A Coupled Compressive Sensing Scheme for Unsourced Multiple Access

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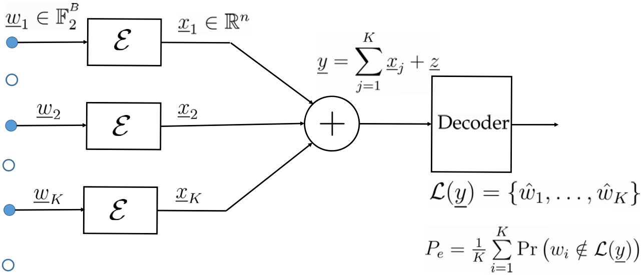
Uncoordinated and Unsourced Multiple Access

- K active users out of K_{tot} total users $K \in [25:300]$, K_{tot} is very large
- Each user has a B-bit message. B is small ≈ 100
- N channel uses available $N \approx 30,000$



Objective

Design a coding scheme minimizing the required SNR P such that

- Low complexity encoding and decoding complexities
- Prob. of decoding error per user $P_e \leq \epsilon \in [0.05, 0.1]$

Differences From Traditional Information Theoretic MAC

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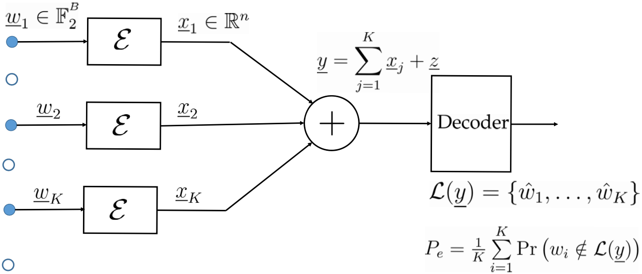

- Uncoordinated: Resource allocation not allowed
- Unsourced: Decoding done upto permutation of messages
- Finite block length regime

Prior Work

[Polyanskiy' 17] Gaussian coding for unsourced MAC

- Derived achievability limits via random Gaussian coding
  - ML decoder: exponential complexity in B, K. $\mathcal{O}(N \cdot \binom{2^B}{K}) \approx \mathcal{O}(N 2^{BK})$
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[Ordentlich and Polyanksiy'17] Compute-and-Forward based coding scheme

- Decoding modulo-2 sums
- Low complexity but still large gap to Polyanksiy's bound

Compressed Sensing View

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_N \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{2^B} \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$$

Unsourced nature $\leftrightarrow$ compressed sensing

- $\vec{b} \in \{0, 1\}^{2^B}$, $||\vec{b}||_1 = K$
- $\vec{a}_i \in \mathbb{R}^N$

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Challenges

- Huge sensing matrix: impractical even for $B \approx 100$
- $\vec{b}$ is binary: optimal sensing matrix & decoder design are open problems
- Finding fundamental limits appears to be an open problem

Compressive Sensing and MAC

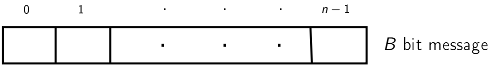
Neighbor Discovery for Wireless Networks [Zhang, Guo'12]

- Each node wishes to identify the network interface addresses (NIAs) of those nodes within a single hop
- Nodes assigned NIAs from address space $\{0, 1, \cdots, N\}$ (e.g. $N = 2^{48} - 1$)
- Strong connection with support recovery problem in compressive sensing
- Deterministic signatures based on second order Reed-Muller codes
- Chirp decoding algorithm - complexity sub-linear in N
- We don't know the gap of this from information theoretic bounds

Coupled Compressive Sensing Scheme

B bit message

Coupled Compressive Sensing Scheme



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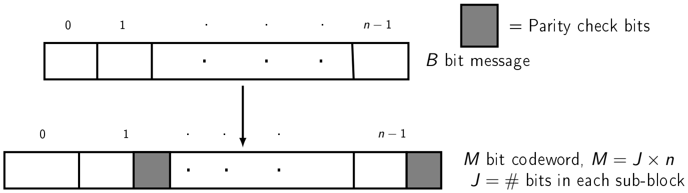

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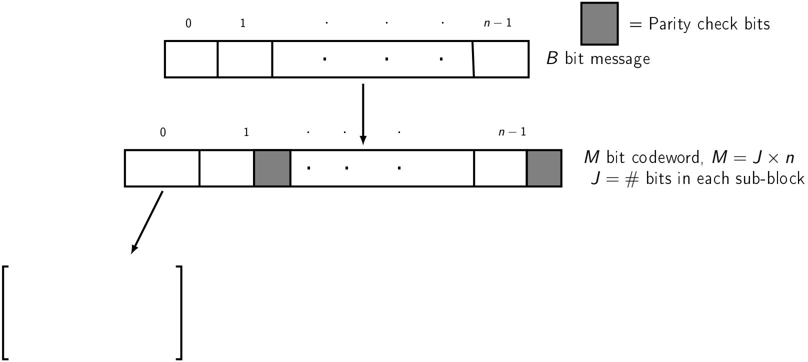

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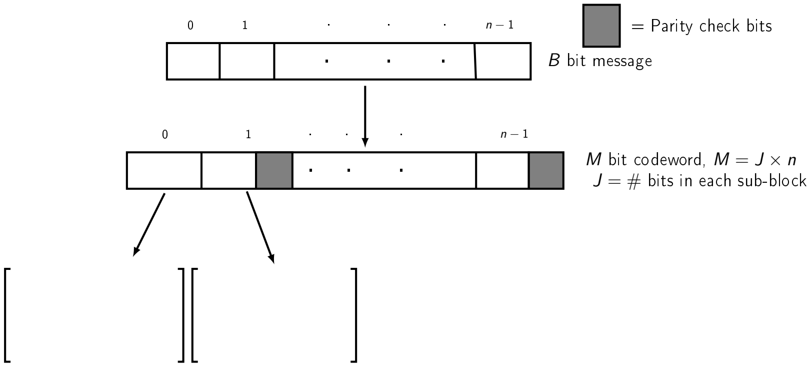

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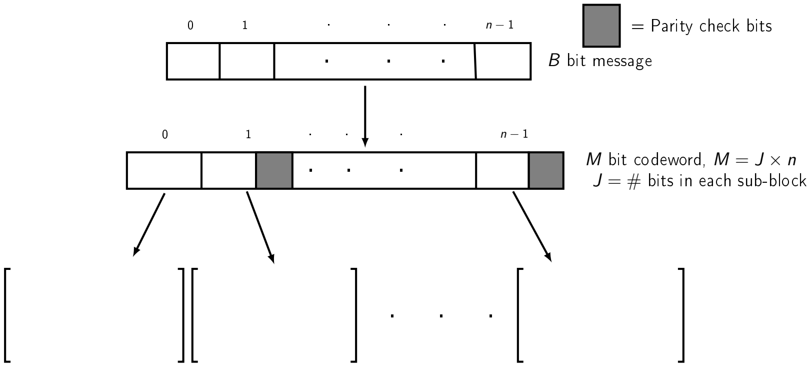

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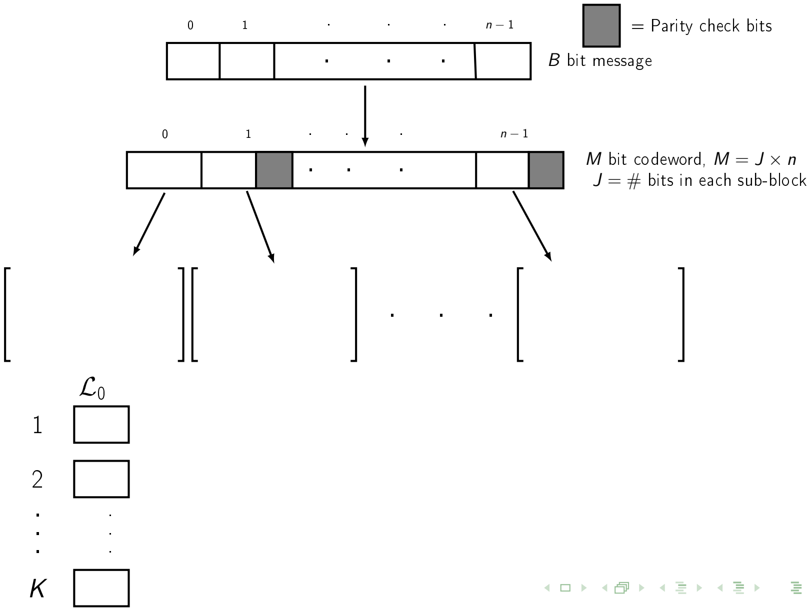

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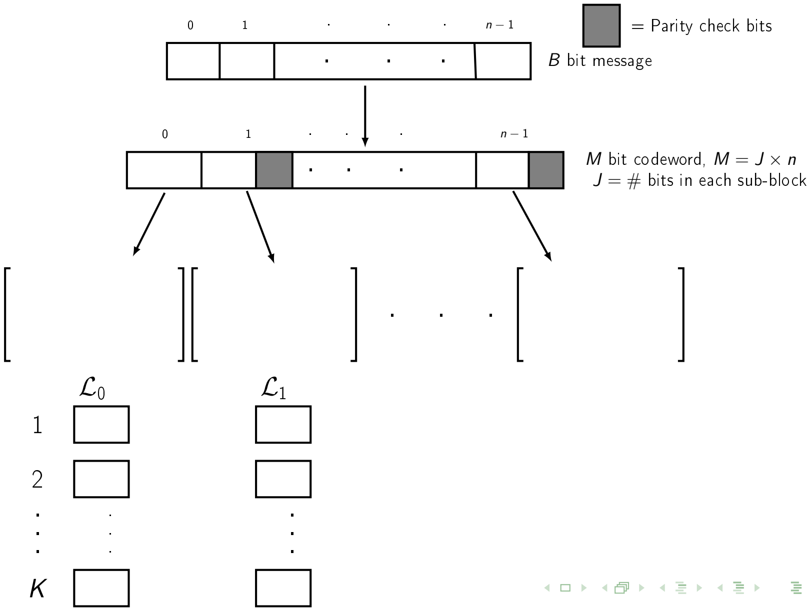

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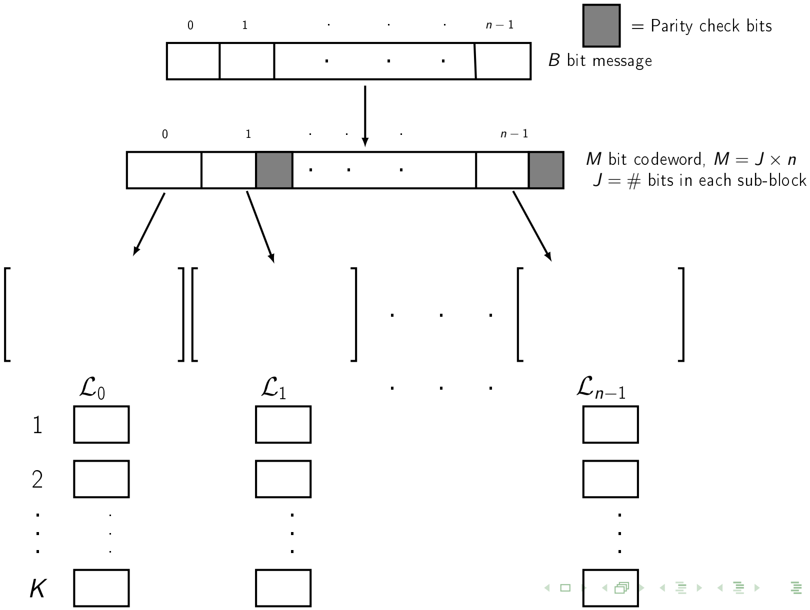

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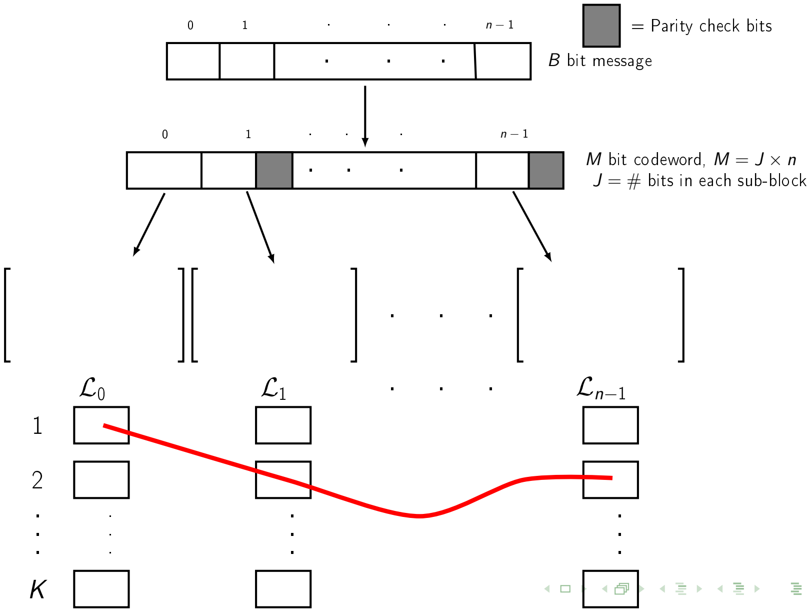
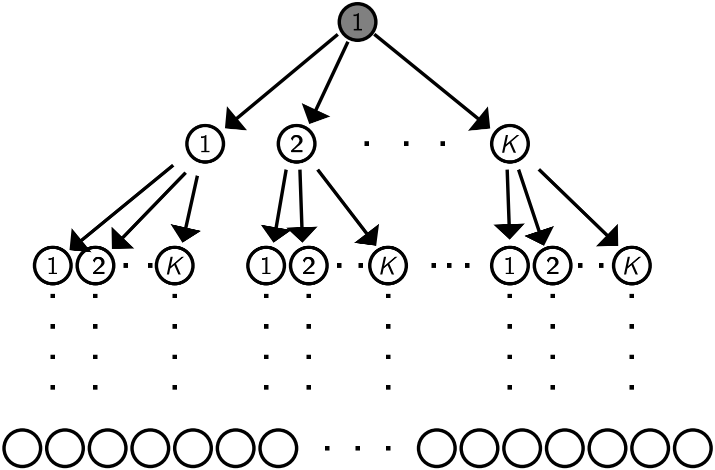


Tree Decoder For Coupled CS



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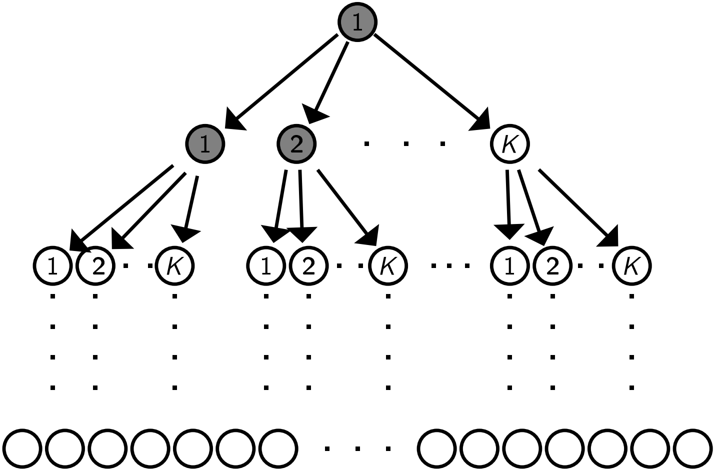

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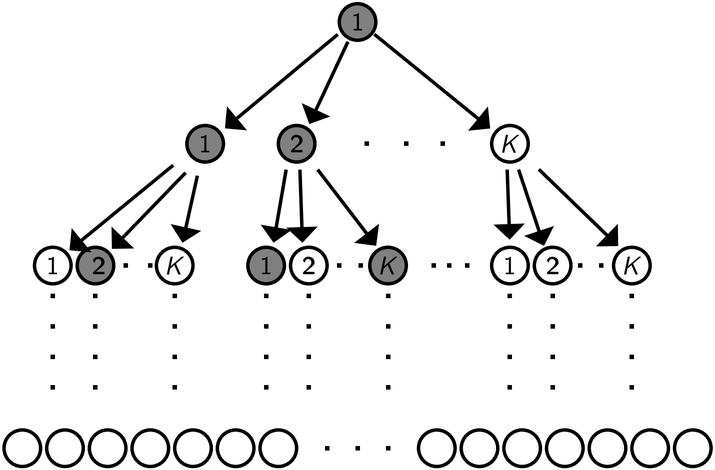

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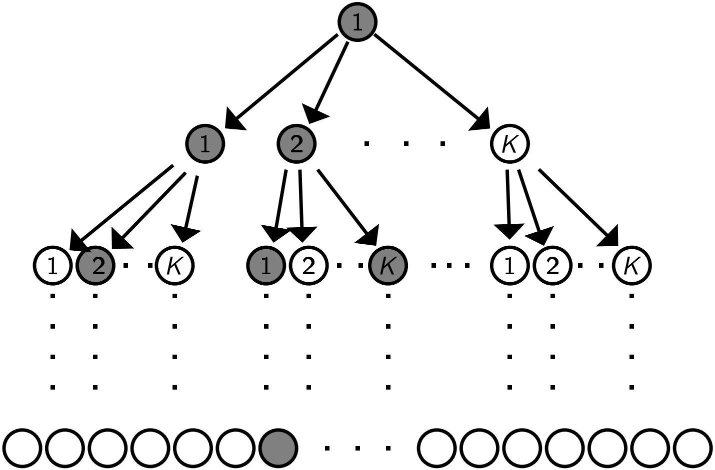

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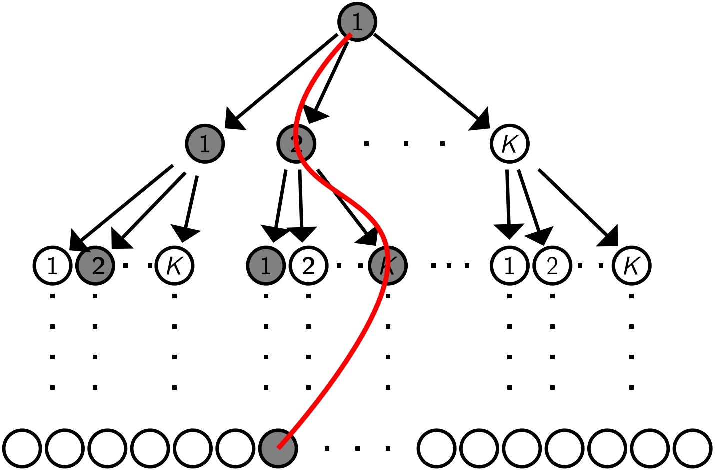

Iterative Extension

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_N \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{2^B} \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$$

- Contribution of columns identified by tree decoder as transmitted vectors is cancelled from the received signal.
- Subsequent iterations: Reduced sparsity sub-problems are solved followed by tree decoding.

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Expressions for $\mathbb{E}[L_i]$ and C

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- $C = K + \sum_{i=1}^{n-2}[(L_i+1)K]$
- $\mathbb{E}[C]$ can be computed using the expression for $\mathbb{E}[L_i]$

Performance Analysis

- $L_i | L_{i-1} \sim B((L_{i-1}+1)K-1, p_i), p_i = \frac{1}{2^{l_i}}, q_i = 1-p_i,$
- $G_{L_{n-1}}(z) = \mathbb{E}[z^{L_{n-1}}] = \sum_{k=0}^{K^{n-1}-1} P(L_{n-1}=k) z^k$

Probability of error for the tree decoder

$$P(L_{n-1} \geq 1) = 1 - G_{L_{n-1}}(0), \qquad \text{where}$$

$$G_{L_{n-1}}(z) = \prod_{i=0}^{n-2} f_{n-1-i}^{K-1}(z),$$

$$f_k(z) = \begin{cases} q_k + p_k f_{k+1}^{K}(z), & 1 \leq k \leq n-1 \\ z^{\frac{1}{K}}, & k = n, \end{cases}$$

Overall probability of error

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- p_{cs} : Probability of error of compressed sensor at sub-block level
- $P_e = 1 - (1 - P(L_{n-1} \geq 1))(1 - p_{cs})^n.$

Optimization of Parity Lengths

$$\begin{aligned} \underset{(l_1, l_2, \ldots l_{n-1})}{\text{minimize}} \quad & \mathbb{E}[C] \\ \text{subject to} \quad & P(L_{n-1} \geq 1) \leq \varepsilon_{\text{tree}}, \\ & \sum_{i=1}^{n-1} l_i = M - B, \\ & l_i \in \{0, 1, \ldots J\} \ \forall \ i \in [1 : n-1]. \end{aligned}$$

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- Geometric programming opt. problem
- Can be solved using any standard convex solver (ex. CVX).

Choice of Parity Lengths

$K = 200, n = 11, J = 15$

$\varepsilon_{\text{tree}}$	$\mathbb{E}[C]$	Parity Lengths
0.006	Infeasible	Infeasible
0.0061930	3.2357×10^{11}	[0, 0, 0, 0, 15, 15, 15, 15, 15, 15]
0.0061931	3357300	[0, 3, 8, 8, 8, 8, 10, 15, 15, 15]
0.0061932	1737000	[0, 4, 8, 8, 8, 8, 9, 15, 15, 15]
0.0061933	926990	[0, 5, 8, 8, 8, 8, 8, 15, 15, 15]
0.0061935	467060	[1, 8, 8, 8, 8, 8, 8, 11, 15, 15]
0.0062	79634	[1, 8, 8, 8, 8, 8, 8, 11, 15, 15]
0.007	7357.8	[6, 8, 8, 8, 8, 8, 8, 8, 13, 15]
0.008	6152.7	[7, 8, 8, 8, 8, 8, 8, 8, 12, 15]
0.02	5022.9	[6, 8, 8, 9, 9, 9, 9, 9, 9, 14]
0.04	4158	[7, 8, 8, 9, 9, 9, 9, 9, 9, 13]
0.6378	3066.3	[9, 9, 9, 9, 9, 9, 9, 9, 9, 9]

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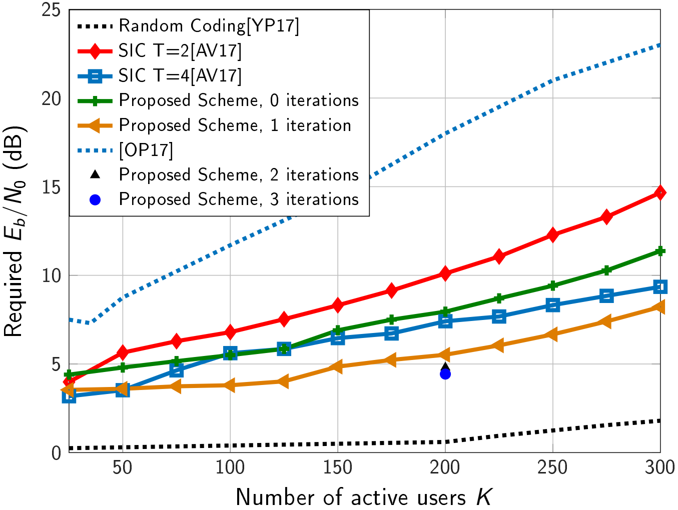
Decoding Algorithm

- Non-negative least squares
- Take top $K + 10$ elements



- $B = 75, N = 22517$
- Only 4.3 dB away from Polyanksiy's achievability result

Conclusion

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- Sub-linear time complexity
- Performance within $\approx$ 4.3dB from the random coding ach. limit

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Open Problems

- A strict FBL lower bound for GMAC?
- Design of optimal sensing matrix for the K-sparse CS problem
- Solve a generic CS problem of huge dimensions using this framework: Bounds for sample and computational complexities.

Questions?

Thank you!