfying

- Each subgraph $\mathcal{G}_{\mathcal{N}_i}$ is connected;
- $\bigcup_{i\in\mathcal{I}}\mathcal{N}_i = \mathcal{V}$;
- $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset, \quad \forall i,j \in \mathcal{I},\ i \neq j.$

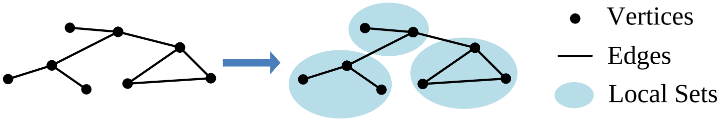

Figure: An illustration of centerless local sets.

Definitions

Local weight

A local weight $\boldsymbol{\varphi}_i \in \mathbb{R}^N$ associated with a centerless local set $\mathcal{N}_i$ satisfies

$$\varphi_i(v) \begin{cases} \geq 0, v \in \mathcal{N}_i \\ = 0, v \notin \mathcal{N}_i \end{cases} \quad \text{and} \quad \sum_{v \in \mathcal{N}_i} \varphi_i(v) = 1.$$

Local measurement

For given centerless local sets and the associated local weights $\{(\mathcal{N}_i, \boldsymbol{\varphi}_i)\}_{i \in \mathcal{I}}$, a set of local measurements for a graph signal $\mathbf{f}$ is $\{f_{\boldsymbol{\varphi}_i}\}_{i \in \mathcal{I}}$, where

$$f_{\boldsymbol{\varphi}_i} \triangleq \langle \mathbf{f}, \boldsymbol{\varphi}_i \rangle = \sum_{v \in \mathcal{N}_i} f(v)\varphi_i(v).$$

Decimation and Local Measurement

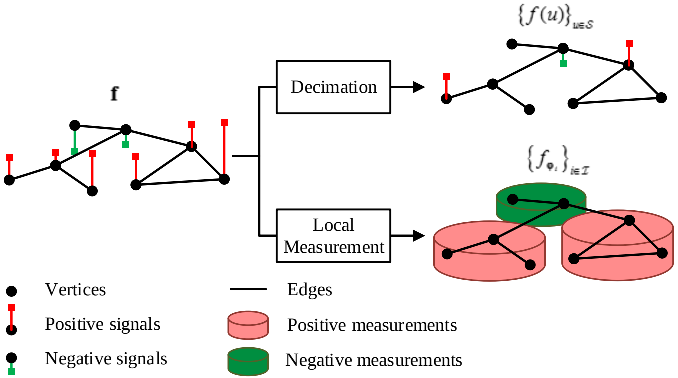

Figure: An illustration of traditional sampling (decimation) scheme versus generalized sampling (local measurement) scheme.

Local Propagation and Its Bound

Local propagation

For given $\{(\mathcal{N}_i, \boldsymbol{\varphi}_i)\}_{i\in\mathcal{I}}$ on $\mathcal{G}(\mathcal{V}, \mathcal{E})$, local propagation $\mathbf{G}$ is defined by

$$\mathbf{G}\mathbf{f} = \mathcal{P}_\omega\left(\sum_{i\in\mathcal{I}} \langle \mathbf{f}, \boldsymbol{\varphi}_i\rangle \boldsymbol{\delta}_{\mathcal{N}_i}\right),$$

where $\mathcal{P}_\omega(\cdot)$ is the projection operator onto $PW_\omega(\mathcal{G})$, and $\boldsymbol{\delta}_{\mathcal{N}_i}$ is defined as

$$\delta_{\mathcal{N}_i}(v) = \begin{cases} 1, & v \in \mathcal{N}_i; \\ 0, & v \notin \mathcal{N}_i. \end{cases}$$

Lemma (bound of local propagation)

For given $\{(\mathcal{N}_i, \boldsymbol{\varphi}_i)\}_{i\in\mathcal{I}}$, $\forall \mathbf{f} \in PW_\omega(\mathcal{G})$, the following inequality holds,

$$\|\mathbf{f} - \mathbf{G}\mathbf{f}\| \le C_{\max}\sqrt{\omega}\|\mathbf{f}\|,$$

where $C_{\max} = \max_{i\in\mathcal{I}} \sqrt{|\mathcal{N}_i| D_i}$, and $|\cdot|$ denotes cardinality.

Table: Iterative Local Measurement Reconstruction.

Input: Graph $\mathcal{G}$, cutoff frequency ω, centerless local sets $\{\mathcal{N}_i\}_{i\in\mathcal{I}}$, local weights $\{\boldsymbol{\varphi}_i\}_{i\in\mathcal{I}}$, local measurements $\{f_{\boldsymbol{\varphi}_i}\}_{i\in\mathcal{I}}$;

Output: Interpolated signal $\mathbf{f}^{(k)}$;

Initialization:

$$\mathbf{f}^{(0)} = \mathcal{P}_\omega\left(\sum_{i\in\mathcal{I}} f_{\boldsymbol{\varphi}_i}\boldsymbol{\delta}_{\mathcal{N}_i}\right);$$

Loop:

$$\mathbf{f}^{(k+1)} = \mathbf{f}^{(k)} + \mathcal{P}_\omega\left(\sum_{i\in\mathcal{I}}(f_{\boldsymbol{\varphi}_i} - \langle \mathbf{f}^{(k)}, \boldsymbol{\varphi}_i\rangle)\boldsymbol{\delta}_{\mathcal{N}_i}\right);$$

Until: The stop condition is satisfied.

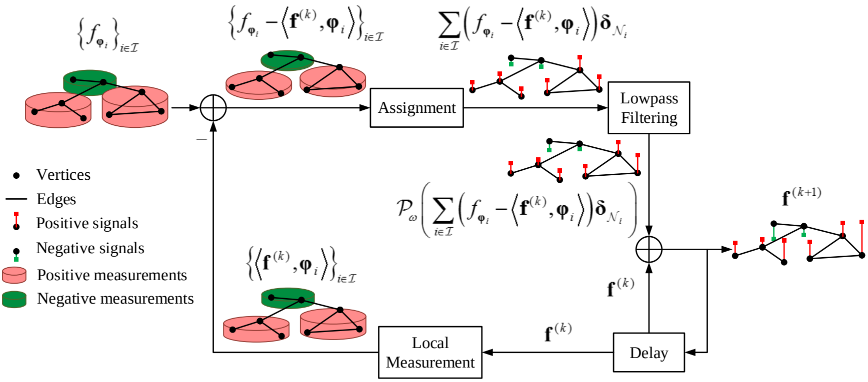

Figure: The procedures of ILMR.

The Convergence of ILMR

Proposition 1 (convergence of ILMR)

For given $\{(\mathcal{N}_i, \boldsymbol{\varphi}_i)\}_{i\in\mathcal{I}}$ and $\omega < 1/C_{\max}^2$, $\forall \mathbf{f} \in PW_\omega(\mathcal{G})$ can be reconstructed from its local measurements $\{f_{\boldsymbol{\varphi}_i}\}_{i\in\mathcal{I}}$ through ILMR, with the error at the kth iteration satisfying

$$\|\mathbf{f}^{(k)} - \mathbf{f}\| \le \gamma^k \|\mathbf{f}^{(0)} - \mathbf{f}\|,$$

where $\gamma = C_{\max}\sqrt{\omega}$.

Remarks

- $\{\boldsymbol{\varphi}_i\}_{i\in\mathcal{I}}$ is even not necessarily known in ILMR if the local measurements come from the result of some repeatable physical operations or black boxes.
- ILMR can be approximately implemented in a localized way. The projection operator $\mathcal{P}_\omega(\cdot)$ can be approximated by a Chebychev polynomial expansion of the Laplacian [1].

[1] D. K. Hammond, P. Vandergheynst, and R. Gribonval, "Wavelets on graphs via spectral graph theory," *Appl. Comput. Harmonic Anal.*, vol. 30, no. 2, pp. 129-150, 2011.

Gaussian Noise and Optimal Local Weights

Proposition 2 (expected error under Gaussian noise)

For given $\{(\mathcal{N}_i, \boldsymbol{\varphi}_i)\}_{i\in\mathcal{I}}$, the original signal $\mathbf{f} \in PW_\omega(\mathcal{G})$, assuming the noise associated with vertex v follows independent Gaussian distribution $\mathcal{N}(0, \sigma^2(v))$, if $\omega < 1/C_{\max}^2$, the reconstruction error of ILMR in the kth iteration satisfies

$$\mathrm{E}\left\{\|\tilde{\mathbf{f}}^{(k)} - \mathbf{f}\|\right\} \le \frac{1}{1-\gamma}\sqrt{\frac{2}{\pi}}\sum_{i\in\mathcal{I}}\sqrt{|\mathcal{N}_i|}\sigma_i + \mathcal{O}\left(\gamma^{k+1}\right),$$

where σ_i is the variance of the zero-mean Gaussian equivalent noise n_i of $\mathcal{N}_i$,

$$\sigma_i^2 = \sum_{v\in\mathcal{N}_i}\sigma^2(v)\varphi_i^2(v).$$

Corollary 1 (optimal local weights)

For given $\{\mathcal{N}_i\}_{i\in\mathcal{I}}$, if the noises associated with the vertices are independent and $n(v) \sim \mathcal{N}(0, \sigma^2(v))$, then the optimal local weights $\{\boldsymbol{\varphi}_i\}_{i\in\mathcal{I}}$ are

$$\varphi_i(v) = \frac{(\sigma^2(v))^{-1}}{\sum_{v\in\mathcal{N}_i}(\sigma^2(v))^{-1}}\delta_{\mathcal{N}_i}(v).$$

Independent and Identical Distributed Gaussian Noise

Corollary 2 (expected error under i.i.d. Gaussian noise)

For given centerless local sets $\{\mathcal{N}_i\}_{i\in\mathcal{I}}$ and the associated weights $\varphi_i(v) = 1/|\mathcal{N}_i|$ for $v \in \mathcal{N}_i$, the original signal $\mathbf{f} \in PW_\omega(\mathcal{G})$, assuming the noise associated with each vertex follows *i.i.d* Gaussian distribution $\mathcal{N}(0, \sigma^2)$, if $\omega < 1/C_{\max}^2$, the expected reconstruction error of ILMR in the kth iteration satisfies

$$\mathrm{E}\left\{\|\tilde{\mathbf{f}}^{(k)} - \mathbf{f}\|\right\} \leq \frac{|\mathcal{I}|\sigma}{1-\gamma}\sqrt{\frac{2}{\pi}} + \mathcal{O}\left(\gamma^{k+1}\right).$$

Table: A greedy method to partition centerless local sets with maximal cardinality.

Input: Graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, Maximal cardinality N_{max};

Output: Centerless local sets $\{\mathcal{N}_i\}_{i \in \mathcal{I}}$;

Initialization: $i = 0$;

Loop Until: $\mathcal{V} = \emptyset$

1) Find one vertex with the smallest degree in $\mathcal{G}$, $u = \arg\min_{v \in \mathcal{V}} d_{\mathcal{G}}(v)$;
2) $i = i + 1$, $\mathcal{N}_i = \{u\}$;
3) Obtain the neighbor set of $\mathcal{N}_i$, $\mathcal{S}_i = \{v \in \mathcal{G} | v \sim w, w \in \mathcal{N}_i, v \notin \mathcal{N}_i\}$;

Loop Until: $|\mathcal{N}_i| = N_{\mathsf{max}}$ or $\mathcal{S}_i = \emptyset$

4) Find one vertex with the smallest degree in $\mathcal{S}_i$, $u = \arg\min_{v \in \mathcal{S}_i} d_{\mathcal{G}}(v)$;
5) $\mathcal{N}_i = \mathcal{N}_i \cup \{u\}$;
6) Update $\mathcal{S}_i = \{v \in \mathcal{G} | v \sim w, w \in \mathcal{N}_i, v \notin \mathcal{N}_i\}$;

End Loop

7) Remove the edges, $\mathcal{E} = \mathcal{E} \backslash \{(p, q) | p \in \mathcal{N}_i, q \in \mathcal{V}\}$;
8) Remove the vertices, $\mathcal{V} = \mathcal{V} \backslash \mathcal{N}_i$ and $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$;

End Loop

Local Weight Choices

The Minnesota road graph is used in the experiments, which has 2640 vertices and 6604 edges. The centerless local sets are generated by the greedy algorithm. Five kinds of local weights are tested.

1. uniform weight, where $\varphi_i(v) = 1/|\mathcal{N}_i|, \forall v \in \mathcal{N}_i$;
2. random weight, where
$$\varphi_i(v) = \frac{\varphi'_i(v)}{\sum_{u\in\mathcal{N}_i} \varphi'_i(u)}, \quad \forall v \in \mathcal{N}_i, \varphi'_i(u) \sim \mathcal{U}(0,1);$$
3. Kronecker delta weight, where $\boldsymbol{\varphi}_i = \boldsymbol{\delta}_u$ for a random $u \in \mathcal{N}_i$;
4. the optimal weight, where
$$\varphi_i(v) = \frac{(\sigma^2(v))^{-1}}{\sum_{v\in\mathcal{N}_i} (\sigma^2(v))^{-1}} \delta_{\mathcal{N}_i}(v);$$
5. the optimal Kronecker delta weight, where $\boldsymbol{\varphi}_i = \boldsymbol{\delta}_u$ for
$$u = \arg\min_{u\in\mathcal{N}_i} \sigma^2(u).$$

Convergence of ILMR

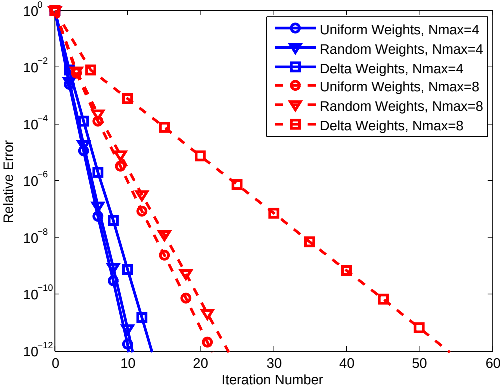

Figure: The convergence behavior of ILMR for various division of centerless local sets and different local weights (cases 1, 2, and 3).

Optimal Local Weights for Gaussian Noise

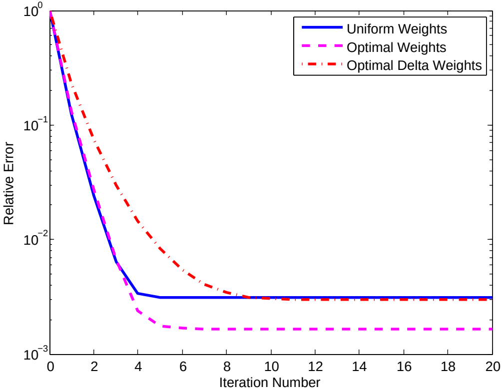

Figure: The convergence curves of reconstruction with uniform weights, the optimal weights, and optimal Kronecker delta weights when independent zero-mean Gaussian noise is added to each vertex (cases 1, 4, and 5).

Performance against i.i.d. Gaussian Noise

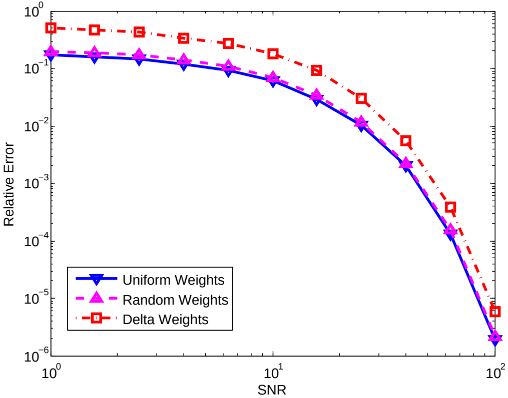

Figure: Relative errors of ILMR under different SNRs with various choices of local weights (cases 1, 2, and 3). The noise associated with each vertex is i.i.d. Gaussian.

Thank you!

Xiaohan Wang: wangxiaohan11@mails.tsinghua.edu.cn

Yuantao Gu: gyt@tsinghua.edu.cn

Gu-Lab: http://gu.ee.tsinghua.edu.cn